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Preface

Here are a set of problems for my Calculus II notes. These problems do not have any solutions available on this site. These are intended mostly for instructors who might want a set of problems to assign for turning in. I try to put up both practice problems (with solutions available) and these problems at the same time so that both will be available to anyone who wishes to use them.

Outline

Here is a list of sections for which problems have been written.

Integration Techniques
- Integration by Parts
- Integrals Involving Trig Functions
- Trig Substitutions
- Partial Fractions
- Integrals Involving Roots
- Integrals Involving Quadratics
- Using Integral Tables
- Integration Strategy
- Improper Integrals
- Comparison Test for Improper Integrals
- Approximating Definite Integrals

Applications of Integrals
- Arc Length
- Surface Area
- Center of Mass
- Hydrostatic Pressure and Force
- Probability

Parametric Equations and Polar Coordinates
- Parametric Equations and Curves
- Tangents with Parametric Equations
- Area with Parametric Equations
- Arc Length with Parametric Equations
- Surface Area with Parametric Equations
- Polar Coordinates
- Tangents with Polar Coordinates
- Area with Polar Coordinates
- Arc Length with Polar Coordinates
- Surface Area with Polar Coordinates
- Arc Length and Surface Area Revisited

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Sequences and Series

Sequences
More on Sequences
Series – The Basics
Series – Convergence/Divergence
Series – Special Series
Integral Test
Comparison Test/Limit Comparison Test
Alternating Series Test
Absolute Convergence
Ratio Test
Root Test
Strategy for Series
Estimating the Value of a Series
Power Series
Power Series and Functions
Taylor Series
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Binomial Series

Vectors

Vectors – The Basics
Vector Arithmetic
Dot Product
Cross Product

Three Dimensional Space

The 3-D Coordinate System
Equations of Lines
Equations of Planes
Quadric Surfaces
Functions of Several Variables
Vector Functions
Calculus with Vector Functions
Tangent, Normal and Binormal Vectors
Arc Length with Vector Functions
Curvature
Velocity and Acceleration
Cylindrical Coordinates
Spherical Coordinates
Integration Techniques

Introduction
Here are a set of problems for which no solutions are available. The main intent of these problems is to have a set of problems available for any instructors who are looking for some extra problems.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have problems written for them.

Integration by Parts
Integrals Involving Trig Functions
Trig Substitutions
Partial Fractions
Integrals Involving Roots
Integrals Involving Quadratics
Using Integral Tables
Integration Strategy
Improper Integrals
Comparison Test for Improper Integrals
Approximating Definite Integrals

Integration by Parts
Evaluate each of the following integrals.

1. $\int 8te^{7t} \, dt$

2. $\int_{2}^{3} (1 - 3x)\sin\left(\frac{1}{2}x\right) \, dx$

3. $\int_{-1}^{2} we^{4w} \, dw$

4. $\int_{1}^{3} (2 - x)^{2} \ln(4x) \, dx$

5. $\int (6 + 3z)\cos(1 + 4z) \, dz$
6. \( \int 2y^2 \cos(9y) \, dy \)

7. \( \int (3z + z^2) \sin(z) \, dz \)

8. \( \int \sqrt{x^3} \ln(\sqrt{x}) \, dx \)

9. \( \int (2w^2 - w)e^{7w-1} \, dw \)

10. \( \int 9t \sec^2(2t) \, dt \)

11. \( \int_{0}^{\pi/2} e^{-x} \sin(4x) \, dx \)

12. \( \int 8 \tan^{-1}(2y) \, dy \)

13. \( \int e^{6t} \cos(2t) \, dt \)

14. \( \int -3 \sin^{-1}(10x) \, dx \)

15. \( \int e^{3-z} \sin(2+z) \, dz \)

16. \( \int_{-1}^{0} 2x^7 e^{1+x^9} \, dx \)

17. \( \int 9t^{11} \cos(1-t^6) \, dt \)

18. \( \int \frac{x^7}{\sqrt{x^4 + 1}} \, dx \)

19. \( \int (5 + x^4) \sin\left(\frac{1}{2}x\right) \, dx \)

20. \( \int 2z^5 e^{1-z} \, dz \)

21. \( \int (5 + 2w^3 - w^4) \cos(3w) \, dw \)

---

**Integrals Involving Trig Functions**

Evaluate each of the following integrals.
1. \( \int \cos^5(2t) \sin^2(2t) \, dt \)
2. \( \int \cos^3(12x) \, dx \)
3. \( \int \cos^2(z) \sin^4(z) \, dz \)
4. \( \int_{\frac{\pi}{4}}^{\pi} \sin^5\left(\frac{\pi}{4}w\right) \cos^6\left(\frac{\pi}{4}w\right) \, dw \)
5. \( \int_{0}^{\pi} \cos^{11}(5z) \sin^3(5z) \, dz \)
6. \( \int \sin^2(7x) \, dx \)
7. \( \int_{0}^{\pi} \tan^3(8x) \sec^3(8x) \, dx \)
8. \( \int \sec^8\left(\frac{1}{2}t\right) \tan^5\left(\frac{1}{2}t\right) \, dt \)
9. \( \int \sec^2(9z) \tan^3(9z) \, dz \)
10. \( \int_{\frac{\pi}{4}}^{\pi} \sec^6(10t) \tan^4(10t) \, dt \)
11. \( \int \tan^{12}(2w) \sec^6(2w) \, dw \)
12. \( \int \cot^2(3x) \csc^6(3x) \, dx \)
13. \( \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^3\left(\frac{\pi}{4}w\right) \cot^3\left(\frac{\pi}{4}w\right) \, dw \)
14. \( \int \csc^4(6w) \, dw \)
15. \( \int \csc^{12}(x) \cot^5(x) \, dx \)
16. \( \int \cot(x) \, dx \)
17. \( \int \cot^3(x) \, dx \)
18. ∫\csc(x)\,dx
19. ∫csc^3(x)\,dx
20. ∫^4_2 \sin(8x)\cos(15x)\,dx
21. ∫\cos(2x)\cos(4x)\,dx
22. ∫\sin(13z)\sin(9z)\,dz
23. ∫\frac{\cos^5(2t)}{\sin^3(2t)}\,dt
24. ∫\frac{\sin^3(2-x)}{\cos^5(2-x)}\,dx
25. ∫\sec^6\left(\frac{1}{2}z\right)\,dz
26. ∫\frac{\tan^5(x)}{\sec^2(x)}\,dx
27. ∫\frac{1+9\cos^5(8w)}{\sin^2(8w)}\,dw
28. ∫(3+7\cos^3(x))\sin^2(x)\,dx
29. ∫\sin^3(9y)\sec^2(9y)\,dy
30. ∫\tan^5(z)\cos^5(z)\,dz
31. ∫\left[\tan(2t) - \sin^3(2t)\right]\sec^3(2t)\,dt

**Trig Substitutions**

For problems 1 – 15 use a trig substitution to eliminate the root.
Calculus II

1. $\sqrt{64t^2} + 1$
2. $\sqrt{4z^2} - 49$
3. $\sqrt{7 - w^2}$
4. $(16 - 81x^2)^{\frac{2}{3}}$
5. $\sqrt{6 + 9y^2}$
6. $(1 - 8z^2)^{\frac{1}{2}}$
7. $\sqrt{9 - 16(3x - 1)^2}$
8. $(11 + (t^2 + 1)^{\frac{2}{3}}$
9. $\sqrt{144(z + 8)^2 - 3}$
10. $\sqrt{4x^2 - 24x + 43}$
11. $(2z^2 - 24z + 36)^{\frac{1}{2}}$
12. $\sqrt{4 - 10t - 5t^2}$
13. $\sqrt{9\sin^2(4t) - 1}$
14. $\sqrt{36 - 9e^{3z}}$
15. $\sqrt{x + 16}$

For problems 16 – 42 use a trig substitution to evaluate the given integral.

16. $\int 3x^3\sqrt{16 - x^2} \, dx$
17. $\int t^3 \left(25 + 81t^2\right)^{\frac{5}{2}} \, dt$
18. \[ \int_{0}^{\frac{\pi}{2}} \frac{w^3}{\sqrt{1-9w^2}} \, dw \]

19. \[ \int \frac{z^5}{(9z^2 - 25)^\frac{3}{2}} \, dz \]

20. \[ \int_{-3}^{1} \sqrt{49y^2 - 4} \, dy \]

21. \[ \int_{1}^{5} \frac{5}{x^2\sqrt{x^2 + 4}} \, dx \]

22. \[ \int \frac{\sqrt{3 - 4t^2}}{t^2} \, dt \]

23. \[ \int \frac{w^5}{\sqrt{8w^2 + 1}} \, dw \]

24. \[ \int \frac{\sqrt{x^2 - 15}}{x^3} \, dx \]

25. \[ \int \frac{2}{(x-3)^6 \sqrt{-x^2 + 6x - 5}} \, dx \]

26. \[ \int \frac{1}{(z+1)^2 \left(2z^2 + 4z - 34\right)^\frac{3}{2}} \, dz \]

27. \[ \int \frac{\sqrt{4y^2 - 16y + 19}}{(y-2)^6} \, dy \]

28. \[ \int_{9}^{12} \frac{(t-4)^3}{\sqrt{t^2 - 8t + 7}} \, dt \]

29. \[ \int_{0}^{6} \sqrt{5x^2 + 10x + 6} \, dx \]

30. \[ \int x^7 \sqrt{9 - x^4} \, dx \]
41. \[ \int \frac{e^{12t}}{\sqrt{4e^{6t} - 1}} \, dt \]

42. \[ \int \sin(z) \cos^3(z) \sqrt{16 + \cos^2(z)} \, dz \]

**Partial Fractions**

Evaluate each of the following integrals.

1. \[ \int \frac{9}{z^2 - 12z} \, dz \]

2. \[ \int \frac{7x}{x^2 + 14x + 40} \, dx \]

3. \[ \int_{0}^{4} \frac{8y - 1}{2y^2 - 15y - 8} \, dy \]

4. \[ \int \frac{9 - w^2}{(w + 1)(3w - 5)(w + 4)} \, dw \]

5. \[ \int_{1}^{8} \frac{12}{z^3 - 2z^2 - 63z} \, dz \]

6. \[ \int \frac{7x + 2x^2}{(x - 4)(2x + 3)(2x + 1)} \, dx \]

7. \[ \int \frac{4x + 10}{(x - 2)(x - 1)^2} \, dx \]

8. \[ \int_{1}^{2} \frac{24}{t^4 - 6t^3} \, dt \]

9. \[ \int \frac{10z + 2}{(z + 1)^2(z - 3)^2} \, dz \]

10. \[ \int \frac{8w + w^2}{(w - 7)(w^2 + 16)} \, dw \]
11. \[ \int \frac{6y - 7}{(2y+1)(4y^2+1)} \, dy \]

12. \[ \int \frac{8t^3 - 5t^2 + 72t - 10}{(t^2+2)(t^2+9)} \, dt \]

13. \[ \int \frac{16w^3 + 6w^2 + 12w + 21}{(w^2+9)(4w^2+3)} \, dw \]

14. \[ \int \frac{x^4 + 5x^3 + 20x + 16}{x(x^2 + 4)^2} \, dx \]

15. \[ \int \frac{6 - z^2}{2z^2 + z - 21} \, dz \]

16. \[ \int \frac{4x^3 - x}{x^2 - x - 30} \, dx \]

17. \[ \int \frac{8 - t^3}{(t - 3)(t+1)^2} \, dt \]

18. \[ \int \frac{x^6 - 6x^5 + 3x^4 - 10x^3 - 9x^2 + 12x - 27}{x^4 + 3x^2} \, dx \]

**Integrals Involving Roots**

Evaluate each of the following integrals.

