Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Integration Techniques

Integration by Parts

1. Evaluate \( \int 4x \cos(2 - 3x) \, dx \).

Hint: Remember that we want to pick \( u \) and \( dv \) so that upon computing \( du \) and \( v \) and plugging everything into the Integration by Parts formula the new integral is one that we can do.

Step 1
The first step here is to pick \( u \) and \( dv \). We want to choose \( u \) and \( dv \) so that when we compute \( du \) and \( v \) and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for \( u \) and \( dv \) should work for us.

\[
\begin{align*}
  u &= 4x \\
  dv &= \cos(2-3x) \, dx
\end{align*}
\]

Step 2
Next we need to compute \( du \) (by differentiating \( u \)) and \( v \) (by integrating \( dv \)).

\[
\begin{align*}
  u &= 4x & \Rightarrow & & du &= 4 \, dx \\
  dv &= \cos(2-3x) \, dx & \Rightarrow & & v &= -\frac{1}{3} \sin(2-3x)
\end{align*}
\]

Step 3
Plugging \( u, du, v \) and \( dv \) into the Integration by Parts formula gives,

\[
\int 4x \cos(2-3x) \, dx = (4x)\left(-\frac{1}{3} \sin(2-3x)\right) - \int -\frac{4}{3} \sin(2-3x) \, dx
\]

\[
= -\frac{4}{3} x \sin(2-3x) + \frac{4}{3} \int \sin(2-3x) \, dx
\]

Step 4
Okay, the new integral we get is easily doable and so all we need to do to finish this problem out is do the integral.

\[
\int 4x \cos(2-3x) \, dx = -\frac{4}{3} x \sin(2-3x) + \frac{4}{3} \cos(2-3x) + C
\]

---

2. Evaluate \( \int_{0}^{2} (2 + 5x) e^{\frac{1}{x}} \, dx \).

Hint : Remember that we want to pick \( u \) and \( dv \) so that upon computing \( du \) and \( v \) and plugging everything into the Integration by Parts formula the new integral is one that we can do.

Also, don’t forget that the limits on the integral won’t have any effect on the choices of \( u \) and \( dv \).

Step 1
The first step here is to pick \( u \) and \( dv \). We want to choose \( u \) and \( dv \) so that when we compute \( du \) and \( v \) and plugging everything into the Integration by Parts formula the new integral we get is one that we can do.

With that in mind it looks like the following choices for \( u \) and \( dv \) should work for us.

\[
\begin{align*}
  u &= 2 + 5x \\
  dv &= e^{\frac{1}{x}} \, dx
\end{align*}
\]
Next we need to compute $du$ (by differentiating $u$) and $v$ (by integrating $dv$).

\[
\begin{align*}
    u &= 2 + 5x & \Rightarrow \quad du &= 5dx \\
    dv &= e^{\frac{1}{3}x} \, dx & \Rightarrow \quad v &= 3e^{\frac{1}{3}x}
\end{align*}
\]

Step 3
We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging $u$, $du$, $v$ and $dv$ into the Integration by Parts formula gives,

\[
\int \left(2 + 5x\right)e^{\frac{1}{3}x} = \left(2 + 5x\right)\left(3e^{\frac{1}{3}x}\right) - \int 5\left(3e^{\frac{1}{3}x}\right) dx = 3e^{\frac{1}{3}x} \left(2 + 5x\right) - 15\int e^{\frac{1}{3}x} dx
\]

Step 4
Okay, the new integral we get is easily doable so let’s evaluate it to get,

\[
\int \left(2 + 5x\right)e^{\frac{1}{3}x} = 3e^{\frac{1}{3}x} \left(2 + 5x\right) - 45e^{\frac{1}{3}x} + c = 15xe^{\frac{1}{3}x} - 39e^{\frac{1}{3}x} + c
\]

Step 5
The final step is then to take care of the limits.

\[
\int_0^0 \left(2 + 5x\right)e^{\frac{1}{3}x} \, dx = \left[15xe^{\frac{1}{3}x} - 39e^{\frac{1}{3}x}\right]_0^0 = -39 - 51e^2 = -415.8419
\]

Do not get excited about the fact that the lower limit is larger than the upper limit. This can happen on occasion and in no way affects how the integral is evaluated.

3. Evaluate $\int \left(3t + t^2\right)\sin\left(2t\right) \, dt$ .

Hint: Remember that we want to pick $u$ and $dv$ so that upon computing $du$ and $v$ and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

Step 1
The first step here is to pick $u$ and $dv$. We want to choose $u$ and $dv$ so that when we compute $du$ and $v$ and plugging everything into the Integration by Parts formula the new integral we get is one that we can do, or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for $u$ and $dv$ should work for us.

\[
\begin{align*}
    u &= 3t + t^2 & dv &= \sin\left(2t\right) \, dt
\end{align*}
\]
Step 2
Next we need to compute $du$ (by differentiating $u$) and $v$ (by integrating $dv$).

$$u = 3t + t^2 \quad \rightarrow \quad du = (3 + 2t)\,dt$$

$$dv = \sin(2t)\,dt \quad \rightarrow \quad v = -\frac{1}{2} \cos(2t)$$

Step 3
Plugging $u$, $du$, $v$ and $dv$ into the Integration by Parts formula gives,

$$\int \left(3t + t^2\right) \sin(2t)\,dt = -\frac{1}{2} \left(3t + t^2\right) \cos(2t) + \frac{1}{4} \int (3 + 2t) \cos(2t)\,dt$$

Step 4
Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the $t$’s have gone down by one we are heading in the right direction.

So, here are the choices for $u$ and $dv$ for the new integral.

$$u = 3 + 2t \quad \rightarrow \quad du = 2dt$$

$$dv = \cos(2t)\,dt \quad \rightarrow \quad v = \frac{1}{2} \sin(2t)$$

Step 5
Okay, all we need to do now is plug these new choices of $u$ and $dv$ into the new integral we got in Step 3 and finish the problem out.

$$\int \left(3t + t^2\right) \sin(2t)\,dt = -\frac{1}{2} \left(3t + t^2\right) \cos(2t) + \frac{1}{4} \left[\frac{1}{2} (3 + 2t) \sin(2t) - \int \sin(2t)\,dt\right]$$

$$= -\frac{1}{2} \left(3t + t^2\right) \cos(2t) + \frac{1}{4} \left[\frac{3}{2} (3 + 2t) \sin(2t) + \frac{1}{2} \cos(2t)\right] + c$$

$$= -\frac{1}{2} \left(3t + t^2\right) \cos(2t) + \frac{1}{4} \left(3 + 2t\right) \sin(2t) + \frac{1}{4} \cos(2t) + c$$

4. Evaluate $\int 6 \tan^{-1}\left(\frac{x}{a}\right)\,dx$.

Hint: Be careful with your choices of $u$ and $dv$ here. If you think about it there is really only one way that the choice can be made here and have the problem be workable.

Step 1
The first step here is to pick $u$ and $dv$.

Note that if we choose the inverse tangent for $dv$ the only way to get $v$ is to integrate $dv$ and so we would need to know the answer to get the answer and so that won’t work for us. Therefore, the only real choice for the inverse tangent is to let it be $u$.

So, here are our choices for $u$ and $dv$. 

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Calculus II

\[ u = 6 \tan^{-1} \left( \frac{s}{w} \right) \quad dv = dw \]

Don’t forget the \( dw \)! The differential \( dw \) still needs to be put into the \( dv \) even though there is nothing else left in the integral.

Step 2
Next we need to compute \( du \) (by differentiating \( u \)) and \( v \) (by integrating \( dv \)).

\[ u = 6 \tan^{-1} \left( \frac{s}{w} \right) \quad \rightarrow \quad du = 6 \frac{-\frac{s}{w^2}}{1 + \left( \frac{s}{w} \right)^2} dw = 6 \frac{-\frac{s}{w^2}}{1 + \frac{s^2}{w^2}} dw \]

\[ dv = dw \quad \rightarrow \quad v = w \]

Step 3
In order to complete this problem we’ll need to do some rewrite on \( du \) as follows,

\[ du = \frac{-48}{w^2 + 64} dw \]

Plugging \( u, du, v \) and \( dv \) into the Integration by Parts formula gives,

\[ \int 6 \tan^{-1} \left( \frac{s}{w} \right) dw = 6w \tan^{-1} \left( \frac{s}{w} \right) + 48 \int \frac{w}{w^2 + 64} dw \]

Step 4
Okay, the new integral we get is easily doable (with the substitution \( u = 64 + w^2 \)) and so all we need to do to finish this problem out is do the integral.

\[ \int 6 \tan^{-1} \left( \frac{s}{w} \right) dw = 6w \tan^{-1} \left( \frac{s}{w} \right) + 24 \ln \left| w^2 + 64 \right| + c \]

5. Evaluate \( \int e^{z^2} \cos \left( \frac{1}{3} z \right) dz \).

Hint : This is one of the few integration by parts problems where either function can go on \( u \) and \( dv \). Be careful however to not get locked into an endless cycle of integration by parts.

Step 1
The first step here is to pick \( u \) and \( dv \).

In this case we can put the exponential in either the \( u \) or the \( dv \) and the cosine in the other. It is one of the few problems where the choice doesn’t really matter.

For this problem we’ll use the following choices for \( u \) and \( dv \).

\[ u = \cos \left( \frac{1}{3} z \right) \quad dv = e^{z^2} dz \]

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Step 2
Next we need to compute \( du \) (by differentiating \( u \)) and \( v \) (by integrating \( dv \)).

\[
\begin{align*}
  u &= \cos \left( \frac{1}{4} z \right) & \Rightarrow & & du &= -\frac{1}{4} \sin \left( \frac{1}{4} z \right) dz \\
  dv &= e^{2z} dz & \Rightarrow & & v &= \frac{1}{2} e^{2z}
\end{align*}
\]

Step 3
Plugging \( u, \ du, \ v \) and \( dv \) into the Integration by Parts formula gives,

\[
\int e^{2z} \cos \left( \frac{1}{4} z \right) dz = \frac{1}{2} e^{2z} \cos \left( \frac{1}{4} z \right) + \frac{1}{8} \int e^{2z} \sin \left( \frac{1}{4} z \right) dz
\]

Step 4
We’ll now need to do integration by parts again and to do this we’ll use the following choices.

\[
\begin{align*}
  u &= \sin \left( \frac{1}{4} z \right) & \Rightarrow & & du &= \frac{1}{4} \cos \left( \frac{1}{4} z \right) dz \\
  dv &= e^{2z} dz & \Rightarrow & & v &= \frac{1}{2} e^{2z}
\end{align*}
\]

Step 5
Plugging these into the integral from Step 3 gives,

\[
\int e^{2z} \cos \left( \frac{1}{4} z \right) dz = \frac{1}{2} e^{2z} \cos \left( \frac{1}{4} z \right) + \frac{1}{16} e^{2z} \sin \left( \frac{1}{4} z \right) - \frac{1}{64} \int e^{2z} \cos \left( \frac{1}{4} z \right) dz
\]

\[
\int e^{2z} \cos \left( \frac{1}{4} z \right) dz = \frac{1}{2} e^{2z} \cos \left( \frac{1}{4} z \right) + \frac{1}{16} e^{2z} \sin \left( \frac{1}{4} z \right) - \frac{1}{64} \int e^{2z} \cos \left( \frac{1}{4} z \right) dz
\]

Step 6
To finish this problem all we need to do is some basic algebraic manipulation to get the identical integrals on the same side of the equal sign.

\[
\int e^{2z} \cos \left( \frac{1}{4} z \right) dz = \frac{1}{2} e^{2z} \cos \left( \frac{1}{4} z \right) + \frac{1}{16} e^{2z} \sin \left( \frac{1}{4} z \right) - \frac{1}{64} \int e^{2z} \cos \left( \frac{1}{4} z \right) dz
\]

\[
\int e^{2z} \cos \left( \frac{1}{4} z \right) dz + \frac{1}{64} \int e^{2z} \cos \left( \frac{1}{4} z \right) dz = \frac{1}{2} e^{2z} \cos \left( \frac{1}{4} z \right) + \frac{1}{16} e^{2z} \sin \left( \frac{1}{4} z \right)
\]

\[
\frac{65}{64} \int e^{2z} \cos \left( \frac{1}{4} z \right) dz = \frac{1}{2} e^{2z} \cos \left( \frac{1}{4} z \right) + \frac{1}{16} e^{2z} \sin \left( \frac{1}{4} z \right)
\]

Finally, all we need to do is move the coefficient on the integral over to the right side.

\[
\int e^{2z} \cos \left( \frac{1}{4} z \right) dz = \frac{65}{64} e^{2z} \cos \left( \frac{1}{4} z \right) + \frac{4}{65} e^{2z} \sin \left( \frac{1}{4} z \right) + c
\]

6. Evaluate \( \int_{0}^{\pi} x^2 \cos \left( 4x \right) dx \).

Hint : Remember that we want to pick \( u \) and \( dv \) so that upon computing \( du \) and \( v \) and plugging everything into the Integration by Parts formula the new integral is one that we can do (or at least will be easier to deal with).

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Also, don’t forget that the limits on the integral won’t have any effect on the choices of \( u \) and \( dv \).

**Step 1**
The first step here is to pick \( u \) and \( dv \). We want to choose \( u \) and \( dv \) so that when we compute \( du \) and \( v \) and plugging everything into the Integration by Parts formula the new integral we get is one that we can do, or will at least be an integral that will be easier to deal with.

With that in mind it looks like the following choices for \( u \) and \( dv \) should work for us.

\[
\begin{align*}
\text{Step 2} \\
\text{Step 3} \\
\text{Step 4} \\
\text{Step 5}
\end{align*}
\]

Next we need to compute \( du \) (by differentiating \( u \)) and \( v \) (by integrating \( dv \)).

\[
\begin{align*}
 u &= x^2 & \rightarrow & & du = 2x \, dx \\
 dv &= \cos(4x) \, dx & \rightarrow & & v = \frac{1}{4} \sin(4x)
\end{align*}
\]

We can deal with the limits as we do the integral or we can just do the indefinite integral and then take care of the limits in the last step. We will be using the later way of dealing with the limits for this problem.

So, plugging \( u, du, v \) and \( dv \) into the Integration by Parts formula gives,

\[
\int x^2 \cos(4x) \, dx = \frac{1}{4} x^2 \sin(4x) - \frac{1}{2} \int x \sin(4x) \, dx
\]

Now, the new integral is still not one that we can do with only Calculus I techniques. However, it is one that we can do another integration by parts on and because the power on the \( x \)'s have gone down by one we are heading in the right direction.

So, here are the choices for \( u \) and \( dv \) for the new integral.

\[
\begin{align*}
\text{Step 5}
\end{align*}
\]

Okay, all we need to do now is plug these new choices of \( u \) and \( dv \) into the new integral we got in Step 3 and evaluate the integral.

\[
\begin{align*}
\int x^2 \cos(4x) \, dx &= \frac{1}{4} x^2 \sin(4x) - \frac{1}{2} \left[ -\frac{1}{4} x \cos(4x) + \frac{1}{4} \int \cos(4x) \, dx \right] \\
&= \frac{1}{4} x^2 \sin(4x) - \frac{1}{2} \left[ -\frac{1}{4} x \cos(4x) + \frac{1}{16} \sin(4x) \right] + c \\
&= \frac{1}{4} x^2 \sin(4x) + \frac{1}{4} x \cos(4x) - \frac{1}{16} \sin(4x) + c
\end{align*}
\]
Step 6
The final step is then to take care of the limits.

\[
\int_0^\pi x^2 \cos(4x) \, dx = \left( \frac{1}{4} x^2 \sin(4x) + \frac{1}{8} x \cos(4x) - \frac{1}{32} \sin(4x) \right) \bigg|_0^\pi = \frac{\pi^2}{8}
\]

7. Evaluate \( \int t^7 \sin(2t^4) \, dt \).

Hint: Be very careful with your choices of \( u \) and \( dv \) for this problem. It looks a lot like previous practice problems but it isn’t!

