CALCULUS II

Practice Problems
Sequences and Series

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Preface

Here are a set of practice problems for my Calculus II notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a number of places on the site.

1. If you’d like a pdf document containing the solutions go to the note page for the section you’d like solutions for and select the download solutions link from there. Or,

2. Go to the download page for the site http://tutorial.math.lamar.edu/download.aspx and select the section you’d like solutions for and a link will be provided there.

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Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.
Sequences and Series

Introduction

Here are a set of practice problems for the Sequences and Series chapter of my Calculus II notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a number of places on the site.

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Here is a list of topics in this chapter that have practice problems written for them.

- Sequences
- More on Sequences
- Series – The Basics
- Series – Convergence/Divergence
- Series – Special Series
- Integral Test
- Comparison Test/Limit Comparison Test
- Alternating Series Test
- Absolute Convergence
- Ratio Test
- Root Test
- Strategy for Series
- Estimating the Value of a Series
- Power Series
- Power Series and Functions
- Taylor Series
- Applications of Series
- Binomial Series
Sequences

For problems 1 & 2 list the first 5 terms of the sequence.

1. \[ \left\{ \frac{4n}{n^2 - 7} \right\}_{n=0}^{\infty} \]

2. \[ \left\{ \frac{(-1)^{n+1}}{2n + (-3)^n} \right\}_{n=2}^{\infty} \]

For problems 3 – 6 determine if the given sequence converges or diverges. If it converges what is its limit?

3. \[ \left\{ \frac{n^2 - 7n + 3}{1 + 10n - 4n^2} \right\}_{n=3}^{\infty} \]

4. \[ \left\{ \frac{(-1)^{n-2} n^2}{4 + n^3} \right\}_{n=0}^{\infty} \]

5. \[ \left\{ \frac{e^{5n}}{3 - e^{2n}} \right\}_{n=1}^{\infty} \]

6. \[ \left\{ \frac{\ln(n + 2)}{\ln(1 + 4n)} \right\}_{n=1}^{\infty} \]

More on Sequences

For each of the following problems determine if the sequence is increasing, decreasing, not monotonic, bounded below, bounded above and/or bounded.

1. \[ \left\{ \frac{1}{4n} \right\}_{n=1}^{\infty} \]

2. \[ \left\{ n(-1)^{n+2} \right\}_{n=0}^{\infty} \]
Series – The Basics

For problems 1 – 3 perform an index shift so that the series starts at \( n = 3 \).

1. \( \sum_{n=1}^{\infty} \left( n2^n - 3^{1-n} \right) \)

2. \( \sum_{n=1}^{\infty} \frac{4-n}{n^2 + 1} \)

3. \( \sum_{n=2}^{\infty} \frac{(-1)^{n-3}(n+2)}{5^{l+2n}} \)

4. Strip out the first 3 terms from the series \( \sum_{n=1}^{\infty} \frac{2^{-n}}{n^2 + 1} \).

5. Given that \( \sum_{n=0}^{\infty} \frac{1}{n^3 + 1} = 1.6865 \) determine the value of \( \sum_{n=2}^{\infty} \frac{1}{n^3 + 1} \).

Series – Convergence/Divergence

For problems 1 & 2 compute the first 3 terms in the sequence of partial sums for the given series.

1. \( \sum_{n=1}^{\infty} n2^n \)
2. \[ \sum_{n=3}^{\infty} \frac{2n}{n+2} \]

For problems 3 & 4 assume that the \( n\text{th} \) term in the sequence of partial sums for the series \( \sum_{n=0}^{\infty} a_n \) is given below. Determine if the series \( \sum_{n=0}^{\infty} a_n \) is convergent or divergent. If the series is convergent determine the value of the series.

3. \( s_n = \frac{5 + 8n^2}{2 - 7n^2} \)

4. \( s_n = \frac{n^2}{5 + 2n} \)

For problems 5 & 6 show that the series is divergent.

5. \[ \sum_{n=0}^{\infty} \frac{3n e^n}{n^2 + 1} \]

6. \[ \sum_{n=5}^{\infty} \frac{6 + 8n + 9n^2}{3 + 2n + n^2} \]

**Series – Special Series**

For each of the following series determine if the series converges or diverges. If the series converges give its value.

1. \[ \sum_{n=0}^{\infty} 3^{2+n} 2^{1-3n} \]

2. \[ \sum_{n=1}^{\infty} \frac{5}{6n} \]

3. \[ \sum_{n=1}^{\infty} \frac{(-6)^{3-n}}{8^{2-n}} \]

4. \[ \sum_{n=1}^{\infty} \frac{3}{n^3 + 7n + 12} \]
Integral Test

For each of the following series determine if the series converges or diverges.

1. \(\sum_{n=1}^{\infty} \frac{1}{n^2}\)

2. \(\sum_{n=0}^{\infty} \frac{2}{3+5n}\)

3. \(\sum_{n=2}^{\infty} \frac{1}{(2n+7)^3}\)

4. \(\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}\)

5. \(\sum_{n=3}^{\infty} \frac{3}{n^2-3n+2}\)

Comparison Test / Limit Comparison Test

For each of the following series determine if the series converges or diverges.

1. \(\sum_{n=1}^{\infty} \left(\frac{1}{n^2+1}\right)^2\)
Alternating Series Test

For each of the following series determine if the series converges or diverges.

1. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{7 + 2n} \]

2. \[ \sum_{n=0}^{\infty} \frac{(-1)^{n+3}}{n^3 + 4n + 1} \]
3. \[ \sum_{n=0}^{\infty} \frac{1}{(-1)^n (2^n + 3^n)} \]

4. \[ \sum_{n=0}^{\infty} \frac{(-1)^{n+6} n}{n^2 + 9} \]

5. \[ \sum_{n=4}^{\infty} \frac{(-1)^{n+2} (1-n)}{3n-n^2} \]

**Absolute Convergence**

For each of the following series determine if they are absolutely convergent, conditionally convergent or divergent.

1. \[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^3 + 1} \]

2. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n-3}}{\sqrt{n}} \]

3. \[ \sum_{n=3}^{\infty} \frac{(-1)^{n+1} (n+1)}{n^3 + 1} \]

**Ratio Test**

For each of the following series determine if the series converges or diverges.

1. \[ \sum_{n=1}^{\infty} \frac{3^{1-2n}}{n^2 + 1} \]

2. \[ \sum_{n=0}^{\infty} \frac{(2n)!}{5n+1} \]

3. \[ \sum_{n=2}^{\infty} \frac{(-2)^{1+3n} (n+1)}{n^2 5^{1+n}} \]
4. \[ \sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!} \]

5. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{6n + 7} \]

**Root Test**

For each of the following series determine if the series converges or diverges.

1. \[ \sum_{n=1}^{\infty} \left( \frac{3n+1}{4-2n} \right)^{2n} \]

2. \[ \sum_{n=0}^{\infty} \frac{n^{1-3n}}{4^{2n}} \]

3. \[ \sum_{n=4}^{\infty} \frac{(-5)^{1+2n}}{2^{5n-3}} \]

**Strategy for Series**

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of “new” problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

**Estimating the Value of a Series**

1. Use the Integral Test and \( n = 10 \) to estimate the value of \[ \sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2} \].

2. Use the Comparison Test and \( n = 20 \) to estimate the value of \[ \sum_{n=3}^{\infty} \frac{1}{n^3 \ln(n)} \].
3. Use the Alternating Series Test and \( n = 16 \) to estimate the value of \( \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + 1} \).

4. Use the Ratio Test and \( n = 8 \) to estimate the value of \( \sum_{n=1}^{\infty} \frac{3^{1+n}}{n^{3+2n}} \).

**Power Series**

For each of the following power series determine the interval and radius of convergence.

1. \( \sum_{n=0}^{\infty} \frac{1}{(-3)^{2n}} \frac{(4x - 12)^n}{(n^2 + 1)} \)

2. \( \sum_{n=0}^{\infty} \frac{n^{2n+1}}{4^{3n}} (2x + 17)^n \)

3. \( \sum_{n=0}^{\infty} \frac{n+1}{(2n+1)!} (x - 2)^n \)

4. \( \sum_{n=0}^{\infty} \frac{4^{1+2n}}{5^{n+1}} (x + 3)^n \)

5. \( \sum_{n=1}^{\infty} \frac{6^n}{n} (4x - 1)^{n-1} \)

**Power Series and Functions**

For problems 1 – 3 write the given function as a power series and give the interval of convergence.

1. \( f(x) = \frac{6}{1 + 7x^4} \)

2. \( f(x) = \frac{x^3}{3 - x^2} \)

3. \( f(x) = \frac{3x^2}{5 - 2\sqrt{x}} \)
4. Give a power series representation for the derivative of the following function.

\[ g(x) = \frac{5x}{1 - 3x^5} \]

5. Give a power series representation for the integral of the following function.

\[ h(x) = \frac{x^4}{9 + x^2} \]

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**Taylor Series**

For problems 1 & 2 use one of the Taylor Series derived in the notes to determine the Taylor Series for the given function.

1. \( f(x) = \cos(4x) \) about \( x = 0 \)

2. \( f(x) = x^6 e^{2x^3} \) about \( x = 0 \)

For problem 3 – 6 find the Taylor Series for each of the following functions.

3. \( f(x) = e^{-6x} \) about \( x = -4 \)

4. \( f(x) = \ln(3 + 4x) \) about \( x = 0 \)

5. \( f(x) = \frac{7}{x^3} \) about \( x = -3 \)

6. \( f(x) = 7x^2 - 6x + 1 \) about \( x = 2 \)

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**Applications of Series**

1. Determine a Taylor Series about \( x = 0 \) for the following integral.

\[ \int \frac{e^x - 1}{x} \,dx \]
2. Write down $T_2(x)$, $T_3(x)$ and $T_4(x)$ for the Taylor Series of $f(x) = e^{-x}$ about $x = -4$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-8, -2]$.

3. Write down $T_3(x)$, $T_4(x)$ and $T_5(x)$ for the Taylor Series of $f(x) = \ln(3 + 4x)$ about $x = 0$. Graph all three of the Taylor polynomials and $f(x)$ on the same graph for the interval $[-\frac{1}{2}, 2]$.

**Binomial Series**

For problems 1 & 2 use the Binomial Theorem to expand the given function.

1. $(4 + 3x)^5$

2. $(9 - x)^4$

For problems 3 and 4 write down the first four terms in the binomial series for the given function.

3. $(1 + 3x)^{-6}$

4. $\sqrt[3]{8 - 2x}$