1. \[ \int \frac{5}{4 - \sqrt{3} - z} \, dz \]

2. \[ \int \frac{1}{x + 4\sqrt{x - 3}} \, dx \]

3. \[ \int \frac{4}{t - \sqrt{t^2 + 7} + 1} \, dt \]
Integrals Involving Quadratics

Evaluate each of the following integrals.

1. \( \int \frac{x+1}{x^2+10x+18} \, dx \)
2. \( \int \frac{15}{3y^2-4y+4} \, dy \)
3. \( \int \frac{12-9z}{1-4z-4z^2} \, dz \)
4. \( \int \frac{3-7t}{(t^2+12t+40)^2} \, dt \)
5. \( \int \frac{11w+4}{(3+6w-w^2)^2} \, dw \)
6. \( \int \frac{3}{(2x^2+10x+4)^2} \, dx \)

Integration Strategy

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of “new” problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.
**Improper Integrals**

Determine if each of the following integrals converge or diverge. If the integral converges determine its value.

1. \[ \int_{4}^{\infty} 2 - 4x + 6x^2 \, dx \]

2. \[ \int_{0}^{5} \frac{1}{4w - 20} \, dw \]

3. \[ \int_{-1}^{2} \frac{3}{\sqrt{4 - 2z}} \, dz \]

4. \[ \int_{-\infty}^{0} x e^{2+3x} \, dx \]

5. \[ \int_{0}^{\infty} x e^{2+3x} \, dx \]

6. \[ \int_{2}^{\infty} \frac{1}{x^2 + 1} \, dx \]

7. \[ \int_{0}^{3} \frac{1}{z^2 - 4z} \, dz \]

8. \[ \int_{-\infty}^{1} \frac{x}{x^2 + 1} \, dx \]

9. \[ \int_{-1}^{2} \frac{1}{y^2 - 2y - 3} \, dy \]

10. \[ \int_{-\infty}^{0} \cos(w) \, dw \]

11. \[ \int_{10}^{\infty} \frac{1}{(5 - 2z)^2} \, dz \]

12. \[ \int_{-\infty}^{\infty} \frac{z^3}{z^4 + 1} \, dz \]
13. \[ \int_4^5 \frac{1}{6y-2} \, dy \]

14. \[ \int_1^5 \frac{1}{\sqrt[3]{w-2}} \, dw \]

15. \[ \int_{-2}^1 \frac{e^x}{x^2} \, dx \]

16. \[ \int_{-\infty}^\infty x^2 e^{-x^3} \, dx \]

17. \[ \int_{-\infty}^{\infty} \frac{y}{y^2 + 1} \, dy \]

18. \[ \int_0^{\frac{3}{2}} \frac{w^3}{\sqrt{9-w^3}} \, dw \]

19. \[ \int_{-3}^{1} \frac{1}{w^2 + 2w} \, dw \]

20. \[ \int_0^{\infty} \frac{e^x}{x^2} \, dx \]

21. \[ \int_0^{\infty} \frac{1}{z \left[ \ln(z) \right]^2} \, dz \]

22. \[ \int_0^{\infty} \frac{1}{w-1} \, dw \]

**Comparison Test for Improper Integrals**

Use the Comparison Test to determine if the following integrals converge or diverge.

1. \[ \int_4^\infty \frac{1}{\sqrt[5]{z-2}} \, dz \]
2. \[ \int_{0}^{\infty} \frac{w}{\sqrt{w^3 + 2}} \, dw \]

3. \[ \int_{2}^{\infty} \frac{1}{(2w+3)^4} \, dw \]

4. \[ \int_{12}^{\infty} \frac{y^2 - 4y + 2}{y-7} \, dy \]

5. \[ \int_{2}^{\infty} \frac{1}{\ln(x)} \, dx \quad \text{Hint: Sketch the graph of } y = x \text{ and } y = \ln(x) \text{ on the same axis system.} \]

6. \[ \int_{2}^{\infty} \frac{\sqrt{z} - 4\sin^2(z)}{z^3} \, dz \]

7. \[ \int_{20}^{\infty} \frac{\sqrt{2x} + \sin^2(x)}{\sqrt{x} - \cos^2(x)} \, dx \]

8. \[ \int_{0}^{\infty} \frac{ze^{-z}}{z^3 + 1} \, dz \]

**Approximating Definite Integrals**

For each of the following integrals use the given value of \( n \) to approximate the value of the definite integral using:

(a) the Midpoint Rule,
(b) the Trapezoid Rule, and
(c) Simpson’s Rule.

Use at least 6 decimal places of accuracy for your work.

1. \[ \int_{-2}^{4} \sin\left(x^2 + 2\right) \, dx \quad \text{using } n = 6 \]

2. \[ \int_{0}^{4} \sqrt{x^4 + 6} \, dx \quad \text{using } n = 6 \]

3. \[ \int_{1}^{5} e^{\cos(x)} \, dx \quad \text{using } n = 8 \]
4. \[ \int_{3}^{5} \frac{1}{1 - \ln(x)} \, dx \] using \( n = 6 \)

5. \[ \int_{-3}^{1} \sin(x) \cos(x^2) \, dx \] using \( n = 8 \)

**Applications of Integrals**

**Introduction**

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Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have problems written for them.

- Arc Length
- Surface Area
- Center of Mass
- Hydrostatic Pressure and Force
- Probability

**Arc Length**

1. Set up, but do not evaluate, an integral for the length of \( y = 14 - 9x \), \(-22 \leq y \leq 31\) using,

   \[ (a) \, ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]

   \[ (b) \, ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \]

2. Set up, but do not evaluate, an integral for the length of \( x = e^{2y} \), \(-1 \leq y \leq 0\) using,

   \[ (a) \, ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \]
3. Set up, but do not evaluate, an integral for the length of \( y = \tan(2x) \), \( 0 \leq x \leq \frac{\pi}{3} \) using,

\[
\text{ (a) } ds = \sqrt{1 + \left[ \frac{dy}{dx} \right]^2} \, dx
\]

\[
\text{ (b) } ds = \sqrt{1 + \left[ \frac{dx}{dy} \right]^2} \, dy
\]

4. Set up, but do not evaluate, an integral for the length of \( \frac{x^2}{16} + 9y^2 = 1 \).

5. For \( x = 6y + 1 \), \( -2 \leq y \leq 8 \)

\( \text{ (a) } \) Use an integral to find the length of the curve.

\( \text{ (b) } \) Verify your answer from part (a) geometrically.

6. Determine the length of \( y = \frac{1}{4}x + 2 \), \( 0 \leq x \leq 9 \).

7. Determine the length of \( y = (8x + 3)^{\frac{3}{2}} \), \( 11^{\frac{3}{2}} \leq y \leq 27^{\frac{3}{2}} \).

8. Determine the length of \( x = (10 - 2y)^{\frac{3}{2}} \), \( -1 \leq y \leq 2 \).

9. Determine the length of \( x = 2 + (y - 1)^{\frac{3}{2}} \), \( 2 \leq y \leq 5 \).

10. Determine the length of \( y = (3x + 2)^{\frac{3}{2}} \), \( 1 \leq x \leq 4 \).

**Surface Area**

1. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating \( y = 7x + 2 \), \( -5 \leq y \leq 0 \) about the x-axis using,

\[
\text{ (a) } ds = \sqrt{1 + \left[ \frac{dy}{dx} \right]^2} \, dx
\]

\[
\text{ (b) } ds = \sqrt{1 + \left[ \frac{dx}{dy} \right]^2} \, dy
\]
2. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating 
\( y = 1 + 2x^3 \), \( 0 \leq x \leq 1 \) about the \( x \)-axis using,

(a) \( ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)

(b) \( ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \)

3. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating 
\( x = e^{2y} \), \(-1 \leq y \leq 0\) about the \( y \)-axis using,

(a) \( ds = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \)

(b) \( ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy \)

4. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating 
\( y = \cos\left(\frac{1}{2}x\right) \), \( 0 \leq x \leq \pi \) about

(a) the \( x \)-axis

(b) the \( y \)-axis.

5. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating 
\( x = \sqrt{3+7y} \), \( 0 \leq y \leq 1 \) about

(a) the \( x \)-axis

(b) the \( y \)-axis.

6. Find the surface area of the object obtained by rotating \( y = \frac{1}{4}\sqrt{6x+2} \), \( \frac{\sqrt{2}}{2} \leq y \leq \frac{\sqrt{2}}{2} \) about the \( x \)-axis.

7. Find the surface area of the object obtained by rotating \( y = 4-x \), \( 1 \leq x \leq 6 \) about the \( y \)-axis.

8. Find the surface area of the object obtained by rotating \( x = 2y^2 + 5 \), \(-1 \leq x \leq 2 \) about the \( y \)-axis.

9. Find the surface area of the object obtained by rotating \( x = 1 - y^2 \), \( 0 \leq y \leq 3 \) about the \( x \)-axis.

10. Find the surface area of the object obtained by rotating \( x = e^{2y} \), \(-1 \leq y \leq 0 \) about the \( y \)-axis.

11. Find for the surface area of the object obtained by rotating \( y = \cos\left(\frac{1}{2}x\right) \), \( 0 \leq x \leq \pi \) about the \( x \)-axis.
**Center of Mass**

Find the center of mass for each of the following regions.

1. The region bounded by \( y = x^3 \), \( x = -2 \) and the \( x \)-axis.

2. The triangle with vertices (-2, -2), (4, -2) and (4,4).

3. The region bounded by \( y = (x - 2)^3 \) and \( y = 4 \).

4. The region bounded by \( y = \cos(x) \) and the \( x \)-axis between \( -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \).

5. The region bounded by \( y = x^2 \) and \( y = x - 6 \).

6. The region bounded by \( y = e^{2x} \) and the \( x \)-axis between \(-1 \leq x \leq 1 \).

7. The region bounded by \( y = e^{2x} \) and \( y = -\cos(\pi x) \) between \(-\frac{1}{2} \leq x \leq \frac{1}{2} \).

**Hydrostatic Pressure and Force**

Find the hydrostatic force on the following plates submerged in water as shown in each image. In each case consider the top of the blue “box” to be the surface of the water in which the plate is submerged. Note as well that the dimensions in many of the images will not be perfectly to scale in order to better fit the plate in the image. The lengths given in each image are in meters.

1. 