Step 1
The first step here is to pick \( u \) and \( dv \) and, in this case, we’ll need to be careful how we chose them.

If we follow the model of many of the examples/practice problems to this point it is tempting to let \( u = t^7 \) and to let \( dv = \sin(2t^4) \).

However, this will lead to some real problems. To compute \( v \) we’d have to integrate the sine and because of the \( t^4 \) in the argument this is not possible. In order to integrate the sine we would have to have a \( t^3 \) in the integrand as well in order to a substitution as shown below,

\[
\int t^3 \sin(2t^4) \, dt = \frac{1}{8} \int \sin(w) \, dw = -\frac{1}{8} \cos(2t^4) + c \quad w = 2t^4
\]

Now, this may seem like a problem, but in fact it’s not a problem for this particular integral. Notice that we actually have \( 7 \) \( t \)'s in the integral and there is no reason that we can’t split them up as follows,

\[
\int t^7 \sin(2t^4) \, dt = \int t^4 t^3 \sin(2t^4) \, dt
\]

After doing this we can now choose \( u \) and \( dv \) as follows,

\[
u = t^4 \quad \quad dv = t^3 \sin(2t^4) \, dt
\]

Step 2
Next we need to compute \( du \) (by differentiating \( u \)) and \( v \) (by integrating \( dv \)).

\[
u = t^4 \quad \quad \rightarrow \quad \quad du = 4t^3 \, dt
\]

\[
dv = t^3 \sin(2t^4) \, dt \quad \rightarrow \quad \quad v = -\frac{1}{8} \cos(2t^4)
\]

Step 3
Plugging \( u, \, du, \, v \) and \( dv \) into the Integration by Parts formula gives,

\[
\int t^7 \sin(2t^4) \, dt = -\frac{1}{8} t^4 \cos(2t^4) + \frac{1}{2} \int t^3 \cos(2t^4) \, dt
\]
Step 4
At this point, notice that the new integral just requires the same Calculus I substitution that we used to find $v$. So, all we need to do is evaluate the new integral and we’ll be done.

$$\int t^7 \sin(2t^4) \, dt = \frac{-1}{8} t^4 \cos(2t^4) + \frac{1}{16} \sin(2t^4) + c$$

Do not get so locked into patterns for these problems that you end up turning the patterns into “rules” on how certain kinds of problems work. Most of the easily seen patterns are also easily broken (as this problem has shown).

Because we (as instructors) tend to work a lot of “easy” problems initially they also tend to conform to the patterns that can be easily seen. This tends to lead students to the idea that the patterns will always work and then when they run into one where the pattern doesn’t work they get in trouble. So, be careful!

Note as well that we’re not saying that patterns don’t exist and that it isn’t useful to recognize them. You just need to be careful and understand that there will, on occasion, be problems where it will look like a pattern you recognize, but in fact will not quite fit the pattern and another approach will be needed to work the problem.

Alternate Solution
Note that there is an alternate solution to this problem. We could use the substitution $w = 2t^4$ as the first step as follows.

$$w = 2t^4 \quad \rightarrow \quad dw = 8t^3 \, dt \quad \& \quad t^4 = \frac{1}{2} w$$

$$\int t^7 \sin(2t^4) \, dt = \int t^4 t^3 \sin(2t^4) \, dt = \int \left(\frac{1}{2} w\right) \left(\frac{1}{8}\right) \sin(w) \, dw = \int \frac{1}{16} w \sin(w) \, dw$$

We won’t avoid integration by parts as we can see here, but it is somewhat easier to see it this time. Here is the rest of the work for this problem.

$$u = \frac{1}{16} w \quad \rightarrow \quad du = \frac{1}{16} \, dw$$

$$dv = \sin(w) \, dw \quad \rightarrow \quad v = -\cos(w)$$

$$\int t^7 \sin(2t^4) \, dt = -\frac{1}{16} \cos(w) + \frac{1}{16} \int \cos(w) \, dw = -\frac{1}{16} \cos(w) + \frac{1}{16} \sin(w) + c$$

As the final step we just need to substitution back in for $w$.

$$\int t^7 \sin(2t^4) \, dt = -\frac{1}{8} t^4 \cos(2t^4) + \frac{1}{16} \sin(2t^4) + c$$

8. Evaluate $\int y^6 \cos(3y) \, dy$.

Hint: Doing this with “standard” integration by parts would take a fair amount of time so maybe this would be a good candidate for the “table” method of integration by parts.
Step 1
Okay, with this problem doing the “standard” method of integration by parts (i.e. picking \( u \) and \( dv \) and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.

\[
\begin{array}{ccc}
\text{ } & \text{cos}(3y) & + \\
y^6 & \frac{1}{3} \sin(3y) & - \\
6y^5 & \frac{1}{3} \cos(3y) & + \\
30y^4 & -\frac{1}{9} \cos(3y) & + \\
120y^3 & -\frac{1}{27} \sin(3y) & - \\
360y^2 & \frac{1}{81} \cos(3y) & + \\
720y & -\frac{1}{243} \sin(3y) & - \\
720 & -\frac{1}{729} \cos(3y) & + \\
0 & \frac{1}{2187} \sin(3y) & - \\
\end{array}
\]

Step 2
Here’s the integral for this problem,

\[
\int y^6 \cos(3y) \, dy = \left( y^6 \right) \left( \frac{1}{3} \sin(3y) \right) - \left( 6y^5 \right) \left( -\frac{1}{9} \cos(3y) \right) + \left( 30y^4 \right) \left( -\frac{1}{27} \sin(3y) \right) - \left( 120y^3 \right) \left( \frac{1}{81} \cos(3y) \right) + \left( 360y^2 \right) \left( -\frac{1}{243} \sin(3y) \right) - \left( 720y \right) \left( -\frac{1}{729} \cos(3y) \right) + \left( 720 \right) \left( -\frac{1}{2187} \sin(3y) \right) + c
\]

\[
= \frac{1}{3} y^6 \sin(3y) + \frac{2}{9} y^5 \cos(3y) - \frac{10}{9} y^4 \sin(3y) - \frac{40}{27} y^3 \cos(3y) + \frac{40}{243} y^2 \sin(3y) + \frac{80}{81} y \cos(3y) - \frac{80}{243} \sin(3y) + c
\]

9. Evaluate \( \int \left( 4x^3 - 9x^2 + 7x + 3 \right) e^{-x} \, dx \).

Hint: Doing this with “standard” integration by parts would take a fair amount of time so maybe this would be a good candidate for the “table” method of integration by parts.

Step 1
Okay, with this problem doing the “standard” method of integration by parts (i.e. picking \( u \) and \( dv \) and using the formula) would take quite a bit of time. So, this looks like a good problem to use the table that we saw in the notes to shorten the process up.

Here is the table for this problem.
Step 2
Here’s the integral for this problem,

\[
\int (4x^3 - 9x^2 + 7x + 3) e^{-x} \, dx = \left(4x^3 - 9x^2 + 7x + 3\right)(-e^{-x}) - \left(12x^2 - 18x + 7\right)(e^{-x}) \\
+ \left(24x - 18\right)(-e^{-x}) - (24)(e^{-x}) + c \\
= -e^{-x} \left(4x^3 - 9x^2 + 7x + 3\right) - e^{-x} \left(12x^2 - 18x + 7\right) \\
- e^{-x} \left(24x - 18\right) - 24e^{-x} + c \\
= -e^{-x} \left(4x^3 + 3x^2 + 13x + 16\right)
\]

\textbf{Integrals Involving Trig Functions}

1. Evaluate \( \int \sin^3 \left(\frac{2}{5}x\right) \cos^4 \left(\frac{2}{5}x\right) \, dx \).

Hint: Pay attention to the exponents and recall that for most of these kinds of problems you’ll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the sine is odd and so we can strip one of them out.

\[
\int \sin^3 \left(\frac{2}{5}x\right) \cos^4 \left(\frac{2}{5}x\right) \, dx = \int \sin^2 \left(\frac{2}{5}x\right) \cos^4 \left(\frac{2}{5}x\right) \sin \left(\frac{2}{5}x\right) \, dx
\]

Step 2
Now we can use the trig identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to convert the remaining sines to cosines.

\[
\int \sin^3 \left(\frac{2}{5}x\right) \cos^4 \left(\frac{2}{5}x\right) \, dx = \int \left(1 - \cos^2 \left(\frac{2}{5}x\right)\right) \cos^4 \left(\frac{2}{5}x\right) \sin \left(\frac{2}{5}x\right) \, dx
\]

Step 3
We can now use the substitution \( u = \cos \left(\frac{2}{5}x\right) \) to evaluate the integral.
\[ \int \sin^3\left(\frac{2}{3}x\right) \cos^4\left(\frac{2}{3}x\right) \, dx = -\frac{1}{2} \int (1-u^2) u^4 \, du = \frac{1}{2} \int u^4 - u^6 \, du = \frac{1}{2} \left( \frac{1}{5} u^5 - \frac{1}{7} u^7 \right) + c \]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don’t forget to substitute back in for \( u \)!

\[ \int \sin^3\left(\frac{2}{3}x\right) \cos^4\left(\frac{2}{3}x\right) \, dx = \frac{1}{15} \cos^7\left(\frac{2}{3}x\right) - \frac{4}{15} \cos^5\left(\frac{2}{3}x\right) + c \]

2. Evaluate \( \int \sin^8\left(3z\right) \cos^5\left(3z\right) \, dz \).

Hint: Pay attention to the exponents and recall that for most of these kinds of problems you’ll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the cosine is odd and so we can strip one of them out.

\[ \int \sin^8\left(3z\right) \cos^5\left(3z\right) \, dz = \int \sin^8\left(3z\right) \cos^4\left(3z\right) \cos\left(3z\right) \, dz \]

Step 2
Now we can use the trig identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to convert the remaining cosines to sines.

\[ \int \sin^8\left(3z\right) \cos^5\left(3z\right) \, dz = \int \sin^8\left(3z\right) \left[ \cos^2\left(3z\right) \right]^2 \cos\left(3z\right) \, dz = \int \sin^8\left(3z\right) \left[ 1 - \sin^2\left(3z\right) \right]^2 \cos\left(3z\right) \, dz \]

Step 3
We can now use the substitution \( u = \sin\left(3z\right) \) to evaluate the integral.

\[ \int \sin^8\left(3z\right) \cos^5\left(3z\right) \, dz = \frac{1}{3} \int u^8 \left[ 1 - u^2 \right]^2 \, du = \frac{1}{3} \int u^8 - 2u^{10} + u^{12} \, du = \frac{1}{3} \left( \frac{1}{9} u^9 - \frac{2}{11} u^{11} + \frac{1}{13} u^{13} \right) + c \]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.
Step 4
Don’t forget to substitute back in for $u$!

\[ \int \sin^8(3z)\cos^5(3z)\,dz = \frac{1}{7} \sin^9(3z) - \frac{2}{35} \sin^{11}(3z) + \frac{1}{115} \sin^{13}(3z) + c \]

3. Evaluate $\int \cos^4(2t)\,dt$.

Hint: Pay attention to the exponents and recall that for most of these kinds of problems you’ll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that we only have even exponents and so we’ll need to use half-angle and double-angle formulas to reduce this integral into one that we can do.

Also, do not get excited about the fact that we don’t have any sines in the integrand. Sometimes we will not have both trig functions in the integrand. That doesn’t mean that that we can’t use the same techniques that we used in this section.

So, let’s start this problem off as follows.

\[ \int \cos^4(2t)\,dt = \int \left( \cos^2(2t) \right)^2 \,dt \]

Step 2
Now we can use the half-angle formula to get,

\[ \int \cos^4(2t)\,dt = \int \left[ \frac{1}{2} (1 + \cos(4t)) \right]^2 \,dt = \int \frac{1}{4} (1 + 2 \cos(4t) + \cos^2(4t)) \,dt \]

Step 3
We’ll need to use the half-angle formula one more time on the third term to get,

\[ \int \cos^4(2t)\,dt = \frac{1}{4} \int 1 + 2 \cos(4t) + \frac{1}{2} \left[ 1 + \cos(8t) \right] \,dt \]

\[ = \frac{1}{4} \int \frac{3}{2} + 2 \cos(4t) + \frac{1}{2} \cos(8t) \,dt \]

Step 4
Now all we have to do is evaluate the integral.

\[ \int \cos^4(2t)\,dt = \frac{1}{4} \left( \frac{3}{2} t + \frac{1}{2} \sin(4t) + \frac{1}{8} \sin(8t) \right) + c = \frac{3}{8} t + \frac{1}{4} \sin(4t) + \frac{1}{16} \sin(8t) + c \]
4. Evaluate \( \int_{\pi}^{2\pi} \cos^3 \left( \frac{1}{2} w \right) \sin^5 \left( \frac{1}{2} w \right) \, dw \).

Hint: Pay attention to the exponents and recall that for most of these kinds of problems you’ll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
We have two options for dealing with the limits. We can deal with the limits as we do the integral or we can evaluate the indefinite integral and take care of the limits in the last step. We’ll use the latter method of dealing with the limits for this problem.

In this case notice that both exponents are odd. This means that we can either strip out a cosine and convert the rest to sines or strip out a sine and convert the rest to cosines. Either are perfectly acceptable solutions. However, the exponent on the cosine is smaller and so there will be less conversion work if we strip out a cosine and convert the remaining cosines to sines.

Here is that work.

\[
\int \cos^3 \left( \frac{1}{2} w \right) \sin^5 \left( \frac{1}{2} w \right) \, dw = \int \cos^2 \left( \frac{1}{2} w \right) \sin^5 \left( \frac{1}{2} w \right) \cos \left( \frac{1}{2} w \right) \, dw \\
= \int \left( 1 - \sin^2 \left( \frac{1}{2} w \right) \right) \sin^5 \left( \frac{1}{2} w \right) \cos \left( \frac{1}{2} w \right) \, dw
\]

Step 2
We can now use the substitution \( u = \sin \left( \frac{1}{2} w \right) \) to evaluate the integral.

\[
\int \cos^3 \left( \frac{1}{2} w \right) \sin^5 \left( \frac{1}{2} w \right) \, dw = 2 \int \left( 1 - u^2 \right) u^5 \, du \\
= 2 \left( \frac{u^6}{6} - \frac{u^8}{8} \right) + C
\]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3
Don’t forget to substitute back in for \( u \)!

\[
\int \cos^3 \left( \frac{1}{2} w \right) \sin^5 \left( \frac{1}{2} w \right) \, dw = \frac{1}{4} \sin^6 \left( \frac{1}{2} w \right) - \frac{1}{4} \sin^8 \left( \frac{1}{2} w \right) + C
\]

Step 4
Now all we need to do is deal with the limits.

\[
\int_{\pi}^{2\pi} \cos^3 \left( \frac{1}{2} w \right) \sin^5 \left( \frac{1}{2} w \right) \, dw = \left. \left( \frac{1}{4} \sin^6 \left( \frac{1}{2} w \right) - \frac{1}{4} \sin^8 \left( \frac{1}{2} w \right) \right) \right|_{\pi}^{2\pi} = \frac{-1}{12}
\]

Alternate Solution
As we noted above we could just have easily stripped out a sine and converted the rest to cosines if we’d wanted to. We’ll not put that work in here, but here is the indefinite integral that you should have gotten had you done it that way.

\[ \int \cos^3 \left( \frac{1}{2} w \right) \sin^5 \left( \frac{1}{2} w \right) dw = -\frac{1}{2} \cos^4 \left( \frac{1}{2} w \right) + \frac{7}{3} \cos^6 \left( \frac{1}{2} w \right) - \frac{1}{3} \cos^8 \left( \frac{1}{2} w \right) + c \]

Note as well that regardless of which approach we use to doing the indefinite integral the value of the definite integral will be the same.

