Precise Definition: We say \( \lim_{x \to a} f(x) = L \) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that whenever \( 0 < |x - a| < \delta \) then \( |f(x) - L| < \varepsilon \).

“Working” Definition: We say \( \lim_{x \to a} f(x) = L \) if we can make \( f(x) \) as close to \( L \) as we want by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) without letting \( x = a \).

Right hand limit: \( \lim_{x \to a^+} f(x) = L \). This has the same definition as the limit except it requires \( x > a \).

Left hand limit: \( \lim_{x \to a^-} f(x) = L \). This has the same definition as the limit except it requires \( x < a \).

Relationship between the limit and one-sided limits
\[
\lim_{x \to a^-} f(x) = L \Rightarrow \lim_{x \to a} f(x) = L \quad \text{provided \( f(x) \) is continuous at \( a \)} \]
\[
\lim_{x \to a^+} f(x) = L \Rightarrow \lim_{x \to a} f(x) = L \quad \text{provided \( f(x) \) is continuous at \( a \)} \]

There is a similar definition for \( \lim_{x \to x_0} f(x) = L \) except we require \( x \) large or negative.

Infinite Limit: We say \( \lim_{x \to a} f(x) = \infty \) if we can make \( f(x) \) arbitrarily large (and positive) by taking \( x \) sufficiently close to \( a \) (on either side of \( a \)) without letting \( x = a \).

There is a similar definition for \( \lim_{x \to a} f(x) = -\infty \) except we make \( f(x) \) arbitrarily large and negative.

Basic Limit Evaluations at \( \pm \infty \)

Note: \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).

Calculus Techniques

L'Hospital's Rule
If \( \lim_{x \to a} f(x) = 0 \) or \( \lim_{x \to a} g(x) = 0 \) then,
\[
\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}
\]

Polynomials at Infinity
\( p(x) \) and \( q(x) \) are polynomials. To compute
\[
\lim_{x \to \pm \infty} \frac{p(x)}{q(x)} = \frac{\text{leading coefficient of } p(x)}{\text{leading coefficient of } q(x)}
\]

Piecewise Function
\[
\lim_{x \to a} f(x) = \begin{cases} \frac{x^2 + 5}{1 - 3x} & \text{if } x > -2 \\ \frac{x^2 - 9}{x - 2} & \text{if } x < 2 \end{cases}
\]

Intermediate Value Theorem
Suppose that \( f(x) \) is continuous on \([a, b]\) and let \( M \) be any number between \( f(a) \) and \( f(b) \).
Then there exists a number \( c \) such that \( a < c < b \) and \( f(c) = M \).

Some Continuous Functions

1. \( f(x) = x^a \) for all \( x \).
2. \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).
3. \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).
4. \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).
5. \( \text{sgn}(a) = 1 \) if \( a > 0 \) and \( \text{sgn}(a) = -1 \) if \( a < 0 \).