2.
1. Let,

\[ f(x) = \begin{cases} 
\frac{3}{4}(2x-x^2) & \text{if } 0 \leq x \leq 2 \\
0 & \text{otherwise}
\end{cases} \]

(a) Show that \( f(x) \) is a probability density function.
(b) Find \( P(X \leq 0.25) \).
(c) Find \( P(X \geq 1.4) \).
(d) Find \( P(0.1 \leq X \leq 1.2) \).
(e) Determine the mean value of \( X \).

2. Let,

\[ f(x) = \begin{cases} 
\frac{4}{\ln(3)(4x+x^2)} & \text{if } 1 \leq x \leq 6 \\
0 & \text{otherwise}
\end{cases} \]

(a) Show that \( f(x) \) is a probability density function.
(b) Find \( P(X \leq 1) \).
(c) Find \( P(X \geq 5) \).
(d) Find \( P(1 \leq X \leq 5) \).
(e) Determine the mean value of \( X \).

3. Let,

\[ f(x) = \begin{cases} 
\frac{1}{10}\left(1+\sin\left(\frac{\pi x}{2}\right)\right) & \text{if } 0 \leq x \leq 10 \\
0 & \text{otherwise}
\end{cases} \]

(a) Show that \( f(x) \) is a probability density function.
(b) Find \( P(X \leq 3) \).
(c) Find \( P(X \geq 5) \).
(d) Find \( P(2.5 \leq X \leq 7) \).
(e) Determine the mean value of \( X \).
4. The probability density function of the life span of a battery is given by the function below, where \( t \) is in years.

\[
f(t) = \begin{cases} 
1.25e^{-0.8t} & \text{if } t \geq 0 \\
0 & \text{if } t < 0 
\end{cases}
\]

(a) Verify that \( f(t) \) is a probability density function.

(b) What is the probability that a battery will have a life span less than 10 months?

(c) What is the probability that a battery will have a life span more than 2 years?

(d) What is the probability that a battery will have a life span between 1.5 and 4 years?

(e) Determine the mean value of the life span of the batteries.

5. The probability density function of the successful outcome from some experiment is given by the function below, where \( t \) is in minutes.

\[
f(t) = \begin{cases} 
36te^{-6t} & \text{if } t \geq 0 \\
0 & \text{if } t < 0 
\end{cases}
\]

(a) Verify that \( f(t) \) is a probability density function.

(b) What is the probability of a successful outcome happening in less than 12 minutes?

(c) What is the probability of a successful outcome happening after 25 minutes?

(d) What is the probability of a successful outcome happening between 10 and 75 minutes?

(e) What is the mean time of a successful outcome from the experiment?

6. Determine the value of \( c \) for which the function below will be a probability density function.

\[
f(x) = \begin{cases} 
c \left(12x^4 - x^5\right) & \text{if } 0 \leq x \leq 12 \\
0 & \text{otherwise} 
\end{cases}
\]

7. Use the function below for this problem.

\[
f(x) = \begin{cases} 
c e^{-\frac{1}{a}x} & \text{if } x \geq 0 \\
0 & \text{if } x < 0 
\end{cases}
\]

(a) Determine the value of \( c \) for which this function will be a probability density function.

(b) Using the value of \( c \) found in the first part determine the mean value of the probability density function.
**Introduction**

Here are a set of problems for which no solutions are available. The main intent of these problems is to have a set of problems available for any instructors who are looking for some extra problems.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have problems written for them.

- Parametric Equations and Curves
- Tangents with Parametric Equations
- Arc Length with Parametric Equations
- Area with Parametric Equations
- Surface Area with Parametric Equations
- Polar Coordinates
- Tangents with Polar Coordinates
- Arc Length with Polar Coordinates
- Area with Polar Coordinates
- Surface Area with Polar Coordinates
- Arc Length and Surface Area Revisited

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**Parametric Equations and Curves**

For problems 1 – 9 eliminate the parameter for the given set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on $x$ and $y$.

1. $x = 2 + t$ \quad $y = t^2 - 4t + 7$

2. $x = 2 + t$ \quad $y = t^2 - 4t + 7$ \quad $-3 \leq t \leq 1$

3. $x = 1 - t^2$ \quad $y = 3 + 2t$

4. $x = 1 - t^2$ \quad $y = 3 + 2t$ \quad $-2 \leq t \leq 3$

5. $x = \frac{1}{3}\sqrt{t}$ \quad $y = t - \sqrt{t} - 6$ \quad $t \geq 0$

6. $x = \frac{1}{3}\sqrt{t}$ \quad $y = t - \sqrt{t} - 6$ \quad $8 \leq t \leq 20$

7. \( x = -6 \cos(4t) \quad y = 2 \sin(4t) \quad -\frac{\pi}{8} \leq t \leq \frac{\pi}{8} \)

8. \( x = 1 - 3 \sin\left(\frac{1}{3}t\right) \quad y = 4 \cos\left(\frac{1}{3}t\right) \)

9. \( x = 6 - 7e^{-2t} \quad y = 4 + 3e^{-2t} \)

10. Answer each of the questions about the following set of parametric equations

    \( x = 3 \cos(at) \quad y = 3 \sin(at) \quad 0 \leq t \leq 2\pi \)

    (a) Sketch the graph of the parametric curve for \( a = 1 \).
    (b) Sketch the graph of the parametric curve for \( a = 6 \).
    (c) Sketch the graph of the parametric curve for \( a = \frac{1}{3} \).
    (d) In general, for \( a > 0 \), how does the value of \( a \) affect the graph of the parametric curve?

For problems 11 – 21 the path of a particle is given by the set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.
(ii) Limits on \( x \) and \( y \).
(iii) A range of \( t \)'s for a single trace of the parametric curve.
(iv) The number of traces of the curve the particle makes if an overall range of \( t \)'s is provided in the problem.

11. \( x = 6 \cos\left(\frac{1}{3}t\right) \quad y = 2 + \sin\left(\frac{1}{3}t\right) \quad 0 \leq t \leq 75\pi \)

12. \( x = 7 - 3 \sin(2t) \quad y = 4 + 2 \cos(2t) \)

13. \( x = 6 \cos^2(3t) \quad y = 2 - 3 \sin^2(3t) \quad -\frac{\pi}{6} \leq t \leq 3\pi \)

14. \( x = \sqrt{2 + \cos^2\left(\frac{1}{3}t\right)} \quad y = \frac{1}{3} \sin\left(\frac{1}{3}t\right) \)

15. \( x = 6 - \sin^3(4t) \quad y = 2 \sin(4t) \quad -127\pi \leq t \leq 201\pi \)

16. \( x = 3 + \cos\left(\frac{1}{6}t\right) \quad y = 4 + \cos^2\left(\frac{1}{6}t\right) \quad -90\pi \leq t \leq 216\pi \)

17. \( x = e^{-4t} \quad y = 2e^{12t} \)

18. \( x = 1 + e^{3t} \quad y = e^{6t} \quad -1 \leq t \leq 6 \)

19. \( x = 1 - \ln(t) \quad y = \left[\ln(t)\right]^2 \quad t > 0 \)
20. \( x = \cos\left(\frac{1}{2}t\right) \quad y = \sec\left(\frac{1}{2}t\right) \quad -\pi < t < \pi \)

21. \( x = \sin(2t) \quad y = \sin^2(2t) - 4\sin(2t) \quad -\frac{\pi}{2} \leq t \leq \frac{\pi}{4} \)

For problems 22 – 27 write down a set of parametric equations for the given equation that meets the given extra conditions (if any).

22. \( x = \sin(3 - x^2) + \cos^2(x) \)

23. \( y = \frac{6\cos(x) - 8}{x^2 + 9x} \)

24. \( x^2 + y^2 = 100 \) and the parametric curve resulting from the parametric equations should be at \((0, 10)\) when \( t = 0 \) and the curve should have a clockwise rotation.

25. \( x^2 + y^2 = 100 \) and the parametric curve resulting from the parametric equations should be at \((0, 10)\) when \( t = 0 \) and the curve should have a counter clockwise rotation.

26. \( \frac{x^2}{25} + y^2 = 1 \) and the parametric curve resulting from the parametric equations should be at \((-5, 0)\) when \( t = 0 \) and the curve should have a counter clockwise rotation.

27. \( \frac{x^2}{25} + y^2 = 1 \) and the parametric curve resulting from the parametric equations should be at \((-5, 0)\) when \( t = 0 \) and the curve should have a clockwise rotation.

28. Eliminate the parameter for the following set of parametric equations and identify the resulting equation.

\[
\begin{align*}
\frac{dy}{dx} & = h + a \cos(\omega t) = x_k = b \sin(\omega t) \\
& \frac{d^2y}{dx^2} 
\end{align*}
\]

**Tangents with Parametric Equations**

For problems 1 – 3 compute \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) for the given set of parametric equations.

1. \( x = 7t^2 - 9t \quad y = t^6 + 2t^2 \)

2. \( x = \tan(2t) - 12 \quad y = 3\sin(2t) + \sec(2t) + 4t \)
3. \( x = \ln \left( 3t^2 \right) + 8t \) \quad \( y = \ln \left( t^4 \right) - 6 \ln \left( t^2 \right) \)

For problems 4 – 7 find the equation of the tangent line(s) to the given set of parametric equations at the given point.

4. \( x = t^3 + \cos \left( \pi t \right) \) \quad \( y = 4t + \sin \left( 2t + 6 \right) \) at \( t = -3 \)

5. \( x = t^2 + 2t - 1 \) \quad \( y = t^3 + 7t^2 + 8t \) at \( t = 1 \)

6. \( x = 6 - e^{t^2 - 9t^2} \) \quad \( y = t^3 + 3t^2 - 18t + 2 \) at \( (5, 2) \)

7. \( x = 6 \sin \left( \frac{\pi}{2} t \right) \) \quad \( y = t^2 + 2t - 8 \) at \( (-6, 7) \)

For problems 8 and 9 find the values of \( t \) that will have horizontal or vertical tangent lines for the given set of parametric equations.

8. \( x = t^3 - 5t^2 + t + 1 \) \quad \( y = t^4 + 8t^3 + 3t^2 \)

9. \( x = 7t^2 + e^{2-t^2} \) \quad \( y = 10 \sin \left( \frac{1}{2} t \right) - 1 \)

**Area with Parametric Equations**

For problems 1 – 3 determine the area of the region below the parametric curve given by the set of parametric equations. For each problem you may assume that each curve traces out exactly once from right to left for the given range of \( t \). For these problems you should only use the given parametric equations to determine the answer.

1. \( x = t^2 + 5t - 1 \) \quad \( y = 40 - t^2 \) \quad \(-2 \leq t \leq 5 \)

2. \( x = 3 \cos^2 \left( t \right) - \sin^2 \left( t \right) \) \quad \( y = 6 + \cos \left( t \right) \) \quad \(-\frac{\pi}{2} \leq t \leq 0 \)

3. \( x = e^{4t} - 2 \) \quad \( y = 4 + e^{2t} - e^{4t} \) \quad \(-6 \leq t \leq 1 \)

**Arc Length with Parametric Equations**

For all the problems in this section you should only use the given parametric equations to determine the answer.