5. Evaluate \( \int \sec^6 (3y) \tan^2 (3y) \, dy \).

Hint: Pay attention to the exponents and recall that for most of these kinds of problems you’ll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the secant is even and so we can strip two of them out.

\[ \int \sec^6 (3y) \tan^2 (3y) \, dy = \int \sec^4 (3y) \tan^2 (3y) \sec^2 (3y) \, dy \]

Step 2
Now we can use the trig identity \( \tan^2 \theta + 1 = \sec^2 \theta \) to convert the remaining secants to tangents.

\[ \int \sec^6 (3y) \tan^2 (3y) \, dy = \int \left[ \sec^2 (3y) \right]^2 \tan^2 (3y) \sec^2 (3y) \, dy \]

\[ = \int \left[ \tan^2 (3y) + 1 \right]^2 \tan^2 (3y) \sec^2 (3y) \, dy \]

Step 3
We can now use the substitution \( u = \tan (3y) \) to evaluate the integral.

\[ \int \sec^6 (3y) \tan^2 (3y) \, dy = \frac{1}{3} \int \left[ u^2 + 1 \right]^2 u^2 \, du \]

\[ = \frac{1}{3} \int u^6 + 2u^4 + u^2 \, du = \frac{1}{3} \left( \frac{1}{7} u^7 + \frac{1}{2} u^5 + \frac{1}{3} u^3 \right) + c \]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don’t forget to substitute back in for \( u! \)

\[ \int \sec^6 (3y) \tan^2 (3y) \, dy = \frac{1}{77} \tan^7 (3y) + \frac{7}{16} \tan^5 (3y) + \frac{1}{6} \tan^3 (3y) + c \]
6. Evaluate $\int \tan^3(6x) \sec^{10}(6x) \, dx$.

Hint: Pay attention to the exponents and recall that for most of these kinds of problems you’ll need to use trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I substitution).

Step 1
The first thing to notice here is that the exponent on the tangent is odd and we’ve got a secant in the problems and so we can strip one of each of them out.

$$\int \tan^3(6x) \sec^{10}(6x) \, dx = \int \tan^2(6x) \sec^9(6x) \tan(6x) \sec(6x) \, dx$$

Step 2
Now we can use the trig identity $\tan^2 \theta + 1 = \sec^2 \theta$ to convert the remaining tangents to secants.

$$\int \tan^3(6x) \sec^{10}(6x) \, dx = \int \left[ \sec^2(6x) - 1 \right] \sec^9(6x) \tan(6x) \sec(6x) \, dx$$

Note that because the exponent on the secant is even we could also have just stripped two of them out and converted the rest of them to tangents. However, that conversion process would have been significantly more work than the path that we chose here.

Step 3
We can now use the substitution $u = \sec(6x)$ to evaluate the integral.

$$\int \tan^3(6x) \sec^{10}(6x) \, dx = \frac{1}{2} \int \left[ u^2 - 1 \right] u^9 \, du$$

$$= \frac{1}{6} \int u^{11} - u^9 \, du = \frac{1}{6} \left( \frac{1}{12} u^{12} - \frac{1}{10} u^{10} \right) + c$$

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don’t forget to substitute back in for $u$!

$$\int \tan^3(6x) \sec^{10}(6x) \, dx = \frac{1}{72} \sec^{12}(6x) - \frac{1}{60} \sec^{10}(6x) + c$$

7. Evaluate $\int_0^\pi \tan^7(z) \sec^3(z) \, dz$. 

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Hint: Pay attention to the exponents and recall that for most of these kinds of problems you’ll need to use
trig identities to put the integral into a form that allows you to do the integral (usually with a Calc I
substitution).

Step 1
We have two options for dealing with the limits. We can deal with the limits as we do the integral or we
can evaluate the indefinite integral and take care of the limits in the last step. We’ll use the latter method
of dealing with the limits for this problem.

The first thing to notice here is that the exponent on the tangent is odd and we’ve got a secant in the
problems and so we can strip one of each of them out and use the trig identity \( \tan^2 \theta + 1 = \sec^2 \theta \)
to convert the remaining tangents to secants.

\[
\int \tan^7(z) \sec^3(z) \, dz = \int \tan^6(z) \sec^2(z) \tan(z) \sec(z) \, dz
\]
\[
= \int \left[ \tan^2(z) \right]^3 \sec^2(z) \tan(z) \sec(z) \, dz
\]
\[
= \int \left[ \sec^2(z) - 1 \right]^3 \sec^2(z) \tan(z) \sec(z) \, dz
\]

Step 2
We can now use the substitution \( u = \sec(z) \) to evaluate the integral.

\[
\int \tan^7(z) \sec^3(z) \, dz = \int \left[ u^2 - 1 \right]^3 u^2 \, du
\]
\[
= \int u^8 - 3u^6 + 3u^4 - u^2 \, du = \frac{1}{9} u^9 - \frac{3}{7} u^7 + \frac{3}{5} u^5 - \frac{1}{3} u^3 + c
\]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that
you recall substitution well enough to fill in the details if you need to. If you are rusty on
substitutions you should probably go back to the Calculus I practice problems and practice on the
substitutions.

Step 3
Don’t forget to substitute back in for \( u \! \! \! 

\[
\int \tan^7(z) \sec^3(z) \, dz = \frac{1}{9} \sec^9(z) - \frac{3}{7} \sec^7(z) + \frac{3}{5} \sec^5(z) - \frac{1}{3} \sec^3(z) + c
\]

Step 4
Now all we need to do is deal with the limits.

\[
\int_0^\pi \tan^7(z) \sec^3(z) \, dz = \left[ \frac{1}{9} \sec^9(z) - \frac{3}{7} \sec^7(z) + \frac{3}{5} \sec^5(z) - \frac{1}{3} \sec^3(z) \right]_0^\pi
\]
\[
= \frac{2}{315} \left( 8 + 13\sqrt{2} \right) = 0.1675
\]
8. Evaluate \( \int \cos(3t)\sin(8t) \, dt \).

Step 1
There really isn’t all that much to this problem. All we have to do is use the formula given in this section for reducing a product of a sine and a cosine into a sum. Doing this gives,

\[
\int \cos(3t)\sin(8t) \, dt = \frac{1}{2} \left[ \sin(8t - 3t) + \sin(8t + 3t) \right] \, dt = \frac{1}{2} \int \sin(5t) + \sin(11t) \, dt
\]

Make sure that you pay attention to the formula! The formula given in this section listed the sine first instead of the cosine. Make sure that you used the formula correctly!

Step 2
Now all we need to do is evaluate the integral.

\[
\int \cos(3t)\sin(8t) \, dt = \frac{1}{2} \left( -\frac{1}{5} \cos(5t) - \frac{1}{11} \cos(11t) \right) + c = \frac{-1}{10} \cos(5t) - \frac{1}{22} \cos(11t) + c
\]

9. Evaluate \( \int_{\frac{1}{3}}^{3} \sin(8x)\sin(x) \, dx \).

Step 1
There really isn’t all that much to this problem. All we have to do is use the formula given in this section for reducing a product of a sine and a cosine into a sum. Doing this gives,

\[
\int_{\frac{1}{3}}^{3} \sin(8x)\sin(x) \, dx = \int_{\frac{1}{3}}^{3} \frac{1}{2} \left[ \cos(8x - x) - \cos(8x + x) \right] \, dx = \frac{1}{2} \int_{\frac{1}{3}}^{3} \cos(7x) - \cos(9x) \, dx
\]

Step 2
Now all we need to do is evaluate the integral.

\[
\int_{\frac{1}{3}}^{3} \sin(8x)\sin(x) \, dx = \frac{1}{2} \left[ \frac{1}{7} \sin(7x) - \frac{1}{9} \sin(9x) \right]_{\frac{1}{3}}^{3} = \frac{1}{14} \sin(21) - \frac{1}{18} \sin(27) - \frac{1}{12} \sin(7) + \frac{1}{18} \sin(9) = -0.0174
\]

Make sure your calculator is set to radians if you computed a decimal answer!

10. Evaluate \( \int \cot(10z) \csc^4(10z) \, dz \).

Hint: Even though no examples of products of cotangents and cosecants were done in the notes for this section you should know how to do them. Ask yourself how you would do the problem if it involved tangents and secants instead and you should be able to see how to do this problem as well.
Step 1
Other than the obvious difference in the actual functions there is no practical difference in how this problem and one that had tangents and secants would work. So, all we need to do is ask ourselves how this would work if it involved tangents and secants and we’ll be able to work this on as well.

We can first notice here is that the exponent on the cotangent is odd and we’ve got a cosecant in the problems and so we can strip the (only) cotangent and one of the secants out.

\[ \int \cot(10z) \csc^4(10z) \, dz = \int \csc^3(10z) \, \cot(10z) \csc(10z) \, dz \]

Step 2
Normally we would use the trig identity \( \cot^2 \theta + 1 = \csc^2 \theta \) to convert the remaining cotangents to cosecants. However, in this case there are no remaining cotangents to convert and so there really isn’t anything to do at this point other than to use the substitution \( u = \csc(10z) \) to evaluate the integral.

\[ \int \cot(10z) \csc^4(10z) \, dz = -\frac{1}{10} \int u^3 \, du = -\frac{1}{40} u^4 + c \]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 3
Don’t forget to substitute back in for \( u! \)

\[ \int \cot(10z) \csc^4(10z) \, dz = -\frac{1}{40} \csc^4(10z) + c \]

11. Evaluate \( \int \csc^6 \left( \frac{w}{4} \right) \cot^4 \left( \frac{w}{4} \right) \, dw \).

Hint: Even though no examples of products of cotangents and cosecants were done in the notes for this section you should know how to do them. Ask yourself how you would do the problem if it involved tangents and secants instead and you should be able to see how to do this problem as well.

Step 1
Other than the obvious difference in the actual functions there is no practical difference in how this problem and one that had tangents and secants would work. So, all we need to do is ask ourselves how this would work if it involved tangents and secants and we’ll be able to work this on as well.

We can first notice here is that the exponent on the cosecant is even and so we can strip out two of them.

\[ \int \csc^6 \left( \frac{w}{4} \right) \cot^4 \left( \frac{w}{4} \right) \, dw = \int \csc^4 \left( \frac{w}{4} \right) \cot^4 \left( \frac{w}{4} \right) \csc^2 \left( \frac{w}{4} \right) \, dw \]

Step 2
Now we can use the trig identity \( \cot^2 \theta + 1 = \csc^2 \theta \) to convert the remaining cosecants to cotangents.
\[
\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)dw = \int \left[\csc^2\left(\frac{1}{4}w\right)\right]^2 \csc^4\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)dw
\]
\[
= \int \left[\cot^2\left(\frac{1}{4}w\right)+1\right]^2 \csc^4\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)dw
\]

**Step 3**

Now we can use the substitution \( u = \cot\left(\frac{1}{4}w\right) \) to evaluate the integral.

\[
\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)dw = -4\int \left[u^2 + 1\right]^2 u^4 du
\]
\[
= -4\int u^8 + 2u^6 + u^4 du = -4\left(\frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5\right) + C
\]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

**Step 4**

Don’t forget to substitute back in for \( u \)!

\[
\int \csc^6\left(\frac{1}{4}w\right)\cot^4\left(\frac{1}{4}w\right)dw = -\frac{4}{9}\cot^9\left(\frac{1}{4}w\right) - \frac{8}{7}\cot^7\left(\frac{1}{4}w\right) - \frac{4}{5}\cot^5\left(\frac{1}{4}w\right) + C
\]

---

12. Evaluate \( \int \frac{\sec^4(2t)}{\tan^9(2t)} dt \).

**Hint**: How would you do this problem if it were a product?

**Step 1**

If this were a product of secants and tangents we would know how to do it. The same ideas work here, except that we have to pay attention to only the numerator. We can’t strip anything out of the denominator (in general) and expect it to work the same way. We can only strip things out of the numerator.

So, let’s notice here is that the exponent on the secant is even and so we can strip out two of them.

\[
\int \frac{\sec^4(2t)}{\tan^9(2t)} dt = \int \frac{\sec^2(2t)}{\tan^9(2t)} \sec^2(2t) dt
\]

**Step 2**

Now we can use the trig identity \( \tan^2\theta + 1 = \sec^2\theta \) to convert the remaining secants to tangents.

\[
\int \frac{\sec^4(2t)}{\tan^9(2t)} dt = \int \frac{\tan^2(2t) + 1}{\tan^9(2t)} \sec^2(2t) dt
\]
Step 3
Now we can use the substitution \( u = \tan(2t) \) to evaluate the integral.

\[
\int \frac{\sec^4(2t)}{\tan^9(2t)} \, dt = \frac{1}{2} \int \frac{u^2 + 1}{u^9} \, du = \frac{1}{2} \int u^7 + u^{-9} \, du = \frac{1}{2} \left[ -\frac{1}{6} u^{-6} - \frac{1}{8} u^{-8} \right] + c
\]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 4
Don’t forget to substitute back in for \( u \)!

\[
\int \frac{\sec^4(2t)}{\tan^9(2t)} \, dt = -\frac{1}{12} \tan^6(2t) - \frac{1}{16} \tan^4(2t) + c = -\frac{1}{12} \cot^6(2t) - \frac{1}{16} \cot^8(2t) + c
\]

13. Evaluate \( \int \frac{2 + 7 \sin^3(z)}{\cos^2(z)} \, dz \).

Hint : How would you do this problem if it were a product?

Step 1
Because of the sum in the numerator it makes some sense (hopefully) to maybe split the integrand (and then the integral) up into two as follows.

\[
\int \frac{2 + 7 \sin^3(z)}{\cos^2(z)} \, dz = \int \frac{2}{\cos^2(z)} + \frac{7 \sin^3(z)}{\cos^2(z)} \, dz = \int \frac{2}{\cos^2(z)} \, dz + \int \frac{7 \sin^3(z)}{\cos^2(z)} \, dz
\]

Step 2
Now, the first integral looks difficult at first glance, but we can easily rewrite this in terms of secants at which point it becomes a really easy integral.

For the second integral again, think about how we would do that if it was a product instead of a quotient. In that case we would simply strip out a sine.

\[
\int \frac{2 + 7 \sin^3(z)}{\cos^2(z)} \, dz = \int 2 \sec^2(z) \, dz + 7 \int \frac{\sin^2(z)}{\cos^2(z)} \sin(z) \, dz
\]

Step 3
As noted above the first integral is now very easy (which we’ll do in the next step) and for the second integral we can use the trig identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to convert the remaining sines in the second integral to cosines.
\[
\int \frac{2 + 7 \sin^3(z)}{\cos^2(z)} \, dz = \int 2 \sec^2(z) \, dz + 7 \int \frac{1 - \cos^2(z)}{\cos^2(z)} \sin(z) \, dz
\]

Step 4
Now we can use the substitution \( u = \cos(z) \) to evaluate the second integral. The first integral doesn’t need any extra work.

\[
\int \frac{2 + 7 \sin^3(z)}{\cos^2(z)} \, dz = 2 \tan(z) - 7 \int \frac{1 - u^2}{u^2} \, du
\]

\[
= 2 \tan(z) - 7 \int u^{-2} - 1 \, du = 2 \tan(z) - 7 \left( -u^{-1} - u \right) + c
\]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 5
Don’t forget to substitute back in for \( u \)!

\[
\int \frac{2 + 7 \sin^3(z)}{\cos^2(z)} \, dz = 2 \tan(z) + 7 \frac{1}{\cos(z)} + 7 \cos(z) + c = 2 \tan(z) + 7 \sec(z) + 7 \cos(z) + c
\]

14. Evaluate \( \int \left[ 9 \sin^5(3x) - 2 \cos^3(3x) \right] \csc^4(3x) \, dx \).

Hint : Since this has a mix of trig functions maybe the best option would be to first get it reduced down to just a couple that we know how to deal with.