For problems 1 – 5 determine the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of \( t \)’s.
1. \[ x = 3 + 9t \quad y = 10 - 15t \quad -5 \leq t \leq 8 \]

2. \[ x = 6(3 + t)^{\frac{3}{2}} \quad y = -3t^{\frac{3}{2}} \quad -2 \leq t \leq 1 \]

3. \[ x = 4t^2 - 3 \quad y = 3t \quad 0 \leq t \leq 5 \]

4. \[ x = 3 + t \quad y = 6 + (t - 1)^2 \quad 1 \leq t \leq 3 \]

5. \[ x = t^2 - 1 \quad y = t^4 + 5 \quad 0 \leq t \leq 1 \]

For problems 6 and 7 a particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

6. \[ x = 6 \cos^2(3t) \quad y = 2 - 3 \sin^2(3t) \quad -\frac{31\pi}{18} \leq t \leq \frac{43\pi}{36} \]

7. \[ x = 3 + \cos\left(\frac{1}{6}t\right) \quad y = 4 + \cos^2\left(\frac{1}{6}t\right) \quad -\frac{31\pi}{3} \leq t \leq \frac{17\pi}{12} \]

For problems 8 – 10 set up, but do not evaluate, an integral that gives the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of \( t \)’s.

8. \[ x = t \cos(2t) \quad y = \sin(3t) \quad 2 \leq t \leq 3 \]

9. \[ x = 1 - \sin\left(1 + \sqrt{t}\right) \quad y = \sin\left(e^{-t}\right) \quad 1 \leq t \leq 4 \]

10. \[ x = \ln(t + 2) \quad y = \frac{1}{t + 7} \quad -1 \leq t \leq 2 \]

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**Surface Area with Parametric Equations**

For all the problems in this section you should only use the given parametric equations to determine the answer.

For problems 1 – 4 determine the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of \( t \)’s.

1. Rotate \( x = t^2 - 3 \quad y = 2 + t^2 \quad 0 \leq t \leq 5 \) about the \( x \)-axis.

2. Rotate \( x = -8t \quad y = 6 + t^2 \quad -3 \leq t \leq 0 \) about the \( y \)-axis.
3. Rotate \( x = t^2 \quad y = t^4 - 2 \quad 0 \leq t \leq 2 \) about the \( y \)-axis.

4. Rotate \( x = 2 + t \quad y = 4e^{-\frac{1}{2}t} \quad -1 \leq t \leq 2 \) about the \( x \)-axis.

For problems 5 – 7 set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of \( t \)'s.

5. Rotate \( x = 2 + e^{\cos(t)} \quad y = 1 + r^2 \quad -2 \leq t \leq 0 \) about the \( x \)-axis.

6. Rotate \( x = \cos^2(t) \quad y = 2\cos(2t) - \sin(t) \quad 0 \leq t \leq 1 \) about the \( y \)-axis.

7. Rotate \( x = t^2 \quad y = \ln\left(3 + e^{-t}\right) \quad 0 \leq t \leq 2 \) about the \( x \)-axis.

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**Polar Coordinates**

1. For the point with polar coordinates \((-9, \frac{3\pi}{4})\) determine three different sets of coordinates for the same point all of which have angles different from \(-\frac{3\pi}{4}\) and are in the range \(-2\pi \leq \theta \leq 2\pi\).

2. For the point with polar coordinates \((7, -\frac{2\pi}{3})\) determine three different sets of coordinates for the same point all of which have angles different from \(\frac{3\pi}{2}\) and are in the range \(-2\pi \leq \theta \leq 2\pi\).

3. The polar coordinates of a point are \((14, 2.48)\). Determine the Cartesian coordinates for the point.

4. The polar coordinates of a point are \((-\frac{3}{10}, -5.29)\). Determine the Cartesian coordinates for the point.

5. The Cartesian coordinate of a point are \((-3, 5)\). Determine a set of polar coordinates for the point.

6. The Cartesian coordinate of a point are \((4, -7)\). Determine a set of polar coordinates for the point.

7. The Cartesian coordinate of a point are \((-3, -12)\). Determine a set of polar coordinates for the point.

For problems 8 and 9 convert the given equation into an equation in terms of polar coordinates.

8. \(7x^2y + 8y = 3 - 6x^2 - 6y^2\)
9. \( \frac{7y}{x^2 + y^2 - 8x} = 9 + y^2 \)

For problems 10 – 12 convert the given equation into an equation in terms of Cartesian coordinates.

10. \( r - \frac{8\sin \theta}{r} = 2\cos \theta \)

11. \( r^3 \csc \theta = 5\cos \theta - 6 \)

12. \( 8 - r = r^2 \sin(2\theta) \)

13. \( r = 2a\cos \theta + 2b\sin \theta \)

For problems 14 – 27 sketch the graph of the given polar equation.

14. \( -7 = r \sin \theta \)

15. \( \theta = \frac{5\pi}{7} \)

16. \( \theta = -\frac{9\pi}{5} \)

17. \( r \cos \theta = 4 \)

18. \( r = 6\sin \theta \)

19. \( r = 100 \)

20. \( r = 24\cos \theta \)

21. \( r = -15\sin \theta \)

22. \( r = 4 + 12\cos \theta \)

23. \( r = 7 - 7\sin \theta \)

24. \( r = 1 + 3\sin \theta \)

25. \( r = 5 - 4\cos \theta \)

26. \( r = 8 + 3\sin \theta \)
27. $r = 1 - \cos \theta$

**Tangents with Polar Coordinates**

1. Find the tangent line to $r = \theta \sin (3\theta)$ at $\theta = \frac{\pi}{2}$.

2. Find the tangent line to $r = \cos (2\theta) - \sin (\theta)$ at $\theta = -\frac{\pi}{4}$.

3. Find the tangent line to $r = \cos (\theta^2 - \theta)$ at $\theta = \pi$.

**Area with Polar Coordinates**

1. Find the area inside the inner loop of $r = 3 + 10 \sin \theta$.

2. Find the area inside the inner loop of $r = 5 + 12 \cos \theta$.

3. Find the area inside the graph of $r = 8 + \cos \theta$ and to the right of the $y$-axis.

4. Find the area inside the graph of $r = 5 - 4 \sin \theta$ and the below the $x$-axis.

5. Find the area that is inside $r = 4$ and outside $r = 4 - 2 \sin \theta$.

6. Find the area that is inside $r = 7 - 3 \cos \theta$ and outside $r = 4$.

7. Find the area that is inside $r = 6 + 6 \cos \theta$ and outside $r = 4 - 3 \cos \theta$.

8. Find the area that is inside $r = 4 + 2 \sin \theta$ and outside $r = 5 - \sin \theta$.

9. Find the area that is inside $r = 5 - \sin \theta$ and outside $r = 4 + 2 \sin \theta$.

10. Find the area that is inside both $r = 6 - 4 \sin \theta$ and $r = 5$.

11. Find the area that is inside both $r = 3 + 2 \cos \theta$ and $r = 3 - \cos \theta$.

**Arc Length with Polar Coordinates**

For problems 1 – 3 determine the length of the given polar curve. For these problems you may assume that the curve traces out exactly once for the given range of $\theta$. 

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1. \( r = \frac{1}{\cos \theta} \), \( 0 \leq \theta \leq \frac{\pi}{3} \)

2. \( r = \theta^2 \), \( 0 \leq \theta \leq 3\pi \)

3. \( r = 6 \cos \theta - 3 \sin \theta \), \( 0 \leq \theta \leq \pi \)

For problems 4 – 6 set up, but do not evaluate, an integral that gives the length of the given polar curve. For these problems you may assume that the curve traces out exactly once for the given range of \( \theta \).

4. \( r = \sin(\theta^2) \), \( 0 \leq \theta \leq \pi \)

5. \( r = \cos(1 + \sin \theta) \), \( 0 \leq \theta \leq 2\pi \)

6. \( r = e^{-\frac{\theta}{4}} \cos \theta \), \( 0 \leq \theta \leq 3\pi \)

**Surface Area with Polar Coordinates**

For problems 1 – 4 set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of \( \theta \).

1. \( r = \cos\left(e^{-\frac{\theta}{4}}\right) \), \( 0 \leq \theta \leq \frac{\pi}{2} \) rotated about the \( x \)-axis.

2. \( r = \theta \sin \theta \), \( 0 \leq \theta \leq \frac{\pi}{2} \) rotated about the \( y \)-axis.

3. \( r = \cos(\theta) \sin(2\theta) \), \( 0 \leq \theta \leq \frac{\pi}{6} \) rotated about the \( x \)-axis.

4. \( r = \theta + \sin \theta \), \( \frac{\pi}{2} \leq \theta \leq \pi \) rotated about the \( y \)-axis.

**Arc Length and Surface Area Revisited**

Problems have not yet been written for this section and probably won’t be to be honest since this is just a summary section.
Introduction

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Sequences
More on Sequences
Series – The Basics
Series – Convergence/Divergence
Series – Special Series
Integral Test
Comparison Test/Limit Comparison Test
Alternating Series Test
Absolute Convergence
Ratio Test
Root Test
Strategy for Series
Estimating the Value of a Series
Power Series
Power Series and Functions
Taylor Series
Applications of Series
Binomial Series

Sequences

For problems 1 – 3 list the first 5 terms of the sequence.

1. \( \left\{ (-1)^n e^{2n} \right\}_{n=1}^{\infty} \)

2. \( \left\{ \frac{6-8n}{n^2+9n} \right\}_{n=0}^{\infty} \)
For problems 4 – 10 determine if the given sequence converges or diverges. If it converges what is its limit?

4. \[ \left\{ \frac{5 + n^3}{2n^2 - 8n + 1} \right\}_{n=0}^{\infty} \]

5. \[ \left\{ \frac{6n^4 + 9n^2}{9n^4 - 8n^2 + 7} \right\}_{n=1}^{\infty} \]

6. \[ \left\{ \frac{(-1)^{n+7}(2 - 8n)}{n^2 + 9} \right\}_{n=2}^{\infty} \]

7. \( \{ \cos(n\pi) \}_{n=0}^{\infty} \)

8. \[ \left\{ \frac{n+1}{\ln(6n)} \right\}_{n=2}^{\infty} \]

9. \[ \left\{ \cos \left( \frac{3}{n+1} \right) \right\}_{n=1}^{\infty} \]

10. \[ \left\{ \ln(4n+1) - \ln(2+7n) \right\}_{n=0}^{\infty} \]

**More on Sequences**

For each of the following problems determine if the sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

1. \[ \left\{ \frac{1}{n^3 + 1} \right\}_{n=1}^{\infty} \]

2. \[ \left\{ e^{3n} \right\}_{n=0}^{\infty} \]
3. \( \left\{ (-3)^n \right\}_{n=0}^{\infty} \)

4. \( \left\{ \sin(n) \right\}_{n=4}^{\infty} \)

5. \( \left\{ \ln\left( \frac{1}{n} \right) \right\}_{n=2}^{\infty} \)

4. \( \left\{ \frac{3-n}{1-3n} \right\}_{n=1}^{\infty} \)

5. \( \left\{ \frac{2n+1}{4n+3} \right\}_{n=0}^{\infty} \)

6. \( \left\{ (1-n)e^n \right\}_{n=3}^{\infty} \)

7. \( \left\{ \frac{n^2+40}{n^2+3n+1} \right\}_{n=1}^{\infty} \)

8. \( \left\{ \frac{5+n}{100,000+n^2} \right\}_{n=0}^{\infty} \)

---

**Series - The Basics**

For problems 1 – 4 perform an index shift so that the series starts at \( n = 4 \).