Step 1
To get started on this problem we should first probably see if we can reduce the integrand down to just sines and cosines. This is easy enough to do simply by recalling the definition of cosecant in terms of sine.

\[
\int \left[ 9 \sin^5(3x) - 2 \cos^3(3x) \right] \csc^4(3x) \, dx = \int \left[ 9 \sin^5(3x) - 2 \cos^3(3x) \right] \frac{1}{\sin^4(3x)} \, dx
\]

\[
= \int 9 \sin(3x) - 2 \frac{\cos^3(3x)}{\sin^4(3x)} \, dx
\]

Step 2
The first integral is simple enough to do without any extra work.
For the second integral again, think about how we would do that if it was a product instead of a quotient. In that case we would simply strip out a cosine.

\[
\int [9 \sin^5(3x) - 2 \cos^3(3x)] \csc^4(3x) \, dx = \int 9 \sin(3x) - 2 \frac{\cos^2(3x)}{\sin^4(3x)} \cos(3x) \, dx
\]

Step 3
For the second integral we can use the trig identity \( \sin^2 \theta + \cos^2 \theta = 1 \) to convert the remaining cosines to sines.

\[
\int [9 \sin^5(3x) - 2 \cos^3(3x)] \csc^4(3x) \, dx = \int 9 \sin(3x) \, dx - 2 \int \frac{1 - \sin^2(3x)}{\sin^4(3x)} \cos(3x) \, dx
\]

Step 4
Now we can use the substitution \( u = \sin(3x) \) to evaluate the second integral. The first integral doesn’t need any extra work.

\[
\int [9 \sin^5(3x) - 2 \cos^3(3x)] \csc^4(3x) \, dx = \int 9 \sin(3x) \, dx - 2 \int \frac{1 - u^2}{u^4} \, du = \int 9 \sin(3x) \, dx - 2 \int u^{-4} - u^{-2} \, du = -3 \cos(3x) - \frac{2}{3} u^{-3} + u^{-1} + c
\]

Note that we’ll not be doing the actual substitution work here. At this point it is assumed that you recall substitution well enough to fill in the details if you need to. If you are rusty on substitutions you should probably go back to the Calculus I practice problems and practice on the substitutions.

Step 5
Don’t forget to substitute back in for \( u \)!

\[
\int [9 \sin^5(3x) - 2 \cos^3(3x)] \csc^4(3x) \, dx = -3 \cos(3x) + \frac{2}{9} \frac{1}{\sin^3(3x)} - \frac{2}{3} \frac{1}{\sin(3x)} + c = -3 \cos(3x) + \frac{2}{9} \csc^3(3x) - \frac{2}{3} \csc(3x) + c
\]

---

**Trig Substitutions**

1. Use a trig substitution to eliminate the root in \( \sqrt{4 - 9z^2} \).
Hint: When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1
The first step is to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

\[ 1 - \sin^2(\theta) = \cos^2(\theta) \]

So, it looks like sine is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint: In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we’ve done the substitution?

Step 2
To get the coefficient on the trig function notice that we need to turn the 9 into a 4 once we’ve substituted the trig function in for \( z \) and squared the substitution out. With that in mind it looks like the substitution should be,

\[ z = \frac{2}{3} \sin(\theta) \]

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3
\[
\sqrt{4 - 9z^2} = \sqrt{4 - 9\left(\frac{2}{3} \sin(\theta)\right)^2} = \sqrt{4 - 9\left(\frac{2}{3}\right)\sin^2(\theta)}
\]
\[
= \sqrt{4 - 4\sin^2(\theta) = 2\sqrt{1 - \sin^2(\theta)}}
\]
\[
= 2\sqrt{\cos^2(\theta)} = 2|\cos(\theta)|
\]

Note that because we don’t know the values of \( \theta \) we can’t determine if the cosine is positive or negative and so cannot get rid of the absolute value bars here.

2. Use a trig substitution to eliminate the root in \( \sqrt{13 + 25x^2} \).

Hint: When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1
The first step is to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

\[ 1 + \tan^2(\theta) = \sec^2(\theta) \]
So, it looks like tangent is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint : In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we’ve done the substitution?

Step 2
To get the coefficient on the trig function notice that we need to turn the 25 into a 13 once we’ve substituted the trig function in for $x$ and squared the substitution out. With that in mind it looks like the substitution should be,

$$x = \frac{\sqrt{13}}{5} \tan(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3
$$\sqrt{13 + 25x^2} = \sqrt{13 + 25\left(\frac{\sqrt{13}}{5} \tan(\theta)\right)^2} = \sqrt{13 + 25\left(\frac{13}{25}\right)\tan^2(\theta)}$$

$$= \sqrt{13 + 13\tan^2(\theta)} = \sqrt{13\left(1 + \tan^2(\theta)\right)}$$

$$= \sqrt{13\sec^2(\theta)} = \sqrt{13|\sec(\theta)|}$$

Note that because we don’t know the values of $\theta$ we can’t determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

3. Use a trig substitution to eliminate the root in $\left(7t^2 - 3\right)^{\frac{5}{2}}$.

Hint : When determining which trig function to use for the substitution recall from the notes in this section that we will use one of three trig identities to convert the sum or difference under the root into a single trig function. Which trig identity is closest to the quantity under the root?

Step 1
First, notice that there really is a root here as the term can be written as,

$$\left(7t^2 - 3\right)^{\frac{5}{2}} = \left[\left(7t^2 - 3\right)^{\frac{1}{2}}\right]^5 = \left[\sqrt{7t^2 - 3}\right]^5$$

Now, we need to figure out which trig function to use for the substitution. To determine this notice that (ignoring the numbers) the quantity under the root looks similar to the identity,

$$\sec^2(\theta) - 1 = \tan^2(\theta)$$
So, it looks like secant is probably the correct trig function to use for the substitution. Now, we need to deal with the numbers on the two terms.

Hint: In order to actually use the identity from the first step we need to get the numbers in each term to be identical upon doing the substitution. So, what would the coefficient of the trig function need to be in order to convert the coefficient of the variable into the constant term once we’ve done the substitution?

Step 2
To get the coefficient on the trig function notice that we need to turn the 7 into a 3 once we’ve substituted the trig function in for \( t \) and squared the substitution out. With that in mind it looks like the substitution should be,

\[
 t = \frac{\sqrt{7}}{\sqrt{3}} \sec(\theta)
\]

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3
\[
\left(7t^2 - 3\right)^{\frac{3}{2}} = \left[\sqrt{7t^2 - 3}\right]^{\frac{3}{2}} = \left[\sqrt{7\left(\frac{\sqrt{7}}{\sqrt{3}} \sec(\theta)\right)^2 - 3}\right]^{\frac{3}{2}} = \left[\sqrt{7\left(\frac{1}{3} \sec^2(\theta)\right) - 3}\right]^{\frac{3}{2}} = \left[\sqrt{3\sec^2(\theta) - 3}\right]^{\frac{3}{2}} = \left[\sqrt{3\sqrt{\tan^2(\theta)} - 1}\right]^{\frac{3}{2}} = \left[\sqrt{3|\tan(\theta)|}\right]^{\frac{3}{2}}
\]

Note that because we don’t know the values of \( \theta \) we can’t determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

4. Use a trig substitution to eliminate the root in \( \sqrt{(w+3)^2 - 100} \).

Hint: Just because this looks a little different from the first couple of problems in this section doesn’t mean that it works any differently. The term under the root still looks vaguely like one of three trig identities we need to use to convert the quantity under the root into a single trig function.

Step 1
Okay, first off we need to acknowledge that this does look a little bit different from the first few problems in this section. However, it isn’t really all that different. We still have a difference between a squared term with a variable in it and a number. This looks similar to the following trig identity (ignoring the coefficients as usual).

\[
\sec^2(\theta) - 1 = \tan^2(\theta)
\]
So, secant is the trig function we’ll need to use for the substitution here and we now need to deal with the numbers on the terms and get the substitution set up.

Hint : Dealing with the numbers in this case is no different than the first few problems in this section.

Step 2
Before dealing with the coefficient on the trig function let’s notice that we’ll be substituting in for \( w + 3 \) in this case since that is the quantity that is being squared in the first term.

So, to get the coefficient on the trig function notice that we need to turn the 1 (i.e. the coefficient of the squared term) into a 100 once we’ve done the substitution. With that in mind it looks like the substitution should be,

\[
w + 3 = 10 \sec(\theta)
\]

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3
\[
\sqrt{(w+3)^2 - 100} = \sqrt{(10\sec(\theta))^2 - 100} = 10\sqrt{\sec^2(\theta)-1} = 10\sqrt{\tan^2(\theta) + 1}
\]

Note that because we don’t know the values of \( \theta \) we can’t determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.
Before dealing with the coefficient on the trig function let’s notice that we’ll be substituting in for $9t - 5$ in this case since that is the quantity that is being squared in the first term.

So, to get the coefficient on the trig function notice that we need to turn the 4 (i.e. the coefficient of the squared term) into a 1 once we’ve done the substitution. With that in mind it looks like the substitution should be,

$$9t - 5 = \frac{1}{2} \tan(\theta)$$

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3

$$\sqrt{4(9t - 5)^2 + 1} = \sqrt{4\left(\frac{1}{2} \tan(\theta)\right)^2 + 1} = \sqrt{4\tan^2(\theta) + 1} = \sqrt{\tan^2(\theta) + 1} = \sec(\theta)$$

Note that because we don’t know the values of $\theta$ we can’t determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

6. Use a trig substitution to eliminate the root in $\sqrt{1 - 4z - 2z^2}$.

Hint: This doesn’t look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1

We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. This clearly does not fit into that form. However, that doesn’t mean that we can’t do some algebraic manipulation on the quantity under the root to get into a form that we can do a trig substitution on.

Because the quantity under the root is a quadratic polynomial we know that we can complete the square on it to turn it into something like what we need for a trig substitution.

Here is the completing the square work.

$$1 - 4z - 2z^2 = -2\left(z^2 + 2z - \frac{1}{2}\right)$$

$$= -2\left(z^2 + 2z + 1 - 1 - \frac{1}{2}\right)$$

$$= -2\left[(z + 1)^2 - \frac{3}{2}\right]$$

$$= 3 - 2(z + 1)^2$$

So, after completing the square the term can be written as,
\[ \sqrt{1 - 4z - 2z^2} = \sqrt{3 - 2(z + 1)^2} \]

Hint: At this point the problem works in the same manner as the previous problems in this section.

Step 2
So, in this case we see that we have a difference of a number and a squared term with a variable in it. This suggests that sine is the correct trig function to use for the substitution.

Now, to get the coefficient on the trig function notice that we need to turn the 2 (i.e. the coefficient of the squared term) into a 3 once we’ve done the substitution. With that in mind it looks like the substitution should be,

\[ z + 1 = \frac{\sqrt{2}}{2} \sin(\theta) \]

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3
\[
\sqrt{1 - 4z - 2z^2} = \sqrt{3 - 2(z + 1)^2} = \sqrt{3 - 2\left(\frac{\sqrt{2}}{2} \sin(\theta)\right)^2} = \sqrt{3 - 3\sin^2(\theta)} = \sqrt{3\cos^2(\theta)} = \sqrt{3} |\cos(\theta)|
\]

Note that because we don’t know the values of \( \theta \) we can’t determine if the cosine is positive or negative and so cannot get rid of the absolute value bars here.

7. Use a trig substitution to eliminate the root in \( \left( x^2 - 8x + 21 \right)^{\frac{3}{2}} \).

Hint: This doesn’t look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1
We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. This clearly does not fit into that form. However, that doesn’t mean that we can’t do some algebraic manipulation on the quantity under the root to get into a form that we can do a trig substitution on.

Because the quantity under the root is a quadratic polynomial we know that we can complete the square on it to turn it into something like what we need for a trig substitution.

Here is the completing the square work.
\( x^2 - 8x + 21 = x^2 - 8x + 16 - 16 + 21 \)
\[ \left[ \frac{1}{2}(-8) \right]^2 = [ -4 ]^2 = 16 \]
\[ = (x - 4)^2 + 5 \]

So, after completing the square the term can be written as,

\[ \left( x^2 - 8x + 21 \right)^{\frac{3}{2}} = \left( (x - 4)^2 + 5 \right)^{\frac{3}{2}} = \left[ \sqrt{(x - 4)^2 + 5} \right]^{3} \]

Note that we also explicitly put the root into the problem as well.

Hint : At this point the problem works in the same manner as the previous problems in this section.

Step 2
So, in this case we see that we have a sum of a squared term with a variable in it and a number. This suggests that tangent is the correct trig function to use for the substitution.

Now, to get the coefficient on the trig function notice that we need to turn the 1 (i.e. the coefficient of the squared term) into a 5 once we’ve done the substitution. With that in mind it looks like the substitution should be,

\[ x - 4 = \sqrt{5} \tan (\theta) \]

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3
\[ \left( x^2 - 8x + 21 \right)^{\frac{3}{2}} = \left[ \sqrt{(x - 4)^2 + 5} \right]^{3} = \left[ \sqrt{\sqrt{5} \tan (\theta) + 5} \right]^{3} \]
\[ = \left[ \sqrt{\sqrt{5} \tan^2 (\theta) + 5} \right]^{3} = \left[ \sqrt{\sqrt{5} \tan^2 (\theta) + 1} \right]^{3} \]
\[ = \left[ \sqrt{\sqrt{5} \sec^2 (\theta)} \right]^{3} = \left[ 5^{\frac{3}{2}} |\sec (\theta)| \right]^{3} \]

Note that because we don’t know the values of \( \theta \) we can’t determine if the secant is positive or negative and so cannot get rid of the absolute value bars here.

8. Use a trig substitution to eliminate the root in \( \sqrt{e^{8x} - 9} \).

Hint : This doesn’t look much like a term that can use a trig substitution. So, the first step should probably be to some algebraic manipulation on the quantity under the root to make it look more like a problem that can use a trig substitution.

Step 1
We know that in order to do a trig substitution we really need a sum or difference of a term with a variable squared and a number. Even though this doesn’t look anything like the “normal” trig substitution problems it is actually pretty close to one. To see this all we need to do is rewrite the term under the root as follows.

\[ \sqrt{e^{8x} - 9} = \sqrt{(e^{4x})^2 - 9} \]

All we did here was take advantage of the basic exponent rules to make it clear that we really do have a difference here of a squared term containing a variable and a number.

Hint : At this point the problem works in the same manner as the previous problems in this section.

Step 2
The form of the quantity under the root suggests that secant is the correct trig function to use for the substitution.

Now, to get the coefficient on the trig function notice that we need to turn the 1 (i.e. the coefficient of the squared term) into a 9 once we’ve done the substitution. With that in mind it looks like the substitution should be,

\[ e^{4x} = 3\sec(\theta) \]

Now, all we have to do is actually perform the substitution and eliminate the root.

Step 3
\[
\sqrt{e^{8x} - 9} = \sqrt{(3\sec(\theta))^2 - 9} = \sqrt{9\sec^2(\theta) - 9} = 3\sqrt{\sec^2(\theta) - 1} = 3\sqrt{\tan^2(\theta)} = 3|\tan(\theta)|
\]

Note that because we don’t know the values of \( \theta \) we can’t determine if the tangent is positive or negative and so cannot get rid of the absolute value bars here.

9. Use a trig substitution to evaluate \( \int \frac{\sqrt{x^2 + 16}}{x^4} \, dx \).

Step 1
In this case it looks like we’ll need the following trig substitution.

\[ x = 4\tan(\theta) \]

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let’s first use the substitution to eliminate the root.

\[ \sqrt{x^2 + 16} = \sqrt{16 \tan^2(\theta) + 16} = 4\sqrt{\sec^2(\theta)} = 4|\sec(\theta)| \]

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

\[ \sqrt{x^2 + 16} = 4\sec(\theta) \]

For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!

\[ dx = 4\sec^2(\theta)\,d\theta \]

Step 3
Now let’s do the actual substitution.

\[ \int \frac{\sqrt{x^2 + 16}}{x^4} \,dx = \int \frac{4\sec(\theta)}{(4\tan(\theta))^4} \,4\sec^2(\theta)\,d\theta = \int \frac{\sec^3(\theta)}{16\tan^4(\theta)} \,d\theta \]

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. In this case the integral looks to be a little difficult to do in terms of secants and tangents so let’s convert the integrand to sines and cosines and see what we get. Doing this gives,

\[ \int \frac{\sqrt{x^2 + 16}}{x^4} \,dx = \frac{1}{16} \int \frac{\cos(\theta)}{\sin^4(\theta)} \,d\theta \]

This is a simple integral to evaluate so here is the integral evaluation.

\[ \int \frac{\sqrt{x^2 + 16}}{x^4} \,dx = \frac{1}{16} \int \frac{\cos(\theta)}{\sin^4(\theta)} \,d\theta \quad u = \sin(\theta) \]

\[ = \frac{1}{16} \int u^{-3} \,du \]

\[ = -\frac{1}{48} u^{-3} + c = -\frac{1}{48} \left[ \sin(\theta) \right]^{-3} + c = -\frac{1}{48} \csc^3(\theta) + c \]

Don’t forget all the “standard” manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don’t recall them you’ll need to go back to the previous section and work some practice problems to get good at them.
Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5
As the final step we just need to go back to $x$’s. To do this we’ll need a quick right triangle. Here is that work.

From the substitution we have,

$$\tan(\theta) = \frac{x}{4}$$

From the right triangle we get,

$$\csc(\theta) = \frac{\sqrt{x^2 + 16}}{x}$$

The integral is then,

$$\int \frac{\sqrt{x^2 + 16}}{x^4} \, dx = -\frac{1}{48} \left[ \frac{\sqrt{x^2 + 16}}{x} \right]^3 + c = -\frac{(x^2 + 16)^{\frac{3}{2}}}{48x^3} + c$$

10. Use a trig substitution to evaluate $\int \sqrt{1 - 7w^2} \, dw$.

Step 1
In this case it looks like we’ll need the following trig substitution.

$$w = \frac{1}{\sqrt{7}} \sin(\theta)$$

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let’s first use the substitution to eliminate the root.

$$\sqrt{1 - 7w^2} = \sqrt{1 - \sin^2(\theta)} = \sqrt{\cos^2(\theta)} = |\cos(\theta)|$$

Next, because we are doing an indefinite integral we will assume that the cosine is positive and so we can drop the absolute value bars to get,

$$\sqrt{1 - 7w^2} = \cos(\theta)$$
For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!

\[ dw = \frac{1}{\sqrt{7}} \cos(\theta) \, d\theta \]

Step 3
Now let’s do the actual substitution.

\[ \int \sqrt{1-7w^2} \, dw = \int \cos(\theta) \left( \frac{1}{\sqrt{7}} \cos(\theta) \right) d\theta = \frac{1}{\sqrt{7}} \int \cos^2(\theta) \, d\theta \]

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

\[ \int \sqrt{1-7w^2} \, dw = \frac{1}{\sqrt{7}} \int \left[ 1 + \cos(2\theta) \right] d\theta = \frac{1}{2\sqrt{7}} \left[ \theta + \frac{1}{2} \sin(2\theta) \right] + c \]

Don’t forget all the “standard” manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don’t recall them you’ll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5
As the final step we just need to go back to \( w \)'s.

To eliminate the the first term (i.e. the \( \theta \)) we can use any of the inverse trig functions. The easiest is to probably just use the original substitution and get a formula involving inverse sine but any of the six trig functions could be used if we wanted to. Using the substitution gives us,

\[ \sin(\theta) = \sqrt{7} \, w \quad \Rightarrow \quad \theta = \sin^{-1}(\sqrt{7} \, w) \]

Eliminating the \( \sin(2\theta) \) requires a little more work. We can’t just use a right triangle as we normally would because that would only give trig functions with an argument of \( \theta \) and we have an argument of \( 2\theta \). However, we could use the double angle formula for sine to reduce this to trig functions with arguments of \( \theta \). Doing this gives,

\[ \int \sqrt{1-7w^2} \, dw = \frac{1}{2\sqrt{7}} \left[ \theta + \sin(\theta) \cos(\theta) \right] + c \]

We can now do the right triangle work.
From the substitution we have,
\[
\sin(\theta) = \frac{\sqrt{7}w}{1} = \frac{\text{opp}}{\text{hyp}}
\]
From the right triangle we get,
\[
\cos(\theta) = \sqrt{1-7w^2}
\]
The integral is then,
\[
\int \sqrt{1-7w^2} \, dw = \frac{1}{2\sqrt{7}} \left[ \sin^{-1} \left( \sqrt{7}w \right) + \sqrt{7}w\sqrt{1-7w^2} \right] + c
\]

11. Use a trig substitution to evaluate \( \int t^3 \left( 3t^2 - 4 \right)^{3/2} \, dt \).

Step 1
First, do not get excited about the exponent in the integrand. These types of problems work exactly the same as those with just a root (as opposed to this case in which we have a root to a power – you do agree that is what we have right?). So, in this case it looks like we’ll need the following trig substitution.
\[
t = \frac{2}{\sqrt{3}} \sec(\theta)
\]
Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let’s first use the substitution to eliminate the root.
\[
\left( 3t^2 - 4 \right)^{3/2} = \left[ \sqrt{3t^2 - 4} \right]^{3/2} = \left[ \sqrt{4\sec^2(\theta) - 4} \right]^{3/2} = \left[ 2\tan^2(\theta) \right]^{3/2} = 32\tan^3(\theta)
\]
Next, because we are doing an indefinite integral we will assume that the tangent is positive and so we can drop the absolute value bars to get,
\[
\left( 3t^2 - 4 \right)^{3/2} = 32\tan^3(\theta)
\]
For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!
\[
dt = \frac{2}{\sqrt{3}} \sec(\theta) \tan(\theta) \, d\theta
\]
Step 3
Now let’s do the actual substitution.

\[ \int r^3 \left( 3t^2 - 4 \right)^{\frac{3}{2}} \, dt = \int \left( \frac{2}{3} \right)^3 \sec^3 (\theta) \left( 32 \tan^5 (\theta) \right) \left( \frac{2}{\sqrt{r}} \sec (\theta) \tan (\theta) \right) \, d\theta \]

\[ = \frac{512}{9} \int \sec^4 (\theta) \tan^6 (\theta) \, d\theta \]

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

\[ \int r^3 \left( 3t^2 - 4 \right)^{\frac{3}{2}} \, dt = \frac{512}{9} \int \left( \tan^2 (\theta) + 1 \right) \tan^6 (\theta) \sec^2 (\theta) \, d\theta \quad u = \tan (\theta) \]

\[ = \frac{512}{9} \int \left( u^2 + 1 \right) u^6 \, du = \frac{512}{9} \int u^8 + u^6 \, du \]

\[ = \frac{512}{9} \left[ \frac{1}{9} \tan^9 (\theta) + \frac{1}{7} \tan^7 (\theta) \right] + c \]

Don’t forget all the “standard” manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don’t recall them you’ll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5
As the final step we just need to go back to \( t \)'s. To do this we’ll need a quick right triangle. Here is that work.

From the substitution we have,

\[ \sec (\theta) = \frac{\sqrt{3} t}{2} \quad \left( = \frac{\text{hyp}}{\text{adj}} \right) \]

From the right triangle we get,

\[ \tan (\theta) = \frac{\sqrt{3t^2 - 4}}{2} \]

The integral is then,
12. Use a trig substitution to evaluate \( \int_{-5}^{-5} \frac{2}{y^4\sqrt{y^2-25}} \, dy \).

Step 1
In this case it looks like we’ll need the following trig substitution.

\[ y = 5 \sec(\theta) \]

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let’s first use the substitution to eliminate the root.

\[ \sqrt{y^2-25} = \frac{25 \sec^2(\theta) - 25}{5 \sqrt{\tan^2(\theta)}} = 5 \tan(\theta) \]

Step 3
Okay, in this case we have limits on \( y \) and so we can get limits on \( \theta \) that will allow us to determine if tangent is positive or negative to allow us to eliminate the absolute value bars.

So, let’s get some limits on \( \theta \).

\[
\begin{align*}
y = -7 : & \quad -7 = 5 \sec(\theta) \quad \rightarrow \quad \sec(\theta) = -\frac{7}{5} \quad \rightarrow \quad \theta = \sec^{-1}\left(-\frac{7}{5}\right) = 2.3664 \\
y = -5 : & \quad -5 = 5 \sec(\theta) \quad \rightarrow \quad \sec(\theta) = -1 \quad \rightarrow \quad \theta = \pi
\end{align*}
\]

So, \( \theta \)’s for this problem are in the range \( 2.3664 \leq \theta \leq \pi \) and these are in the second quadrant. In the second quadrant we know that tangent is negative and so we can drop the absolute value bars provided we add in a minus sign. This gives,

\[ \sqrt{y^2-25} = -5 \tan(\theta) \]

For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!
\[ dy = 5 \sec(\theta) \tan(\theta) \, d\theta \]

**Step 4**

Now let’s do the actual substitution.

\[
\int_{-5}^{\infty} \frac{2}{y^4 \sqrt{y^2 - 25}} \, dy = \int_{\frac{2.3664}{5}}^{\pi} \frac{2}{5^4 \sec^4(\theta) \left(-5 \tan(\theta)\right)} \left(5 \sec(\theta) \tan(\theta)\right) \, d\theta
\]

\[
= -\frac{2}{625} \int_{\frac{2.3664}{5}}^{\pi} \frac{1}{\sec^3(\theta)} \, d\theta
\]

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Also notice that upon doing the substitution we replaced the \( y \) limits with the \( \theta \) limits. This will help with a later step.

**Step 5**

We now need to evaluate the integral. In terms of secants this integral would be pretty difficult, however we a quick change to cosines we get the following integral.

\[
\int_{-5}^{\infty} \frac{2}{y^4 \sqrt{y^2 - 25}} \, dy = -\frac{2}{625} \int_{\frac{2.3664}{5}}^{\pi} \cos^3(\theta) \, d\theta
\]

This should be relatively simple to do so here is the integration work.

\[
\int_{-5}^{\infty} \frac{2}{y^4 \sqrt{y^2 - 25}} \, dy = -\frac{2}{625} \int_{\frac{2.3664}{5}}^{\pi} \left(1 - \sin^2(\theta)\right) \cos(\theta) \, d\theta \quad u = \sin(\theta)
\]

\[
= -\frac{2}{625} \int_{\frac{\sin(\pi)}{\sin(2.3664)}}^{\sin(\frac{\pi}{2})} \left(1 - u^2\right) \, du
\]

\[
= -\frac{2}{625} \left[u - \frac{1}{3} u^3\right]_{0.69986}^{0} = 0.001874
\]

Don’t forget all the “standard” manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don’t recall them you’ll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Also, note that because we converted the limits at every substitution into limits for the “new” variable we did not need to do any back substitution work on our answer!
13. Use a trig substitution to evaluate \[ \int_1^4 2z^5 \sqrt{2 + 9z^2} \, dz \,.
\]

Step 1
In this case it looks like we’ll need the following trig substitution.
\[ z = \frac{\sqrt{2}}{3} \tan(\theta) \]

Now we need to use the substitution to eliminate the root and get set up for actually substituting this into the integral.

Step 2
Let’s first use the substitution to eliminate the root.
\[ \sqrt{2 + 9z^2} = \sqrt{2 + 2\tan^2(\theta)} = \sqrt{2\sec^2(\theta)} = \sqrt{2}\sec(\theta) \]

Step 3
Okay, in this case we have limits on \( z \) and so we can get limits on \( \theta \) that will allow us to determine if tangent is positive or negative to allow us to eliminate the absolute value bars.

So, let’s get some limits on \( \theta \).
\[
\begin{align*}
z = 1: & \quad 1 = \frac{\sqrt{2}}{3} \tan(\theta) \quad \Rightarrow \quad \tan(\theta) = \frac{3}{\sqrt{2}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{3}{\sqrt{2}}\right) = 1.1303 \\
z = 4: & \quad 4 = \frac{\sqrt{2}}{3} \tan(\theta) \quad \Rightarrow \quad \tan(\theta) = \frac{12}{\sqrt{2}} \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{12}{\sqrt{2}}\right) = 1.4535
\end{align*}
\]

So, \( \theta \)’s for this problem are in the range \( 1.1303 \leq \theta \leq 1.4535 \) and these are in the first quadrant. In the first quadrant we know that cosine, and hence secant, is positive and so we can just drop the absolute value bars. This gives,
\[
\sqrt{2 + 9z^2} = \sqrt{2}\sec(\theta)
\]

For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!
\[
dz = \frac{\sqrt{2}}{3}\sec^2(\theta) \, d\theta
\]

Step 4
Now let’s do the actual substitution.
\[
\int_1^4 2z^5 \sqrt{2 + 9z^2} \, dz = \int_{1.1303}^{1.4535} 2\left(\frac{\sqrt{2}}{3}\right)^5 \tan^5(\theta) \left(\sqrt{2}\sec(\theta)\right)\left(\frac{\sqrt{2}}{3}\sec^2(\theta)\right) \, d\theta
\]
\[
= \frac{16\sqrt{2}}{729} \int_{1.1303}^{1.4535} \tan^5(\theta) \sec^3(\theta) \, d\theta
\]
Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Also notice that upon doing the substitution we replaced the \( y \) limits with the \( \theta \) limits. This will help with a later step.

**Step 5**

We now need to evaluate the integral. Here is that work.

\[
\int \frac{2z^5}{\sqrt{2 + 9z^2}} \, dz = \frac{16\sqrt{729}}{729} \int_{1.1303}^{1.4535} \left[ \sec^2(\theta) - 1 \right]^2 \sec^2(\theta) \tan(\theta) \sec(\theta) \, d\theta
\]

\[
= \frac{16\sqrt{729}}{729} \int_{\text{sec}(1.1303)}^{\text{sec}(1.4535)} \left[ u^2 - 1 \right]^2 u^2 \, du
\]

\[
= \frac{16\sqrt{729}}{729} \int_{2.3452}^{8.5444} \left( u^6 - 2u^4 + u^2 \right) \, du
\]

\[
= \frac{16\sqrt{729}}{729} \left[ \frac{1}{7} u^7 - \frac{2}{5} u^5 + \frac{1}{3} u^3 \right]_{2.3452}^{8.5444} = 14182.86074
\]

Don’t forget all the “standard” manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don’t recall them you’ll need to go back to the previous section and work some practice problems to get good at them.

Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Also, note that because we converted the limits at every substitution into limits for the “new” variable we did not need to do any back substitution work on our answer!

---

14. Use a trig substitution to evaluate \( \int \frac{1}{\sqrt{9x^2 - 36x + 37}} \, dx \).

**Step 1**

The first thing we’ll need to do here is complete the square on the polynomial to get this into a form we can use a trig substitution on.

\[
9x^2 - 36x + 37 = 9\left( x^2 - 4x + \frac{37}{9} \right) = 9\left( x^2 - 4x + 4 - 4 + \frac{37}{9} \right) = 9\left( (x - 2)^2 + \frac{1}{9} \right)
\]

\[
= 9(x - 2)^2 + 1
\]

The integral is now,

\[
\int \frac{1}{\sqrt{9x^2 - 36x + 37}} \, dx = \int \frac{1}{\sqrt{9(x-2)^2 + 1}} \, dx
\]
Now we can proceed with the trig substitution.

Step 2
It looks like we’ll need to the following trig substitution.

\[ x - 2 = \frac{1}{3} \tan(\theta) \]

Next let’s eliminate the root.

\[ \sqrt{9(x - 2)^2 + 1} = \sqrt{\tan^2(\theta) + 1} = \sqrt{\sec^2(\theta)} = |\sec(\theta)| \]

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

\[ \sqrt{9(x - 2)^2 + 1} = \sec(\theta) \]

For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!

\[ (1)dx = \frac{1}{3} \sec^2(\theta) \, d\theta \quad \Rightarrow \quad dx = \frac{1}{3} \sec^2(\theta) \, d\theta \]

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3
Now let’s do the actual substitution.

\[ \int \frac{1}{\sqrt{9x^2 - 36x + 37}} \, dx = \int \frac{1}{ \sec(\theta) \left( \frac{1}{3} \sec^2(\theta) \right) } \, d\theta = \frac{1}{3} \int \sec(\theta) \, d\theta \]

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

\[ \int \frac{1}{\sqrt{9x^2 - 36x + 37}} \, dx = \frac{1}{3} \ln |\sec(\theta) + \tan(\theta)| + c \]

Note that this was one of the few trig substitution integrals that didn’t really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of \( \sec(\theta) \) in the previous section and reuse that result here.

Step 5
As the final step we just need to go back to \( x \)'s. To do this we’ll need a quick right triangle. Here is that work.

From the substitution we have,

\[
\tan (\theta) = \frac{3(x-2)}{1} \quad \left( \frac{\text{opp}}{\text{adj}} \right)
\]

From the right triangle we get,

\[
\sec (\theta) = \sqrt{9(x-2)^2 + 1}
\]

The integral is then,

\[
\int \frac{1}{\sqrt{9x^2 - 36x + 37}} \, dx = \frac{1}{3} \ln \left| \sqrt{9(x-2)^2 + 1} + 3(x-2) \right| + c
\]

15. Use a trig substitution to evaluate \( \int \frac{(z+3)^5}{(40-6z-z^2)^2} \, dz \).

Step 1
The first thing we’ll need to do here is complete the square on the polynomial to get this into a form we can use a trig substitution on.

\[
40 - 6z - z^2 = -\left( z^2 + 6z - 40 \right) = -\left( z^2 + 6z + 9 - 9 - 40 \right) = -\left( z + 3 \right)^2 - 49
\]

\[
= 49 - (z + 3)^2
\]

The integral is now,

\[
\int \frac{(z+3)^5}{(40-6z-z^2)^2} \, dz = \int \frac{(z+3)^5}{(49-(z+3)^2)^2} \, dz
\]

Now we can proceed with the trig substitution.

Step 2
It looks like we’ll need to the following trig substitution.

\[
z + 3 = 7 \sin (\theta)
\]

Next let’s eliminate the root.
Next, because we are doing an indefinite integral we will assume that the cosine is positive and so we can drop the absolute value bars to get,

\[
\left(49 - (z + 3)^2\right)^{\frac{1}{2}} = 343 \cos^3(\theta)
\]

For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!

\[
(1) \, dz = 7 \cos(\theta) \, d\theta \quad \Rightarrow \quad dz = 7 \cos(\theta) \, d\theta
\]

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3
Now let’s do the actual substitution.

\[
\int \frac{(z + 3)^5}{(40 - 6z - z^2)^\frac{3}{2}} \, dz = \int \frac{16807 \sin^5(\theta)}{343 \cos^3(\theta)} (7 \cos(\theta)) \, d\theta = 343 \int \frac{\sin^5(\theta)}{\cos^3(\theta)} \, d\theta
\]

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4
We now need to evaluate the integral. Here is that work.

\[
\int \frac{(z + 3)^5}{(40 - 6z - z^2)^\frac{3}{2}} \, dz = 343 \int \frac{\left[1 - \cos^2(\theta)\right]^2}{\cos^3(\theta)} \sin(\theta) \, d\theta \quad u = \cos(\theta)
\]

\[
= -343 \int \frac{[1 - u^2]^2}{u^2} \, du = -343 \int u^{-2} - 2 + u^2 \, du
\]

\[
= -343 \left(-u^{-1} - 2u + \frac{1}{3}u^3\right) + c
\]

\[
= -343 \left(-\frac{1}{\cos(\theta)} - 2\cos(\theta) + \frac{1}{3}\cos^3(\theta)\right) + c
\]

Don’t forget all the “standard” manipulations of the integrand that we often need to do in order to evaluate integrals involving trig functions. If you don’t recall them you’ll need to go back to the previous section and work some practice problems to get good at them.
Every trig substitution problem reduces down to an integral involving trig functions and the majority of them will need some manipulation of the integrand in order to evaluate.

Step 5
As the final step we just need to go back to \( z \)'s. To do this we’ll need a quick right triangle. Here is that work.

From the substitution we have,

\[
\sin(\theta) = \frac{z + 3}{7} \quad \left(= \frac{\text{adj}}{\text{hyp}}\right)
\]

From the right triangle we get,

\[
\cos(\theta) = \frac{\sqrt{49 - (z + 3)^2}}{7}
\]

The integral is then,

\[
\int \frac{(z + 3)^5}{(40 - 6z - z^2)^{1/2}} \, dz = \frac{2401}{\sqrt{49 - (z + 3)^2}} + 98 \frac{\sqrt{49 - (z + 3)^2} - \left(\frac{49 - (z + 3)^2}{3}\right)^{1/2}}{3} + c
\]

16. Use a trig substitution to evaluate \( \int \cos(x) \sqrt{9 + 25\sin^2(x)} \, dx \).

Step 1
Let’s first rewrite the integral a little bit.

\[
\int \cos(x) \sqrt{9 + 25\sin^2(x)} \, dx = \int \cos(x) \sqrt{9 + 25\left[\sin(x)\right]^2} \, dx
\]

Step 2
With the integral written as it is in the first step we can now see that we do have a sum of a number and something squared under the root. We know from the problems done previously in this section that looks like a tangent substitution. So, let’s use the following substitution.

\[
\sin(x) = \frac{1}{3} \tan(\theta)
\]

Do not get excited about the fact that we are substituting one trig function for another. That will happen on occasion with these kinds of problems. Note however, that we need to be careful and make sure that we also change the variable from \( x \) (\( i.e. \) the variable in the original trig function) into \( \theta \) (\( i.e. \) the variable in the new trig function).
Next let’s eliminate the root.

\[
\sqrt{9 + 25\sin^2(x)} = \sqrt{9 + 25\left(\frac{3}{5}\tan(\theta)\right)^2} = \sqrt{9 + 9\tan^2(\theta)} = 3\sqrt{\sec^2(\theta)} = 3\sec(\theta)
\]

Next, because we are doing an indefinite integral we will assume that the secant is positive and so we can drop the absolute value bars to get,

\[
\sqrt{9 + 25\sin^2(x)} = 3\sec(\theta)
\]

For a final substitution preparation step let’s also compute the differential so we don’t forget to use that in the substitution!

\[
\cos(x)\,dx = \frac{3}{5}\sec^2(\theta)\,d\theta
\]

Recall that all we really need to do here is compute the differential for both the right and left sides of the substitution.

Step 3

Now let’s do the actual substitution.

\[
\int \cos(x)\sqrt{9 + 25\sin^2(x)}\,dx = \int \sqrt{9 + 25\sin^2(x)}\,\cos(x)\,dx = \int (3\sec(\theta))\left(\frac{3}{5}\sec^2(\theta)\right)d\theta = \frac{9}{5}\int \sec^3(\theta)\,d\theta
\]

Do not forget to substitute in the differential we computed in the previous step. This is probably the most common mistake with trig substitutions. Forgetting the differential can substantially change the problem, often making the integral very difficult to evaluate.

Step 4

We now need to evaluate the integral. Here is that work.

\[
\int \cos(x)\sqrt{9 + 25\sin^2(x)}\,dx = \frac{9}{10}\left[\sec(\theta)\tan(\theta) + \ln|\sec(\theta) + \tan(\theta)|\right] + c
\]

Note that this was one of the few trig substitution integrals that didn’t really require a lot of manipulation of trig functions to completely evaluate. All we had to really do here was use the fact that we determined the integral of \(\sec^3(\theta)\) in the previous section and reuse that result here.

Step 5

As the final step we just need to go back to \(x\)’s. To do this we’ll need a quick right triangle. Here is that work.
From the substitution we have,
\[
\tan(\theta) = \frac{5\sin(x)}{3} \quad (\text{opp}) \quad \text{adj}
\]

From the right triangle we get,
\[
\sec(\theta) = \frac{\sqrt{9 + 25\sin^2(x)}}{3}
\]

The integral is then,
\[
\int \cos(x)\sqrt{9 + 25\sin^2(x)} \, dx = \frac{\sin(x)\sqrt{9 + 25\sin^2(x)}}{2} + \frac{9}{10} \ln \frac{5\sin(x) + \sqrt{9 + 25\sin^2(x)}}{3} + \mathcal{C}
\]

---

**Partial Fractions**

1. Evaluate the integral \( \int \frac{4}{x^2 + 5x - 14} \, dx \).

Step 1
To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this we’ll need to factor the denominator.

\[
\int \frac{4}{x^2 + 5x - 14} \, dx = \int \frac{4}{(x + 7)(x - 2)} \, dx
\]

The form of the partial fraction decomposition for the integrand is then,
\[
\frac{4}{(x + 7)(x - 2)} = \frac{A}{x + 7} + \frac{B}{x - 2}
\]

Step 2
Setting the numerators equal gives,
\[
4 = A(x - 2) + B(x + 7)
\]

Step 3
We can use the “trick” discussed in the notes to easily get the coefficients in this case so let’s do that. Here is that work.
The partial fraction form of the integrand is then,

\[ \frac{4}{(x + 7)(x - 2)} = \frac{-\frac{4}{9}}{x + 7} + \frac{\frac{4}{9}}{x - 2} \]

Step 4
We can now do the integral.

\[
\int \frac{4}{(x + 7)(x - 2)} \, dx = \int \left( \frac{-\frac{4}{9}}{x + 7} + \frac{\frac{4}{9}}{x - 2} \right) \, dx = \frac{4}{9} \ln |x - 2| - \frac{4}{9} \ln |x + 7| + c
\]

2. Evaluate the integral \( \int \frac{8 - 3t}{10t^2 + 13t - 3} \, dt \).

Step 1
To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this we’ll need to factor the denominator.

\[
\int \frac{8 - 3t}{10t^2 + 13t - 3} \, dt = \int \frac{8 - 3t}{(2t + 3)(5t - 1)} \, dt
\]

The form of the partial fraction decomposition for the integrand is then,

\[ \frac{8 - 3t}{10t^2 + 13t - 3} = \frac{A}{2t + 3} + \frac{B}{5t - 1} \]

Step 2
Setting the numerators equal gives,

\[ 8 - 3t = A(5t - 1) + B(2t + 3) \]

Step 3
We can use the “trick” discussed in the notes to easily get the coefficients in this case so let’s do that. Here is that work.

\[
t = \frac{1}{5} : \quad \frac{22}{5} = \frac{12}{5} B \quad \Rightarrow \quad A = -\frac{26}{17}
\]

\[
t = -\frac{1}{2} : \quad \frac{25}{2} = -\frac{17}{2} A \quad \Rightarrow \quad B = \frac{17}{17}
\]

The partial fraction form of the integrand is then,
We can now do the integral.

\[
\int \frac{8-3t}{10t^2+13t-3} \, dt = \int \left( -\frac{25}{2t+3} + \frac{37}{5t-1} \right) \, dt = \frac{37}{85} \ln |5t-1| - \frac{25}{34} \ln |2t+3| + c
\]

Hopefully you are getting good enough with integration that you can do some of these integrals in your head. Be careful however with both of these integrals. When doing these kinds of integrals in our head it is easy to forget about the substitutions that are technically required to do them and then miss the coefficients from the substitutions that need to show up in the answer.

3. Evaluate the integral \[ \int_{-1}^{0} \frac{w^2+7w}{(w+2)(w-1)(w-4)} \, dw. \]

Step 1
In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

\[
\frac{w^2+7w}{(w+2)(w-1)(w-4)} = \frac{A}{w+2} + \frac{B}{w-1} + \frac{C}{w-4}
\]

Step 2
Setting the numerators equal gives,

\[
w^2+7w = A(w-1)(w-4) + B(w+2)(w-4) + C(w+2)(w-1)
\]

Step 3
We can use the “trick” discussed in the notes to easily get the coefficients in this case so let’s do that. Here is that work.

\[
w = 1: \quad 8 = -9B \quad \Rightarrow \quad A = -\frac{5}{9}
\]

\[
w = 4: \quad 44 = 18C \quad \Rightarrow \quad B = -\frac{8}{9}
\]

\[
w = -2: \quad -10 = 18A \quad \quad \quad C = \frac{22}{9}
\]

The partial fraction form of the integrand is then,

\[
\frac{w^2+7w}{(w+2)(w-1)(w-4)} = -\frac{5}{9} \cdot \frac{1}{w+2} - \frac{8}{9} \cdot \frac{1}{w-1} + \frac{22}{9} \cdot \frac{1}{w-4}
\]
Step 4
We can now do the integral.

\[
\int_{-1}^{0} \frac{w^2 + 7w}{(w+2)(w-1)(w-4)} \, dw = \int_{-1}^{0} -\frac{5}{9} \frac{8}{w+2} - \frac{8}{9} \frac{22}{w-1} + \frac{22}{9} \frac{w}{w-4} \, dw
\]

\[
= \left[ -\frac{5}{9} \ln |w+2| - \frac{8}{9} \ln |w-1| + \frac{22}{9} \ln |w-4| \right]_{-1}^{0}
\]

\[
= \frac{22}{9} \ln (4) + \frac{3}{9} \ln (2) - \frac{22}{9} \ln (5) = \frac{22}{9} \ln (2) - \frac{22}{9} \ln (5)
\]

Note that we used a quick logarithm property to combine the first two logarithms into a single logarithm. You should probably review your logarithm properties if you don’t recognize the one that we used. These kinds of property applications can really simplify your work on occasion if you know them!