1. \( \sum_{n=8}^{\infty} \frac{2+n}{5-n} \)

2. \( \sum_{n=2}^{\infty} \frac{3^{n+2}}{1-4^{1+n+1}} \)

3. \( \sum_{n=0}^{\infty} (-2)^{2-n} e^{ln} \)

4. \( \sum_{n=5}^{\infty} \frac{(-1)^{3+n} n^3}{n^2-2n+1} \)
5. Strip out the first 4 terms from the series \( \sum_{n=0}^{\infty} 3^n 6^{2-n} \).

6. Strip out the first 2 terms from the series \( \sum_{n=3}^{\infty} \frac{4}{n^2 + n + 1} \).

7. Given that \( \sum_{n=4}^{\infty} n 4^{-n} = 0.02257 \) determine the value of \( \sum_{n=1}^{\infty} n 4^{-n} \).

8. Given that \( \sum_{n=3}^{\infty} \frac{n+1}{n^3} = 0.47199 \) determine the value of \( \sum_{n=5}^{\infty} \frac{n+1}{n^3} \).

**Series - Convergence/Divergence**

For problems 1 – 4 compute the first 3 terms in the sequence of partial sums for the given series.

1. \( \sum_{n=0}^{\infty} \frac{1}{1+3^n} \)

2. \( \sum_{n=1}^{\infty} \left(2^n - 3^n\right) \)

3. \( \sum_{n=1}^{\infty} \frac{1+n}{2n} \)

4. \( \sum_{n=0}^{\infty} 10 \)

For problems 5 – 7 assume that the \( n^{\text{th}} \) term in the sequence of partial sums for the series \( \sum_{n=0}^{\infty} a_n \) is given below. Determine if the series \( \sum_{n=0}^{\infty} a_n \) is convergent or divergent. If the series is convergent determine the value of the series.

5. \( s_n = \left(n^2 + 4n\right)e^{-2n} \)

6. \( s_n = \frac{1+2n+3n^2}{4n^2 + 5n + 6} \)
7. \( s_n = \frac{n}{\ln(n + 2)} \)

8. Let \( d_n = \frac{7 - 8n}{4 + 3n} \)
   
   (a) Does the sequence \( \{d_n\}_{n=0}^{\infty} \) converge or diverge?
   
   (b) Does the series \( \sum_{n=0}^{\infty} d_n \) converge or diverge?

9. Let \( d_n = 1 + ne^{-n} \)
   
   (c) Does the sequence \( \{d_n\}_{n=0}^{\infty} \) converge or diverge?
   
   (d) Does the series \( \sum_{n=0}^{\infty} d_n \) converge or diverge?

For problems 10 – 12 show that the series is divergent.

10. \( \sum_{n=1}^{\infty} \frac{9 - 2n^2}{1 + 4n + n^2} \)

11. \( \sum_{n=0}^{\infty} \frac{5^n + 1}{3^n} \)

12. \( \sum_{n=1}^{\infty} \cos(n) \)

**Series – Special Series**

For each of the following series determine if the series converges or diverges. If the series converges give its value.

1. \( \sum_{n=2}^{\infty} \frac{-2}{n^2 + n} \)

2. \( \sum_{n=1}^{\infty} \frac{12}{n} \)
Integral Test

For each of the following series determine if the series converges or diverges.

1. \( \sum_{n=2}^{\infty} \frac{4}{(\sqrt{n})^3} \)

2. \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2} \sqrt{n}} \)
For each of the following series determine if the series converges or diverges.

\[ \sum_{n=2}^{\infty} \frac{1}{2n+1} \]

\[ \sum_{n=0}^{\infty} \frac{8}{(n+10)^2} \]

\[ \sum_{n=0}^{\infty} \frac{1}{n^2 + 1} \]

\[ \sum_{n=2}^{\infty} \frac{\ln(n)}{n} \]

\[ \sum_{n=0}^{\infty} \frac{n^3}{n^4 + 1} \]

\[ \sum_{n=0}^{\infty} \frac{n^3}{(n^4 + 1)^2} \]

\[ \sum_{n=1}^{\infty} \frac{4}{n^2 - n - 6} \]

\[ \sum_{n=1}^{\infty} \frac{9}{n^2 + 5n + 4} \]

\[ \sum_{n=0}^{\infty} ne^{-n} \]

**Comparison Test / Limit Comparison Test**

For each of the following series determine if the series converges or diverges.

\[ \sum_{n=0}^{\infty} \frac{3^n - n}{2^{n+1}} \]

\[ \sum_{n=1}^{\infty} \frac{4n - 3}{2n^5} \]
Calculus II

3. \[ \sum_{n=4}^{\infty} \frac{1}{(2n-1)(n-3)} \]

4. \[ \sum_{n=8}^{\infty} \frac{\ln(n^2)}{n} \]

5. \[ \sum_{n=1}^{\infty} \frac{4n}{(n+1)^3} \]

6. \[ \sum_{n=0}^{\infty} \frac{n-4}{(n^2+1)e^n} \]

7. \[ \sum_{n=2}^{\infty} \frac{\sqrt{2 + \cos^2(5n)}}{\sqrt{n^2 - n - 1}} \]

8. \[ \sum_{n=3}^{\infty} \frac{n-1}{\sqrt{n^3 + n + 3}} \]

9. \[ \sum_{n=0}^{\infty} \frac{3n^2 + 7n - 1}{n^4 - n + 3} \]

10. \[ \sum_{n=1}^{\infty} \frac{(1 - \sin(n))(1 + \sin(n))}{n^2 + 8n + 1} \]

**Alternating Series Test**

For each of the following series determine if the series converges or diverges.

1. \[ \sum_{n=0}^{\infty} \frac{(-1)^{n+7}}{n^2 + 3} \]

2. \[ \sum_{n=0}^{\infty} \frac{(-1)^{n-2}}{3^n + 3n} \]

3. \[ \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^3 + 4n^2 + 8} \]
4. \( \sum_{n=1}^{\infty} \frac{1}{(-2)^n (6n+1)} \)

5. \( \sum_{n=3}^{\infty} \frac{4n \cos(n \pi)}{2n^2 + 1} \)

6. \( \sum_{n=0}^{\infty} \frac{(-1)^{n-10} n^2}{n^3 + n^2 + 4} \)

7. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+5} (2n+1)}{n^2 + 8} \)

**Absolute Convergence**

For each of the following series determine if they are absolutely convergent or conditionally convergent.

1. \( \sum_{n=2}^{\infty} \frac{(-1)^{n-2}}{\sqrt{n} - 1} \)

2. \( \sum_{n=3}^{\infty} \frac{\cos(n \pi)}{n^4} \)

3. \( \sum_{n=0}^{\infty} \frac{(-1)^{n-3} n}{4n^2 + 3} \)

4. \( \sum_{n=1}^{\infty} \frac{(-1)^{n+6} (1 + n^3)}{n^4} \)

5. \( \sum_{n=2}^{\infty} \frac{\cos^3(n)}{n^3 - n} \)

**Ratio Test**

For each of the following series determine if the series converges or diverges.
1. \( \sum_{n=0}^{\infty} \frac{n^3 + n^2}{(n+1)!} \)

2. \( \sum_{n=1}^{\infty} \frac{n + 2}{5^{1-n}(n+1)} \)

3. \( \sum_{n=0}^{\infty} \frac{(2n-1)!}{(3n)!} \)

4. \( \sum_{n=0}^{\infty} \frac{(-2)^{3n}}{3n^2 + 1} \)

5. \( \sum_{n=2}^{\infty} \frac{4^{1+n}n^2}{3^{2+n}(n+3)} \)

6. \( \sum_{n=1}^{\infty} \frac{4}{(-1)^{n+2}(n^2 + n + 1)} \)

7. \( \sum_{n=3}^{\infty} \frac{6^{2n}(n-4)}{4^{3-2n}(2-n^2)} \)

8. \( \sum_{n=2}^{\infty} \frac{(-1)^n(n+1)}{n^2 + 1} \)

**Root Test**

For each of the following series determine if the series converges or diverges.

1. \( \sum_{n=0}^{\infty} \frac{2^{1-3n}}{(-3)^{1-2n}} \)

2. \( \sum_{n=2}^{\infty} \left( \frac{5n^2 - 2n + 1}{3n^2 + n - 3} \right)^{-n} \)

3. \( \sum_{n=1}^{\infty} \frac{10^{3+5n}}{n^{1+n^2}} \)
Strategy for Series

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of “new” problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

Estimating the Value of a Series

1. Use the Integral Test and $n = 8$ to estimate the value of \[ \sum_{n=2}^{\infty} \frac{1}{n (\ln n)^2} \, . \]

2. Use the Integral Test and $n = 14$ to estimate the value of \[ \sum_{n=0}^{\infty} \frac{1}{(2n+1)^{5/3}} \, . \]

3. Use the Comparison Test and $n = 10$ to estimate the value of \[ \sum_{n=0}^{\infty} \frac{3^n - 2}{2^{2n} + 1} \, . \]

4. Use the Comparison Test and $n = 8$ to estimate the value of \[ \sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1} \, . \]

5. Use the Alternating Series Test and $n = 12$ to estimate the value of \[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n + 1} \, . \]

6. Use the Alternating Series Test and $n = 18$ to estimate the value of \[ \sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{3n + 4} \, . \]

7. Use the Ratio Test and $n = 10$ to estimate the value of \[ \sum_{n=1}^{\infty} \frac{(-2)^{1+2n}}{n^2 \cdot 7^n} \, . \]
8. Use the Ratio Test and \( n = 5 \) to estimate the value of \( \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \).

**Power Series**

For each of the following power series determine the interval and radius of convergence.

1. \( \sum_{n=0}^{\infty} \frac{6^{1-n}}{(-2)^{1-2n}} (x + 4)^n \)

2. \( \sum_{n=0}^{\infty} \frac{(10x - 1)^n}{n^{3+n}} \)

3. \( \sum_{n=0}^{\infty} \frac{(3n)!}{(2n-2)!} (6x - 9)^n \)

4. \( \sum_{n=0}^{\infty} \frac{(-1)^n n^2}{4n + 1} (5x + 20)^n \)

5. \( \sum_{n=0}^{\infty} \frac{(-1)^n 8^{1+n}}{n+4} (x - 7)^n \)

6. \( \sum_{n=0}^{\infty} \frac{2^{1+2n}}{(-3)^{1+2n} n^2} (4x + 2)^n \)

7. \( \sum_{n=0}^{\infty} \frac{(x + 12)^{2+n}}{(-16)^{2+n}} \)

**Power Series and Functions**

For problems 1 – 4 write the given function as a power series and give the interval of convergence.

1. \( f(x) = \frac{x}{1 - 8x} \)
2. \( f(x) = \frac{-12x^2}{1+6x^3} \)

3. \( f(x) = \frac{x^7}{8 + x^3} \)

4. \( f(x) = \frac{\sqrt[5]{x^7}}{4 - 3x^2} \)

For problems 5 & 6 give a power series representation for the derivative of the following function.

5. \( g(x) = \frac{x^{10}}{2 - x^2} \)

6. \( g(x) = \frac{9x^5}{1 + 3x^6} \)

For problems 7 & 8 give a power series representation for the integral of the following function.

7. \( h(x) = \frac{7x}{3 - 6x} \)

8. \( h(x) = \frac{x^4}{2 + x^9} \)

**Taylor Series**

For problems 1 – 3 use one of the Taylor Series derived in the notes to determine the Taylor Series for the given function.

1. \( f(x) = \sin(x^4) \) about \( x = 0 \)

2. \( f(x) = 9x^4e^{-12x} \) about \( x = 0 \)

3. \( f(x) = 6x^2 \cos(7x^5) \) about \( x = 0 \)

For problem 4 – 13 find the Taylor Series for each of the following functions.

4. \( f(x) = \sin(x) \) about \( x = \frac{\pi}{2} \)
5. \( f(x) = e^{-8x} \) about \( x = 3 \)

6. \( f(x) = \ln(1-x) \) about \( x = -2 \)

7. \( f(x) = \ln(2+9x) \) about \( x = 1 \)

8. \( f(x) = \frac{1}{(6-x)^4} \) about \( x = 4 \)

9. \( f(x) = \frac{1}{(4+9x)^2} \) about \( x = -2 \)

10. \( f(x) = \sqrt{2+x} \) about \( x = 1 \)

11. \( f(x) = \sqrt{1-4x} \) about \( x = -3 \)

12. \( f(x) = -3x^2 - x + 10 \) about \( x = -8 \)

13. \( f(x) = x^3 + 9x^2 - 10x + 2 \) about \( x = 3 \)

**Applications of Series**

1. Determine a Taylor Series about \( x = 0 \) for the following integral.

\[
\int \frac{\cos(x) - 1}{x} \, dx
\]

2. Write down \( T_2(x) \), \( T_4(x) \) and \( T_6(x) \) for the Taylor Series of \( f(x) = \sin(x) \) about \( x = \frac{3\pi}{2} \). Graph all three of the Taylor polynomials and \( f(x) \) on the same graph for the interval \([ -\pi, 2\pi ] \).

3. Write down \( T_2(x) \), \( T_3(x) \) and \( T_4(x) \) for the Taylor Series of \( f(x) = \ln(1-x) \) about \( x = -2 \). Graph all three of the Taylor polynomials and \( f(x) \) on the same graph for the interval \([ -4, 0 ] \).
4. Write down $T_1(x)$, $T_3(x)$ and $T_5(x)$ for the Taylor Series of $f(x) = \frac{1}{(6-x)^7}$ about $x = 4$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[1,5]$.

5. Write down $T_2(x)$, $T_4(x)$ and $T_6(x)$ for the Taylor Series of $f(x) = \sqrt{2+x}$ about $x = 1$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-2,4]$.

**Binomial Series**

For problems 1 – 3 use the Binomial Theorem to expand the given function.

1. $\left(\frac{1}{2} + \frac{2}{3}x\right)^4$

2. $(1-8x)^5$

3. $(3+7x)^6$

For problems 4 – 6 write down the first four terms in the binomial series for the given function.

4. $(8+4x)^{-4}$

5. $\sqrt{16+25x}$

6. $\sqrt[4]{16-4x}$

**Vectors**

**Introduction**

Here are a set of problems for which no solutions are available. The main intent of these problems is to have a set of problems available for any instructors who are looking for some extra problems.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.
Here is a list of topics in this chapter that have problems written for them.

**Vectors – The Basics**

**Vector Arithmetic**

**Dot Product**

**Cross Product**

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## Vectors – The Basics

1. Describe the difference between \((4, -1)\) and \(\langle 4, -1 \rangle\). Illustrate the difference with a sketch.

For problems 2 – 6 give the vector for the set of points. Find its magnitude and determine if the vector is a unit vector.

2. The line segment from \((6, -2, 3)\) to \((-3, -2, 1)\).

3. The line segment from \(\left(1, -\frac{1}{5}\right)\) to \(\left(\frac{8}{5}, \frac{3}{5}\right)\).

4. The line segment from \((2, -1, 5)\) to \((8, -3, -6)\).

5. The position vector for \((-12, 4, 8)\).

6. The position vector for \((\cos(\theta), \sin(\theta))\) for any angle \(\theta\).

7. The vector \(\vec{v} = \langle -8, -3 \rangle\) starts at the point \(P = (8, 2)\). At what point does the vector end?

8. The vector \(\vec{v} = \langle 0, 5, -3 \rangle\) starts at the point \(P = (-1, 0, 5)\). At what point does the vector end?

9. The vector \(\vec{v} = \langle -8, -3 \rangle\) ends at the point \(P = (8, 2)\). At what point does the vector start?

10. The vector \(\vec{v} = \langle 4, -2, 1 \rangle\) ends at the point \(P = (7, -7, 2)\). At what point does the vector start?

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## Vector Arithmetic

1. Given \(\vec{a} = 3\hat{i} - 9\hat{j}\) and \(\vec{b} = -6\hat{i} + \hat{j}\) compute each of the following.

   (a) \(10\vec{b}\)
Calculus II

(b) \[14\vec{a} + 20\vec{b}\]
(c) \[8\vec{b} - \dfrac{1}{3}\vec{a}\]

2. Given \(\vec{u} = \langle 0, 4, -1 \rangle\) and \(\vec{v} = \langle 6, -2, -7 \rangle\) compute each of the following.
   (a) \(\dfrac{3}{4}\vec{u}\)
   (b) \(-3\vec{u} - 7\vec{v}\)
   (c) \(\|\vec{v} + 10\vec{a}\|\)

3. Given \(\vec{p} = \langle 3, -1, -2 \rangle\) and \(\vec{q} = -\dfrac{1}{3}\vec{i} - \dfrac{1}{2}\vec{k}\) compute each of the following.
   (a) \(2\vec{p}\)
   (b) \(9\vec{q} - 2\vec{p}\)
   (c) \(\|8\vec{p} - 12\vec{q}\|\)

4. Find a unit vector that points in the same direction as \(\vec{a} = \langle 10, -3, 8, -2 \rangle\).

5. Find a unit vector that points in the same direction as \(\vec{w} = -\vec{i} - 6\vec{j}\).

6. Find a unit vector that points in the opposite direction as \(\vec{c} = 2\vec{i} + 7\vec{j} - 5\vec{k}\).

7. Find a unit vector that points in the opposite direction as \(\vec{b} = \langle 0, -3, -11 \rangle\).

8. Find a vector that points in the same direction as \(\vec{p} = 2\vec{i} - 3\vec{j} + \vec{k}\) with a magnitude of \(\dfrac{1}{2}\).

9. Find a vector that points in the opposite direction as \(\vec{a} = \langle -3, -14, 2 \rangle\) with a magnitude of 32.

10. Find a vector that points in the same direction as \(\vec{b} = -3\vec{i} + 2\vec{k}\) with a magnitude that is \(\dfrac{1}{10}\) the magnitude of \(\vec{b}\).

11. Determine if \(\vec{p} = 8\vec{i} - 3\vec{j}\) and \(\vec{q} = 16\vec{i} - 6\vec{j}\) are parallel vectors.

12. Determine if \(\vec{v} = \langle 1, 0, -4 \rangle\) and \(\vec{w} = \langle 9, 3, 1 \rangle\) are parallel vectors.

13. Determine if \(\vec{a} = 10\vec{i} + 8\vec{j} + 20\vec{k}\) and \(\vec{b} = \langle -35, -28, 70 \rangle\) are parallel vectors.

14. Prove the property: \(\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}\).
15. Prove the property: \( \vec{v} + \vec{0} = \vec{v} \).

16. Prove the property: \( 1\vec{v} = \vec{v} \).

17. Prove the property: \( (a + b)\vec{v} = a\vec{v} + b\vec{v} \).

**Dot Product**

For problems 1 – 5 determine the dot product, \( \vec{a} \cdot \vec{b} \).

1. \( \vec{a} = 9\vec{i} - 8\vec{k} , \quad \vec{b} = \langle 3, -2, 1 \rangle \)

2. \( \vec{a} = \langle 4, -1, 0, 5 \rangle , \quad \vec{b} = \langle 3, 0, -10, 6 \rangle \)

3. \( \vec{a} = \vec{i} - 5\vec{j} - 2\vec{k} , \quad \vec{b} = -4\vec{i} + 2\vec{j} + 8\vec{k} \)

4. \( \|\vec{a}\| = \frac{1}{2}, \quad \|\vec{b}\| = \frac{9}{4} \) and the angle between the two vectors is \( \theta = \pi \).

5. \( \|\vec{a}\| = 24 \), \( \|\vec{b}\| = 9 \) and the angle between the two vectors is \( \theta = \frac{2\pi}{7} \).

For problems 6 – 8 determine the angle between the two vectors.

6. \( \vec{p} = 9\vec{i} - \vec{j} , \quad \vec{q} = -3\vec{i} - 6\vec{j} \)

7. \( \vec{a} = \langle 4, 0, -3 \rangle , \quad \vec{b} = 2\vec{i} + 10\vec{j} - 11\vec{k} \)

8. \( \vec{w} = \langle 8, 3, -1, -4 \rangle , \quad \vec{v} = \langle -1, 9, 4, -8 \rangle \)

For problems 9 – 12 determine if the two vectors are parallel, orthogonal or neither.

9. \( \vec{q} = 7\vec{i} - 14\vec{j} - 21\vec{k} , \quad \vec{p} = \langle -4, 8, 12 \rangle \)

10. \( \vec{u} = \langle 5, 0, -2 \rangle , \quad \vec{q} = \langle 4, -7, 10 \rangle \)

11. \( \vec{a} = 9\vec{i} - \vec{j} + 5\vec{k} , \quad \vec{b} = -2\vec{i} + 7\vec{j} + \vec{k} \)

12. \( \vec{v} = \langle -1, 3, 1, 5 \rangle , \quad \vec{w} = \langle -8, 3, -7, -2 \rangle \)
13. Given that \( \mathbf{a} \cdot \mathbf{b} = -6 \), \( \|a\| = 4.3 \) and the angle between \( \mathbf{a} \) and \( \mathbf{b} \) is \( \theta = \frac{\pi}{6} \) determine if \( \mathbf{b} \) is a unit vector or not.

For problems 14 & 15 determine the value of \( b \) for which the two vectors will be orthogonal.

14. \( \mathbf{u} = \langle 3, -1, 6 \rangle \), \( \mathbf{v} = \langle 3, -2b, 1 \rangle \)

15. \( \mathbf{u} = \langle 1 - b, 4, -2 \rangle \), \( \mathbf{v} = \langle b, 6, 3b \rangle \)

16. Given \( \mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \) and \( \mathbf{b} = -3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} \) compute \( \text{proj}_\mathbf{a} \mathbf{b} \).