4. Evaluate the integral \( \int \frac{8}{3x^3 + 7x^2 + 4x} \, dx \).

Step 1
To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this we’ll need to factor the denominator.

\[
\int \frac{8}{3x^3 + 7x^2 + 4x} \, dx = \int \frac{8}{x(3x+4)(x+1)} \, dx
\]

The form of the partial fraction decomposition for the integrand is then,

\[
\frac{8}{x(3x+4)(x+1)} = \frac{A}{x} + \frac{B}{3x+4} + \frac{C}{x+1}
\]

Step 2
Setting the numerators equal gives,

\[
8 = A(3x+4)(x+1) + Bx(x+1) + Cx(3x+4)
\]

Step 3
We can use the “trick” discussed in the notes to easily get the coefficients in this case so let’s do that. Here is that work.

\[
x = -\frac{4}{3} : \quad 8 = \frac{4}{9} B \quad \Rightarrow \quad A = 2
\]

\[
x = -1 : \quad 8 = -C \quad \Rightarrow \quad B = 18
\]

\[
x = 0 : \quad 8 = 4A \quad \Rightarrow \quad C = -8
\]

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The partial fraction form of the integrand is then,

\[ \frac{8}{x(3x + 4)(x + 1)} = \frac{2}{x} + \frac{18}{3x + 4} - \frac{8}{x + 1} \]

Step 4
We can now do the integral.

\[ \int \frac{8}{x(3x + 4)(x + 1)} \, dx = \int \left( \frac{2}{x} + \frac{18}{3x + 4} - \frac{8}{x + 1} \right) \, dx = 2 \ln |x| + 6 \ln |3x + 4| - 8 \ln |x + 1| + c \]

Hopefully you are getting good enough with integration that you can do some of these integrals in your head. Be careful however with the second integral. When doing these kinds of integrals in our head it is easy to forget about the substitutions that are technically required to do them and then miss the coefficients from the substitutions that need to show up in the answer.

5. Evaluate the integral

\[ \int_{2}^{4} \frac{3z^2 + 1}{(z + 1)(z - 5)^2} \, dz \]

Step 1
In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

\[ \frac{3z^2 + 1}{(z + 1)(z - 5)^2} = \frac{A}{z + 1} + \frac{B}{z - 5} + \frac{C}{(z - 5)^2} \]

Step 2
Setting the numerators equal gives,

\[ 3z^2 + 1 = A(z - 5)^2 + B(z + 1)(z - 5) + C(z + 1) \]

Step 3
We can use the “trick” discussed in the notes to easily get two of the coefficients and then we can just pick another value of \( z \) to get the third so let’s do that. Here is that work.

\[
\begin{align*}
  z = -1 : & \quad 4 = 36A \\
  z = 5 : & \quad 76 = 6C \\
  z = 0 : & \quad 1 = 25A - 5B + C = \frac{139}{9} - 5B
\end{align*}
\]

\[ A = \frac{1}{9}, \quad B = \frac{26}{9}, \quad C = \frac{38}{3} \]

The partial fraction form of the integrand is then,
\[
\frac{3z^2 + 1}{(z+1)(z-5)^2} = \frac{1}{9} + \frac{26}{9} \frac{1}{z-5} + \frac{18}{3} \frac{1}{(z-5)^2}
\]

Step 4
We can now do the integral.
\[
\int_2^4 \frac{3z^2 + 1}{(z+1)(z-5)^2} \, dz = \int_2^4 \frac{1}{9} \frac{1}{z+1} + \frac{26}{9} \frac{1}{z-5} + \frac{38}{3} \frac{1}{(z-5)^2} \, dz
\]
\[
= \left( \frac{1}{9} \ln|z+1| + \frac{26}{9} \ln|z-5| - \frac{38}{3} \frac{1}{z-5} \right)_2^4 = \frac{1}{9} \ln(5) - \frac{27}{9} \ln(3) + \frac{26}{9}
\]

6. Evaluate the integral \( \int \frac{4x-11}{x^3-9x^2} \, dx \).

Step 1
To get the problem started off we need the form of the partial fraction decomposition of the integrand. However, in order to get this we’ll need to factor the denominator.
\[
\int \frac{4x-11}{x^3-9x^2} \, dx = \int \frac{4x-11}{x^2(x-9)} \, dx
\]
The form of the partial fraction decomposition for the integrand is then,
\[
\frac{4x-11}{x^2(x-9)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-9}
\]
Step 2
Setting the numerators equal gives,
\[
4x-11 = Ax(x-9) + B(x-9) + Cx^2
\]
Step 3
We can use the “trick” discussed in the notes to easily get two of the coefficients and then we can just pick another value of \( x \) to get the third so let’s do that. Here is that work.
\[
\begin{align*}
   x = 0 : & \quad -11 = -9B \\
   x = 9 : & \quad 25 = 81C \\
   x = 1 : & \quad -7 = -8A - 8B + C = -8A - \frac{267}{81} \\
\end{align*}
\]
\[
A = -\frac{25}{81} \quad \Rightarrow \quad B = \frac{11}{9} \quad C = \frac{25}{81}
\]
The partial fraction form of the integrand is then,
\[
\frac{4x - 11}{x^2(x - 9)} = -\frac{25}{81} \frac{1}{x} + \frac{11}{9} \frac{1}{x^2} + \frac{25}{81} \frac{1}{x - 9}
\]

Step 4
We can now do the integral.

\[
\int \frac{4x - 11}{x^2(x - 9)} \, dx = \int -\frac{25}{81} \frac{1}{x} + \frac{11}{9} \frac{1}{x^2} + \frac{25}{81} \frac{1}{x - 9} \, dx = -\frac{25}{81} \ln |x| - \frac{11}{9} + \frac{25}{81} \ln |x - 9| + c
\]

7. Evaluate the integral \[
\int \frac{z^2 + 2z + 3}{(z - 6)(z^2 + 4)} \, dz .
\]

Step 1
In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

\[
\frac{z^2 + 2z + 3}{(z - 6)(z^2 + 4)} = \frac{A}{z - 6} + \frac{Bz + C}{z^2 + 4}
\]

Step 2
Setting the numerators equal gives,

\[
z^2 + 2z + 3 = A(z^2 + 4) + (Bz + C)(z - 6) = (A + B)z^2 + (-6B + C)z + 4A - 6C
\]

In this case the “trick” discussed in the notes won’t work all that well for us and so we’ll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3
Now, setting the coefficients equal gives the following system.

\[
\begin{align*}
z^2 &: \quad A + B = 1 \\
z^1 &: \quad -6B + C = 2 \quad \Rightarrow \quad B = -\frac{11}{40} \\
z^0 &: \quad 4A - 6C = 3 \quad \Rightarrow \quad C = \frac{7}{20}
\end{align*}
\]

The partial fraction form of the integrand is then,

\[
\frac{z^2 + 2z + 3}{(z - 6)(z^2 + 4)} = \frac{\frac{51}{40}}{z - 6} + \frac{-\frac{11}{40}z + \frac{7}{20}}{z^2 + 4}
\]

Step 4
We can now do the integral.
\[
\int \frac{z^2 + 2z + 3}{(z-6)(z^2+4)}
\, dz = \int \frac{51}{40z-6} - \frac{11}{40z^2+4}
\, dz
\]
\[
= \frac{51}{40} \ln |z-6| - \frac{11}{40} \ln |z^2+4| + \frac{z}{20} \tan^{-1} \left( \frac{z}{2} \right) + c
\]

Note that the second integration needed the substitution \( u = z^2 + 4 \) while the third needed the formula provided in the notes.

8. Evaluate the integral \[
\int \frac{8 + t + 6t^2 - 12t^3}{(3t^2 + 4)(t^2 + 7)}
\, dt
\].

Step 1
In this case the denominator is already factored and so we can go straight to the form of the partial fraction decomposition for the integrand.

\[
\frac{8 + t + 6t^2 - 12t^3}{(3t^2 + 4)(t^2 + 7)} = \frac{A + B}{3t^2 + 4} + \frac{Ct + D}{t^2 + 7}
\]

Step 2
Setting the numerators equal gives,

\[
8 + t + 6t^2 - 12t^3 = (A + B)(t^2 + 7) + (Ct + D)(3t^2 + 4)
\]
\[
= (A + 3C)t^3 + (B + 3D)t^2 + (7A + 4C)t + 7B + 4D
\]

In this case the “trick” discussed in the notes won’t work all that well for us and so we’ll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3
Now, setting the coefficients equal gives the following system.

\[
t^3 : \quad A + 3C = -12 \quad \quad \quad A = 3
\]
\[
t^2 : \quad B + 3D = 6 \quad \quad \quad B = 0
\]
\[
t^1 : \quad 7A + 4C = 1 \quad \quad \quad C = -5
\]
\[
t^0 : \quad 7B + 4D = 8 \quad \quad \quad D = 2
\]

The partial fraction form of the integrand is then,
\[
\frac{8 + t + 6t^2 - 12t^3}{(3t^2 + 4)(t^2 + 7)} = \frac{3t}{3t^2 + 4} + \frac{-5t + 2}{t^2 + 7}
\]

Step 4
We can now do the integral.

\[
\int \frac{8 + t + 6t^2 - 12t^3}{(3t^2 + 4)(t^2 + 7)} \, dt = \int \frac{3t}{3t^2 + 4} + \frac{-5t + 2}{t^2 + 7} \, dt
\]

\[
= \int \frac{3t}{3t^2 + 4} - \frac{5t}{t^2 + 7} + \frac{2}{t^2 + 7} \, dt
\]

\[
= \frac{1}{2} \ln |3t^2 + 4| - \frac{5}{2} \ln |t^2 + 7| + \frac{3}{\sqrt{7}} \tan^{-1} \left( \frac{t}{\sqrt{7}} \right) + c
\]

Note that the first and second integrations needed the substitutions \(u = 3t^2 + 4\) and \(u = t^2 + 7\) respectively while the third needed the formula provided in the notes.

9. Evaluate the integral \(\int \frac{6x^2 - 3x}{(x - 2)(x + 4)} \, dx\).

Hint: Pay attention to the degree of the numerator and denominator!

Step 1
Remember that we can only do partial fractions on a rational expression if the degree of the numerator is less than the degree of the denominator. In this case both the numerator and denominator are both degree 2. This can be easily seen if we multiply the denominator out.

\[
\frac{6x^2 - 3x}{(x - 2)(x + 4)} = \frac{6x^2 - 3x}{x^2 + 2x - 8}
\]

So, the first step is to do long division (we’ll leave it up to you to check our Algebra skills for the long division) to get,

\[
\frac{6x^2 - 3x}{(x - 2)(x + 4)} = 6 + \frac{48 - 15x}{(x - 2)(x + 4)}
\]

Step 2
Now we can do the partial fractions on the second term. Here is the form of the partial fraction decomposition.

\[
\frac{48 - 15x}{(x - 2)(x + 4)} = \frac{A}{x - 2} + \frac{B}{x + 4}
\]
Setting the numerators equal gives,

\[48 - 15x = A(x + 4) + B(x - 2)\]

Step 3
The “trick” will work here easily enough so here is that work.

\[
\begin{align*}
   x = -4: & \quad 108 = -6B \quad \Rightarrow \quad A = 3 \\
   x = 2: & \quad 18 = 6A \quad \Rightarrow \quad B = -18
\end{align*}
\]

The partial fraction form of the second term is then,

\[
\frac{48 - 15x}{(x - 2)(x + 4)} = \frac{3}{x - 2} - \frac{18}{x + 4}
\]

Step 4
We can now do the integral.

\[
\int \frac{6x^2 - 3x}{(x - 2)(x + 4)} \, dx = \int \left( 6 + \frac{3}{x - 2} - \frac{18}{x + 4} \right) \, dx = 6x + 3 \ln |x - 2| - 18 \ln |x + 4| + c
\]

10. Evaluate the integral \[\int \frac{2 + w^4}{w^3 + 9w} \, dw\].

Hint: Pay attention to the degree of the numerator and denominator!

Step 1
Remember that we can only do partial fractions on a rational expression if the degree of the numerator is less than the degree of the denominator. In this case the degree of the numerator is 4 and the degree of the denominator is 3.

So, the first step is to do long division (we’ll leave it up to you to check our Algebra skills for the long division) to get,

\[
\frac{2 + w^4}{w^3 + 9w} = w + \frac{2 - 9w^2}{w(w^2 + 9)}
\]

Step 2
Now we can do the partial fractions on the second term. Here is the form of the partial fraction decomposition.
\[
\frac{2 - 9w^2}{w(w^2 + 9)} = \frac{A}{w} + \frac{Bw + C}{w^2 + 9}
\]

Setting the numerators equal gives,

\[
2 - 9w^2 = A(w^2 + 9) + w(Bw + C) = (A + B)w^2 + Cw + 9A
\]

In this case the “trick” discussed in the notes won’t work all that well for us and so we’ll have to resort to multiplying everything out and collecting like terms as shown above.

Step 3
Now, setting the coefficients equal gives the following system.

\[
\begin{align*}
&\quad w^2 : \quad A + B = -9 & A = -\frac{2}{9} \\
&w^1 : \quad C = 0 & B = -\frac{83}{9} \\
&w^0 : \quad 9A = 2 & C = 0
\end{align*}
\]

The partial fraction form of the second term is then,

\[
\frac{2 - 9w^2}{w(w^2 + 9)} = \frac{-2}{9} w - \frac{83}{9} \frac{w}{w^2 + 9}
\]

Step 4
We can now do the integral.

\[
\int \frac{2 + w^4}{w^5 + 9w} \, dw = \int \left( w + \frac{\frac{2}{9} w - \frac{83}{9} \frac{w}{w^2 + 9}}{w^2 + 9} \right) \, dw = \frac{1}{2} w^2 + \frac{2}{5} \ln |w| - \frac{83}{18} \ln |w^2 + 9| + C
\]

---

**Integrals Involving Roots**

1. Evaluate the integral \( \int \frac{7}{2 + \sqrt{x - 4}} \, dx \).

Step 1
The substitution we’ll use here is,

\[ u = \sqrt{x - 4} \]

Step 2
Now we need to get set up for the substitution. In other words, we need so solve for $x$ and get $dx$.

$$x = u^2 + 4 \quad \Rightarrow \quad dx = 2u \, du$$

Step 3
Doing the substitution gives,

$$\int \frac{7}{2 + \sqrt{x - 4}} \, dx = \int \frac{7}{2 + u} (2u) \, du = \int \frac{14u}{2 + u} \, du$$

Step 4
This new integral can be done with the substitution $v = u + 2$. Doing this gives,

$$\int \frac{7}{2 + \sqrt{x - 4}} \, dx = \int \frac{14(v - 2)}{v} \, dv = \int \frac{14 - 28}{v} \, dv = 14v - 28\ln|v| + c$$

Step 5
The last step is to now do all the back substitutions to get the final answer.

$$\int \frac{7}{2 + \sqrt{x - 4}} \, dx = 14(u + 2) - 28\ln|u + 2| + c = \frac{14(\sqrt{x - 4} + 2)}{2} - 28\ln\sqrt{x - 4} + 2 + c$$

Note that we could have avoided the second substitution if we’d used $u = \sqrt{x - 4} + 2$ for the original substitution.

This often doesn’t work, but in this case because the only extra term in the denominator was a constant it didn’t change the differential work and so would work pretty easily for this problem.