17. Given \( \mathbf{a} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k} \) and \( \mathbf{b} = -3\mathbf{i} - 4\mathbf{j} + 7\mathbf{k} \) compute \( \text{proj}_\mathbf{b} \mathbf{a} \).

18. Given \( \mathbf{p} = \langle 5, -2, 1 \rangle \) and \( \mathbf{q} = \langle 0, 4, 8 \rangle \) compute \( \text{proj}_\mathbf{p} \mathbf{q} \).

19. Given \( \mathbf{u} = \langle 1, 3, 0, -2 \rangle \) and \( \mathbf{w} = \langle -2, 2, 4, 1 \rangle \) compute \( \text{proj}_\mathbf{u} \mathbf{w} \).

20. Determine the direction cosines and direction angles for \( \mathbf{r} = \langle 5, 2, -7 \rangle \).

21. Determine the direction cosines and direction angles for \( \mathbf{r} = \langle \frac{1}{2}, -\frac{3}{4}, \frac{5}{2} \rangle \).

22. Prove the property \( (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w}) \).

23. Prove the property \( \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v} \).

24. Prove the property \( \mathbf{v} \cdot 0 = 0 \).

25. Prove the property \( \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 \).

**Cross Product**

1. If \( \mathbf{w} = \langle 1, 0, -3 \rangle \) and \( \mathbf{v} = \langle 6, -3, -4 \rangle \) compute \( \mathbf{v} \times \mathbf{w} \).

2. If \( \mathbf{w} = \langle 1, 0, -3 \rangle \) and \( \mathbf{v} = \langle 6, -3, -4 \rangle \) compute \( \mathbf{w} \times \mathbf{v} \).

3. If \( \mathbf{a} = 3\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \) and \( \mathbf{b} = \langle 4, -1, -6 \rangle \) compute \( \mathbf{a} \times \mathbf{b} \).
4. Find a vector that is orthogonal to the plane containing the points \( P = (-4, 2, 6) \), \( Q = (-3, 2, 1) \) and \( R = (2, -1, 1) \).

5. Find a vector that is orthogonal to the plane containing the points \( P = (-1, 1, 6) \), \( Q = (-2, 3, 2) \) and \( R = (-2, 4, 5) \).

5. Are the vectors \( \vec{u} = \langle -2, 4, -1 \rangle \), \( \vec{v} = \langle 5, -2, -1 \rangle \) and \( \vec{w} = \langle 3, 4, -3 \rangle \) are in the same plane?

6. Are the vectors \( \vec{u} = \langle 1, -1, 4 \rangle \), \( \vec{v} = \langle 4, 2, -2 \rangle \) and \( \vec{w} = \langle -5, 4, -17 \rangle \) are in the same plane?

7. Determine the value of \( b \) so that the vectors \( \vec{u} = \langle 4, -5, 3 \rangle \), \( \vec{v} = \langle -2, 0, -5 \rangle \) and \( \vec{w} = \langle b, -1, 6 \rangle \) are in the same plane.

### Three Dimensional Space

**Introduction**

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Here is a list of topics in this chapter that have problems written for them.

- **The 3-D Coordinate System**
- **Equations of Lines**
- **Equations of Planes**
- **Quadric Surfaces**
- **Functions of Several Variables**
- **Vector Functions**
- **Calculus with Vector Functions**
- **Tangent, Normal and Binormal Vectors**
- **Arc Length with Vector Functions**
- **Curvature**
- **Velocity and Acceleration**
- **Cylindrical Coordinates**
- **Spherical Coordinates**

The 3-D Coordinate System

1. Give the projection of \( P = (-9, 1, 5) \) onto the three coordinate planes.

2. Give the projection of \( P = (3, -2, -5) \) onto the three coordinate planes.

3. Which of the points \( P = (8, -9, 3) \) and \( Q = (-6, 4, -5) \) is closest to the \( xz \)-plane?

4. Which of the points \( P = (8, -9, 3) \) and \( Q = (-6, 4, -5) \) is closest to the \( xy \)-plane?

5. Which of the points \( P = (5, -4, 3) \) and \( Q = (-6, 3, 9) \) is closest to the \( x \)-axis?

6. Which of the points \( P = (5, -4, 3) \) and \( Q = (-6, 3, 9) \) is closest to the \( y \)-axis?

For problems 7 – 9 list all of the coordinates systems (\( \mathbb{R} \), \( \mathbb{R}^2 \), \( \mathbb{R}^3 \)) that the given equation will have a graph in. Do not sketch the graph.

7. \( 8z + \frac{x + 1}{y^2 + 2} = 4x \)

8. \( \sqrt{y + 2} = 6 \)

9. \( 7y^3 - \frac{2}{x + 1} = xy \)

Equations of Lines

For problems 1 – 4 give the equation of the line in vector form, parametric form and symmetric form.

1. The line through the points \( (7, -3, 1) \) and \( (-2, 1, 4) \).

2. The line through the point \( (1, -5, 0) \) and parallel to the line given by \( \vec{r}(t) = \langle 8 - 3t, -10 + 9t, -1 - t \rangle \).

3. The line through the point \( (1, -7, 14) \) and parallel to the line given by \( x = 6t \), \( y = 9 \), \( z = 8 - 16t \).

4. The line through the point \( (-7, 2, 4) \) and orthogonal to both \( \vec{v} = \langle 0, -9, 1 \rangle \) and \( \vec{w} = 3\vec{i} + \vec{j} - 4\vec{k} \).

For problems 5 – 7 determine if the two lines are parallel, orthogonal or neither.
5. The line given by \( \mathbf{r}(t) = \langle 4 - 7t, -10 + 5t, 21 - 4t \rangle \) and the line given by \( \mathbf{r}(t) = \langle -2 + 3t, 7 + 5t, 5 + t \rangle \).

6. The line through the points \((10, -4, 18)\) and \((5, 6, -7)\) and the line given by \( x = 5 + 3t \), \( y = -6t \), \( z = 1 + 15t \).

7. The line given by \( x = 29 \), \( y = -3 - 6t \), \( z = 12 - t \) and the line given by \( \mathbf{r}(t) = \langle 12 - 14t, 2 + 7t, -10 + 3t \rangle \).

For problems 8 – 10 determine the intersection point of the two lines or show that they do not intersect.

8. The line passing through the points \((0, -9, -1)\) and \((1, 6, -3)\) and the line given by \( \mathbf{r}(t) = \langle -9 - 4t, 10 + 6t, 1 - 2t \rangle \).

9. The line given by \( x = 1 + 6t \), \( y = -1 - 3t \), \( z = 4 + 12t \) and the line given by \( x = 4 + t \), \( y = -10 - 8t \), \( z = 3 - 5t \).

10. The line given by \( \mathbf{r}(t) = \langle 14 + 5t, -3t, 1 + 7t \rangle \) and the line given by \( \mathbf{r}(t) = \langle 3 - 3t, 5 + 2t, -2 + 4t \rangle \).

11. Does the line passing through \((-5, 4, -1)\) and \((-3, -5, 0)\) intersect the \(yz\)-plane? If so, give the point.

12. Does the line given by \( \mathbf{r}(t) = \langle 6 + t, -8 + 14t, 4t \rangle \) intersect the \(xz\)-plane? If so, give the point.

13. Which of the three coordinate planes does the line given by \( x = 16t \), \( y = -4 - 9t \), \( z = 34 \) intersect?

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**Equations of Planes**

For problems 1 – 5 write down the equation of the plane.

1. The plane containing the points \((6, -3, 1)\), \((5, -4, 1)\) and \((3, -4, 0)\).

2. The plane containing the point \((1, -5, 8)\) and orthogonal to the line given by \( x = -3 + 15t \), \( y = 14 - t \), \( z = 9 - 3t \).

3. The plane containing the point \((-8, 3, 7)\) and parallel to the plane given by \( 4x + 8y - 2z = 45 \).

4. The plane containing the point \((2, 0, -8)\) and containing the line given by \( \mathbf{r}(t) = \langle 8t, -1 - 5t, 4 - t \rangle \).
5. The plane containing the two lines given by \( \mathbf{r}(t) = \langle 7 + 5t, 2 + t, 6t \rangle \) and \( \mathbf{r}(t) = \langle 7 - 6t, 2 - 2t, 10t \rangle \).

For problems 6 – 8 determine if the two planes are parallel, orthogonal or neither.

6. The plane given by \(-5x + 3y + 2z = -8\) and the plane given by \(6x - 8z = 15\).

7. The plane given by \(3x + 9y + 7z = -1\) and the plane containing the points \((1, -1, 9), (4, -1, 2)\) and \((-2, 3, 4)\).

8. The plane given by \(-x - 8y + 3z = 6\) and the plane given by \(2x + 2y + 6z = -91\).

For problems 9 – 11 determine where the line intersects the plane or show that it does not intersect the plane.

9. The line given by \( \mathbf{r}(t) = \langle 9 + t, -4 + t, 2 + 5t \rangle \) and the plane given by \(4x - 9y + z = 6\).

10. The line given by \( \mathbf{r}(t) = \langle 2 - 3t, 1 + t, -4 - 2t \rangle \) and the plane given by \(x - 7y - 4z = -1\).

11. The line given by \( x = 8, y = -9t, z = 1 + 10t \) and the plane given by \(8x + 9y + 2z = 17\).

For problems 12 & 13 find the line of intersection of the two planes.

12. Find the line of intersection of the plane given by \(4x + y + 10z = -2\) and the plane given by \(-8x + 2y + 3z = -8\).

13. Find the line of intersection of the plane given by \(x - 10y - 2z = 3\) and the plane given by \(2x - y + z = -13\).

14. Determine if the line given by \(x = 4 + 3t, y = -2, z = 1 + 6t\) and the plane given by \(8x - y + 4z = -3\) are parallel, orthogonal or neither.

**Quadric Surfaces**

Sketch each of the following quadric surfaces.

1. \(z^2 = x^2 + \frac{y^2}{2}\)

2. \(z = 2 + 4x^2 + 6y^2\)

3. \(4x^2 + y^2 + 3z^2 = 1\)
4. \( \frac{x^2}{9} + \frac{y^2}{16} = 1 \)

5. \( y = \frac{x^2}{2} + \frac{z^2}{3} - 7 \)

6. \( 6x^2 + 2z^2 = 1 \)

7. \( x = 12 - \frac{y^2}{4} - 3z^2 \)

8. \( x^2 = 4y^2 + 9z^2 \)

**Functions of Several Variables**

For problems 1 – 6 find the domain of the given function.

1. \( f(x, y) = \sqrt{2x + 4y - 1} \)

2. \( f(x, y) = \ln \left( \frac{1}{x - y} \right) \)

3. \( f(x, y) = \sqrt{\frac{1}{x^3} - \frac{1}{y^5}} \)

4. \( f(x, y, z) = \frac{1}{x + 1} + \frac{1}{y - 1} + \frac{1}{x + y - z} \)

5. \( f(x, y, z) = \ln \left( x^2 + y^2 - 8z \right) \)

6. \( f(x, y) = \sqrt{x + y} - \sqrt{x - 3} \)

For problems 7 – 11 identify and sketch the level curves (or contours) for the given function.

7. \( x^2 - 4z - y = 2 \)

8. \( x - 4z - y^2 = 2 \)

9. \( z^2 + 4x^2 = 1 - 4y^2 \)
10. \( z + 4x^2 = 1 - 4y^2 \)

11. \( 2x - 6y + z = -2 \)

For problems 12 – 14 identify and sketch the traces for the given curves.

12. \( x^2 - 4z - y = 2 \)

13. \( z^2 + 4x^2 = 1 - 4y^2 \)

14. \( 2x - 6y + z = -2 \)

**Vector Functions**

For problems 1 – 3 find the domain of the given vector function.

1. \( \vec{r}(t) = \left\langle \frac{1}{t^2 - 1}, \frac{1}{t + 3}, \frac{1}{t - 6} \right\rangle \)

2. \( \vec{r}(t) = \left\langle \sqrt{t}, \sqrt{t+1}, \sqrt{t+2} \right\rangle \)

3. \( \vec{r}(t) = \left\langle \ln(t + 7), \ln(t - 3) \right\rangle \)

For problems 4 – 8 sketch the graph of the given vector function.

4. \( \vec{r}(t) = \left\langle -4, t + 1 \right\rangle \)

5. \( \vec{r}(t) = \left\langle -2 \cos(t), 5 \sin(t) \right\rangle \)

6. \( \vec{r}(t) = \left\langle \sqrt{t+2}, 1 - t \right\rangle \)

7. \( \vec{r}(t) = \left\langle 2t + 1, t^2 - 1 \right\rangle \)

8. \( \vec{r}(t) = \left\langle t^2 + 4, 6 - t^2 \right\rangle \)

For problems 9 – 12 identify the graph of the vector function without sketching the graph.

9. \( \vec{r}(t) = \left\langle 6, 2 + 8t, -1 + 10t \right\rangle \)
10. \( \mathbf{r}(t) = \langle 12t, 6-8t, 4+7t \rangle \)

11. \( \mathbf{r}(t) = \langle 2t, 6\cos(t), 6\sin(t) \rangle \)

12. \( \mathbf{r}(t) = \langle -2t, 6\cos(t), 6\sin(t) \rangle \)

For problems 13 – 16 write down the equation of the line segment between the two points.

13. The line segment starting at \((4, -7)\) and ending at \((2, 0)\).

14. The line segment starting at \((-1, 2)\) and ending at \((7, -2)\).

15. The line segment starting at \((4, 1, -3)\) and ending at \((-1, 2, 6)\).

16. The line segment starting at \((1, -1, 9)\) and ending at \((4, -7, 10)\).

**Calculus with Vector Functions**

For problems 1 – 6 evaluate the given limit.

1. \( \lim_{t \to 0} \left( \cos(2t)\mathbf{i} - e^{4-t} \mathbf{j} + \left( t^2 + 3t - 9 \right)\mathbf{k} \right) \)

2. \( \lim_{t \to 4} \left( \frac{t-4}{t^2-3t-4}, \frac{t^2-4t}{16-t^2} \right) \)

3. \( \lim_{t \to 0} \left( \frac{\sin(t)}{2t}, \frac{1-\cos(t)}{t}, -3 \right) \)

4. \( \lim_{t \to -8} \left( \frac{e^{t^2-64}}{t+8} - 1, \frac{\sin(t+8)}{t+8}, \frac{\mathbf{j}}{\mathbf{k}} - \mathbf{i} \right) \)

5. \( \lim_{t \to \infty} \left( \frac{5t^2-8t+1}{12+5t^2}, \frac{2+t^3}{1+t^2+t^4} \right) \)

6. \( \lim_{t \to \infty} \left( \ln \left( 1 - \frac{4}{t} \right), \frac{1}{t^2}, 2 \right) \)
For problems 7 – 11 compute the derivative of the given vector function.

7. \( \vec{r}(t) = \left( \sqrt{3t}, \frac{1}{t^3}, \frac{1}{2t} \right) \)

8. \( \vec{r}(t) = \cos(2t) \hat{i} - \sin(2t) \hat{j} + \ln(2t) \hat{k} \)

9. \( \vec{r}(t) = \left( e^{t^2-1}, 4 - \sec(2t), 7 \right) \)

10. \( \vec{r}(t) = \sin(t) \cos(t) \hat{i} - t^3 \ln(t^2) \hat{j} \)

11. \( \vec{r}(t) = \left( \frac{1}{(t^2 - 4)^2}, \frac{t^3}{t^2 + 2}, \frac{t^2 + 2}{t^3} \right) \)

For problems 11 – 14 evaluate the given integral.

11. \( \int \vec{r}(t) \, dt \), where \( \vec{r}(t) = \left( \frac{1}{t} - t^3, \frac{1}{6t} - \frac{8}{t^4} \right) \)

12. \( \int \vec{r}(t) \, dt \), where \( \vec{r}(t) = \left( t^2 - 5 \right) e^{t^3-15} \hat{i} + 4t \sqrt{t^2+1} \hat{j} - \sin^2(5t) \hat{k} \)

13. \( \int_0^1 \vec{r}(t) \, dt \) where \( \vec{r}(t) = \left( t \cos(\pi t), 8t - 2, 12t^3 - e^{2t} \right) \)

14. \( \int_0^1 \vec{r}(t) \, dt \) where \( \vec{r}(t) = \tan(t) \hat{i} - \sin^3(t) \cos^2(t) \hat{j} - 8t \)

**Tangent, Normal and Binormal Vectors**

For problems 1 – 3 find the unit tangent vector for the given vector function.

1. \( \vec{r}(t) = t^2 \hat{i} - \cos(8t) \hat{j} + \sin(8t) \hat{k} \)

2. \( \vec{r}(t) = \left( 8t, 2 - t^6, t^4 \right) \)

3. \( \vec{r}(t) = \left( \ln(6t), e^{1-t}, 5t \right) \)

For problems 4 & 5 find the tangent line to the vector function at the given point.
4. \( \vec{r}(t) = \langle 3 + t^2, t^4, 6 \rangle \) at \( t = -1 \).

5. \( \vec{r}(t) = \langle 2t, \cos^2(t), e^{6t} \rangle \) at \( t = 0 \).

For problems 5 & 6 find the unit normal and the binormal vectors for the given vector function.

5. \( \vec{r}(t) = \langle e^{4t} \sin(t), e^{4t} \cos(t), 2 \rangle \)

6. \( \vec{r}(t) = 2t \hat{i} + \frac{1}{2} t^2 \hat{j} + \ln(t^2) \hat{k} \)

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### Arc Length with Vector Functions

For problems 1 – 3 determine the length of the vector function on the given interval.

1. \( \vec{r}(t) = 4 \cos(2t) \hat{i} + 3t \hat{j} - \sin(2t) \hat{k} \) from \( 0 \leq t \leq 3\pi \).

2. \( \vec{r}(t) = \langle 9 - 2t, 4 + 2t, \sqrt{2} t^2 \rangle \) from \( 0 \leq t \leq 1 \).

3. \( \vec{r}(t) = 2t \hat{i} + \frac{1}{2} t^2 \hat{j} + \ln(t^2) \hat{k} \) from \( 1 \leq t \leq 3 \).

For problems 4 – 6 find the arc length function for the given vector function.

4. \( \vec{r}(t) = \langle 8t, 6 + t, -7t \rangle \)

5. \( \vec{r}(t) = \langle 8t, 4t^2, 3 \rangle \)

6. \( \vec{r}(t) = \langle e^{4t} \sin(t), e^{4t} \cos(t), 2 \rangle \)

7. Determine where on the curve given by \( \vec{r}(t) = \langle 8t, 4t^2, 3 \rangle \) we are after traveling a distance of 4.

8. Determine where on the curve given by \( \vec{r}(t) = \langle e^{4t} \sin(t), e^{4t} \cos(t), 2 \rangle \) we are after traveling a distance of 15.
Curvature

Find the curvature for each the following vector functions.

1. \( \vec{r}(t) = \left( 5t, 1-2t, 4t^2 \right) \)

2. \( \vec{r}(t) = \left( 6, e^{-st}, 3te^{-st} \right) \)

3. \( \vec{r}(t) = \left( \cos(\omega t), t, \sin(\omega t) \right) \)

Velocity and Acceleration

1. An objects acceleration is given by \( \vec{a} = \cos(2t) \vec{i} + 4t^3 \vec{j} + 6\sin(3t) \vec{k} \). The objects initial velocity is \( \vec{v}(0) = 6\vec{i} + 2\vec{j} + 7\vec{k} \) and the objects initial position is \( \vec{r}(0) = \vec{i} - 9\vec{j} + 6\vec{k} \). Determine the objects velocity and position functions.

2. An objects acceleration is given by \( \vec{a} = 10t \vec{i} - 6 \vec{j} + t \cos(t) \vec{k} \). The objects initial velocity is \( \vec{v}(0) = -\vec{i} + 11\vec{j} - \vec{k} \) and the objects initial position is \( \vec{r}(0) = 4\vec{i} + \vec{j} + 10\vec{k} \). Determine the objects velocity and position functions.

3. Determine the tangential and normal components of acceleration for the object whose position is given by \( \vec{r}(t) = \left( 5t, 1-2t, 4t^2 \right) \).

4. Determine the tangential and normal components of acceleration for the object whose position is given by \( \vec{r}(t) = \left( 6, e^{-st}, 3te^{-st} \right) \).

Cylindrical Coordinates

For problems 1 & 2 convert the Cartesian coordinates for the point into Cylindrical coordinates.

1. \((-3, 5, -8)\)

2. \((4, 1, 7)\)
3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates.

\[ \frac{x - y}{x^2 + y^2 + 1} = xyz \]

For problems 4 – 6 convert the equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

4. \( zr^3 \cos(\theta) = 4r + 8 \)

5. \( r^2 - 3 \sin(\theta) = z^3 + \sqrt{r^2} + 1 \)

6. \( \tan(\theta) + 2z = 1 - r^2 \)

For problems 7 – 9 identify the surface generated by the given equation.

7. \( z = -4r, \ z < 0 \)

8. \( 2r + 6 \cos(\theta) + 9 \sin(\theta) = \frac{51}{r} \)

9. \( \theta = \frac{\pi}{3} \)

**Spherical Coordinates**

For problems 1 – 3 convert the Cartesian coordinates for the point into Spherical coordinates.

1. \((6, 2, -8)\)

2. \((-1, 5, 2)\)

3. \((-3, -2, 1)\)

4. Convert the Cylindrical coordinates for the point \((5, 1.294, 6)\) into Spherical coordinates.

5. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

\[ \frac{xz}{y} = 2 - x \]
For problems 6 – 8 convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

6. \( \rho \cos \varphi \sin \varphi \sin \theta = 3 \)

7. \( \rho - \cos \varphi = 2 + \cos^2 \varphi \)

8. \( \tan \varphi (\cos \theta - \sin \theta) = 4 \)

For problems 9 & 10 identify the surface generated by the given equation.

9. \( \cos^2 \varphi - \sin^2 \varphi = 0 \)

10. \( \sin \varphi \cos \theta + \sin \varphi \sin \theta + \cos \varphi = \frac{1}{\rho} \)