2. Evaluate the integral $\int \frac{1}{w + 2\sqrt{1 - w} + 2} \, dw$.

Step 1
The substitution we’ll use here is,

$$u = \sqrt{1 - w}$$

Step 2
Now we need to get set up for the substitution. In other words, we need so solve for $w$ and get $dw$.

$$w = 1 - u^2 \quad \Rightarrow \quad dw = -2u \, du$$

Step 3
Doing the substitution gives,
\[
\int \frac{1}{w + 2\sqrt{1 - w^2} + 2} \, dw = \int \frac{1}{1 - u^2 + 2u + 2} (-2u) \, du = \int \frac{2u}{u^2 - 2u - 3} \, du
\]

Step 4
This integral requires partial fractions to evaluate. Let’s start with the form of the partial fraction decomposition.

\[
\frac{2u}{(u+1)(u-3)} = \frac{A}{u+1} + \frac{B}{u-3}
\]

Setting the coefficients equal gives,

\[
2u = A(u-3) + B(u+1)
\]

Using the “trick” to get the coefficients gives,

\[
u = 3: \quad 6 = 4B \quad \Rightarrow \quad A = \frac{1}{2}
u = -1: \quad -2 = -4A \quad \Rightarrow \quad B = \frac{3}{2}
\]

The integral is then,

\[
\int \frac{2u}{(u+1)(u-3)} \, du = \int \frac{\frac{1}{2}}{u+1} + \frac{\frac{3}{2}}{u-3} \, du = \frac{1}{2} \ln |u+1| + \frac{3}{2} \ln |u-3| + c
\]

Step 5
The last step is to now do all the back substitutions to get the final answer.

\[
\int \frac{1}{w + 2\sqrt{1 - w^2} + 2} \, dw = \frac{1}{2} \ln \sqrt{1 - w^2 + 1} + \frac{3}{2} \ln \sqrt{1 - w^2 - 3} + c
\]

3. Evaluate the integral \[
\int \frac{t - 2}{t - 3\sqrt{2t^2 - 4} + 2} \, dt \]

Step 1
The substitution we’ll use here is,

\[
u = \sqrt{2t - 4}
\]

Step 2
Now we need to get set up for the substitution. In other words, we need so solve for \(t\) and get \(dt\).

\[
t = \frac{1}{2} u^2 + 2 \quad \Rightarrow \quad dt = u \, du
\]

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Step 3
Doing the substitution gives,
\[
\int \frac{t - 2}{t - 3\sqrt{2}t - 4 + 2} \, dt = \int \frac{\frac{1}{2}u^2 + 2 - 2}{\frac{1}{2}u^2 + 2 - 3u + 2}(u) \, du = \int \frac{u^3}{u^2 - 6u + 8} \, du
\]

Step 4
This integral requires partial fractions to evaluate.

However, we first need to do long division on the integrand since the degree of the numerator (3) is higher than the degree of the denominator (2). This gives,
\[
\frac{u^3}{u^2 - 6u + 8} = u + 6 + \frac{28u - 48}{(u - 2)(u - 4)}
\]
The form of the partial fraction decomposition on the third term is,
\[
\frac{28u - 48}{(u - 2)(u - 4)} = \frac{A}{u - 2} + \frac{B}{u - 4}
\]
Setting the coefficients equal gives,
\[
28u - 48 = A(u - 4) + B(u - 2)
\]
Using the “trick” to get the coefficients gives,
\[
\begin{align*}
    u = 4 : & \quad 64 = 2B \quad \Rightarrow \quad A = -4 \\
    u = 2 : & \quad 8 = -2A \quad \Rightarrow \quad B = 32
\end{align*}
\]
The integral is then,
\[
\int \frac{u^3}{u^2 - 6u + 8} \, du = \int u + 6 - \frac{4}{u - 2} + \frac{32}{u - 4} \, du = \frac{1}{2}u^2 + 6u - 4 \ln|u - 2| + 32 \ln|u - 4| + c
\]

Step 5
The last step is to now do all the back substitutions to get the final answer.
\[
\int \frac{u^3}{u^2 - 6u + 8} \, du = \frac{t - 2 + 6\sqrt{2t - 4} - 4 \ln \sqrt{2t - 4} - 2 + 32 \ln \sqrt{2t - 4} - 4 + c}{u^2 - 6u + 8}
\]
Integrals Involving Quadratics

1. Evaluate the integral \[ \int \frac{7}{w^2 + 3w + 3} \, dw \, . \]

**Step 1**
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

\[ \int \frac{7}{w^2 + 3w + 3} \, dw = \int \frac{7}{(w + \frac{3}{2})^2 + \frac{3}{4}} \, dw \]

**Step 2**
From this we can see that the following substitution should work for us.

\[ u = w + \frac{3}{2} \quad \Rightarrow \quad du = dw \]

Doing the substitution gives,

\[ \int \frac{7}{w^2 + 3w + 3} \, dw = \int \frac{7}{u^2 + \frac{3}{4}} \, du \]

**Step 3**
This integral can be done with the formula given at the start of this section.

\[ \int \frac{7}{w^2 + 3w + 3} \, dw = \frac{14}{\sqrt{3}} \tan^{-1} \left( \frac{2u}{\sqrt{3}} \right) + c = \frac{14}{\sqrt{3}} \tan^{-1} \left( \frac{2w + 3}{\sqrt{3}} \right) + c \]

Don’t forget to back substitute in for \( u \)!

2. Evaluate the integral \[ \int \frac{10x}{4x^2 - 8x + 9} \, dx \, . \]

**Step 1**
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

\[ \int \frac{10x}{4x^2 - 8x + 9} \, dx = \int \frac{10x}{4(x-1)^2 + 5} \, dx \]

**Step 2**
From this we can see that the following substitution should work for us.
\[ u = x - 1 \quad \Rightarrow \quad du = dx \quad \& \quad x = u + 1 \]

Doing the substitution gives,

\[ \int \frac{10x}{4x^2 - 8x + 9} \, dx = \int \frac{10(u + 1)}{4u^2 + 5} \, du \]

Step 3
We can quickly do this integral if we split it up as follows,

\[ \int \frac{10x}{4x^2 - 8x + 9} \, dx = \int \frac{10u}{4u^2 + 5} \, du + \int \frac{10}{4u^2 + 5} \, du = \int \frac{10u}{4u^2 + 5} \, du + \frac{5}{2} \int \frac{1}{u^2 + \frac{5}{4}} \, du \]

After a quick rewrite of the second integral we can see that we can do the first with the substitution \( v = 4u^2 + 5 \) and the second is an inverse trig integral we can evaluate using the formula given at the start of the notes for this section.

\[
\int \frac{10x}{4x^2 - 8x + 9} \, dx = \frac{5}{4} \ln|4u^2 + 5| + \frac{5}{2} \tan^{-1} \left( \frac{2u}{\sqrt{5}} \right) + c \\
= \frac{5}{4} \ln|4u^2 + 5| + \sqrt{5} \tan^{-1} \left( \frac{2u}{\sqrt{5}} \right) + c \\
= \frac{5}{4} \ln|4(x-1)^2 + 5| + \sqrt{5} \tan^{-1} \left( \frac{2x-2}{\sqrt{5}} \right) + c
\]

Don’t forget to back substitute in for \( u \)!

3. Evaluate the integral \( \int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} \, dt \).

Step 1
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

\[ \int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} \, dt = \int \frac{2t + 9}{((t - 7)^2 - 3)^{\frac{3}{2}}} \, dt \]

Step 2
From this we can see that the following substitution should work for us.
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\[ u = t - 7 \quad \Rightarrow \quad du = dt \quad \& \quad t = u + 7 \]

Doing the substitution gives,

\[ \int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} dt = \int \frac{2(u + 7) + 9}{(u^2 - 3)^{\frac{3}{2}}} du = \int \frac{2u + 23}{(u^2 - 3)^{\frac{3}{2}}} du \]

Step 3
Next we’ll need to split the integral up as follows,

\[ \int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} dt = \int \frac{2u}{(u^2 - 3)^{\frac{3}{2}}} du + \int \frac{23}{(u^2 - 3)^{\frac{3}{2}}} du \]

The first integral can be done with the substitution \( v = u^2 - 3 \) and the second integral will require the trig substitution \( u = \sqrt{3} \sec \theta \). Here is the substitution work.

\[ \int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} dt = \int v^{-\frac{3}{2}} dv + \int \frac{23}{(3\sec^2 \theta - 3)^{\frac{3}{2}}} \left( \sqrt{3} \sec \theta \tan \theta \right) d\theta \]

\[ = \int v^{-\frac{3}{2}} dv + \int \frac{23 \sqrt{3} \sec \theta \tan \theta}{(3 \sec^2 \theta - 3)^{\frac{3}{2}}} d\theta \]

\[ = \int v^{-\frac{3}{2}} dv + \int \frac{23 \sec \theta}{9 \tan^4 \theta} d\theta \]

\[ = \int v^{-\frac{3}{2}} dv + \frac{23}{9} \int \frac{\cos^3 \theta}{\sin^4 \theta} d\theta \]

Now, for the second integral, don’t forget the manipulations we often need to do so we can do these kinds of integrals. If you need some practice on these kinds of integrals go back to the practice problems for the second section of this chapter and work some of them.

Here is the rest of the integration process for this problem.

\[ \int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} dt = \int v^{-\frac{3}{2}} dv + \frac{23}{9} \int \frac{1 - \sin^2 \theta}{\sin^3 \theta} \cos \theta d\theta \quad w = \sin \theta \]

\[ = \int v^{-\frac{3}{2}} dv + \frac{23}{9} \int w^{-4} - w^{-2} dw \]

\[ = -\frac{2}{3} v^{-\frac{3}{2}} + \frac{23}{9} \left[ -\frac{1}{5} (\sin \theta)^{-3} + (\sin \theta)^{-1} \right] + c \]

Step 4
We now need to do quite a bit of back substitution to get the answer back into \( t \)'s. Let's start with the result of the second integration. Converting the \( \theta \)'s back to \( u \)'s will require a quick right triangle.

From the substitution we have,

\[
\sec \theta = \frac{u}{\sqrt{3}} \quad \left( = \frac{\text{hyp}}{\text{adj}} \right)
\]

From the right triangle we get,

\[
\sin \theta = \frac{\sqrt{u^2 - 3}}{u}
\]

Plugging this into the integral above gives,

\[
\int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} \, dt = \frac{2}{3(u^2 - 3)^{\frac{3}{2}}} - \frac{23u^3}{27(u^2 - 3)^{\frac{3}{2}}} + \frac{23u}{9\sqrt{u^2 - 3}} + c
\]

Note that we also back substituted for the \( v \) in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the \( u \).

\[
\int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} \, dt = \frac{2}{3((t-7)^2 - 3)^{\frac{3}{2}}} - \frac{23(t-7)^3}{27((t-7)^2 - 3)^{\frac{3}{2}}} + \frac{23(t-7)}{9\sqrt{(t-7)^2 - 3}} + c
\]

\[
= \frac{23(t-7) - 18 + 23(t-7)^{\frac{3}{2}}}{9\sqrt{(t-7)^2 - 3} - 27((t-7)^2 - 3)^{\frac{3}{2}}} + c
\]

We'll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don’t get into too much of a hurry when working them.

4. Evaluate the integral \( \int \frac{3z}{(1-4z-2z^2)^{\frac{3}{2}}} \, dz \) .

Step 1
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

\[
\int \frac{3z}{(1-4z-2z^2)^{\frac{3}{2}}} \, dz = \int \frac{3z}{(3-2(z+1)^2)^{\frac{3}{2}}} \, dz
\]
Step 2
From this we can see that the following substitution should work for us.

\[ u = z + 1 \quad \Rightarrow \quad du = dz \quad \& \quad z = u - 1 \]

Doing the substitution gives,

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = \int \frac{3(u - 1)}{(3 - 2u^2)^2} \, du = \int \frac{3u - 3}{(3 - 2u^2)^2} \, du
\]

Step 3
Next we’ll need to split the integral up as follows,

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = \int \frac{3u}{(3 - 2u^2)^2} \, du - \int \frac{3}{(3 - 2u^2)^2} \, du
\]

The first integral can be done with the substitution \( v = 3 - 2u^2 \) and the second integral will require the trig substitution \( u = \frac{\sqrt{3}}{\sqrt{2}} \sin \theta \). Here is the substitution work.

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = -\frac{3}{4} \int v^{-2} \, dv - \int \frac{3}{(3 - 3\sin^2 \theta)^2} \left( \frac{\sqrt{3}}{\sqrt{2}} \cos \theta \right) \, d\theta
\]

\[
= -\frac{3}{4} \int v^{-2} \, dv - \int \frac{3}{(3 \cos^2 \theta)^2} \left( \frac{\sqrt{3}}{\sqrt{2}} \cos \theta \right) \, d\theta
\]

\[
= -\frac{3}{4} \int v^{-2} \, dv - \frac{1}{\sqrt{2}} \int \sec^3 \theta \, d\theta
\]

The second integral for this problem comes down to an integral that was done in the notes for the second section of this chapter and so we’ll just use the formula derived in that section to do this integral.

Here is the rest of the integration process for this problem.

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = -\frac{3}{4} v^{-1} - \frac{1}{2\sqrt{6}} \left[ \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right] + c
\]

Step 4
We now need to do quite a bit of back substitution to get the answer back into \( z \)'s. Let’s start with the result of the second integration. Converting the \( \theta \)'s back to \( u \)'s will require a quick right triangle.
From the substitution we have,
\[ \sin \theta = \frac{\sqrt{2} u}{\sqrt{3}} \quad \left( = \frac{\text{opp}}{\text{hyp}} \right) \]

From the right triangle we get,
\[ \tan \theta = \frac{\sqrt{2} u}{\sqrt{3 - 2u^2}} \quad \& \quad \sec \theta = \frac{\sqrt{3}}{\sqrt{3 - 2u^2}} \]

Plugging this into the integral above gives,
\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = \frac{3}{4(3 - 2u^2)} - \frac{1}{2\sqrt{6}} \ln \frac{\sqrt{3} + \sqrt{2} u}{\sqrt{3 - 2u^2}} + c
\]

Note that we also back substituted for the \( v \) in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the \( u \).

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = \frac{3}{4(3 - 2(z+1)^2)} - \frac{z + 1}{6 - 4(z+1)^2} - \frac{1}{2\sqrt{6}} \ln \frac{\sqrt{3} + \sqrt{2} (z+1)}{\sqrt{3 - 2(z+1)^2}} + c
\]

We’ll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don’t get into too much of a hurry when working them.

---

**Integration Strategy**

Problems have not yet been written for this section.

I was finding it very difficult to come up with a good mix of “new” problems and decided my time was better spent writing problems for later sections rather than trying to come up with a sufficient number of problems for what is essentially a review section. I intend to come back at a later date when I have more time to devote to this section and add problems then.

**Improper Integrals**

1. Determine if the following integral converges or diverges. If the integral converges determine its value.
\[
\int_0^\infty (1 + 2x) e^{-x} \, dx
\]

Hint: Don’t forget that we can’t do the integral as long as there is an infinity in one of the limits!

Step 1
First, we need to recall that we can’t do the integral as long as there is an infinity in one of the limits. Therefore, we’ll need to eliminate the infinity first as follows,

\[
\int_0^\infty (1 + 2x) e^{-x} \, dx = \lim_{t \to \infty} \int_0^t (1 + 2x) e^{-x} \, dx
\]

Note that this step really is needed for these integrals! For some integrals we can use basic logic and “evaluate” at infinity to get the answer. However, many of these kinds of improper integrals can’t be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can “evaluate” at infinity we need to be in the habit of doing this for those that can’t be done that way.

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we need to do integration by parts to evaluate this integral. Here is the integration work.

\[
\begin{align*}
u &= 1 + 2x & du &= 2 \, dx \\
dv &= e^{-x} \, dx & v &= -e^{-x}
\end{align*}
\]

\[
\int (1 + 2x) e^{-x} \, dx = -(1 + 2x) e^{-x} + 2 \int e^{-x} \, dx = -(1 + 2x) e^{-x} - 2e^{-x} + c = -(3 + 2x) e^{-x} + c
\]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let’s take care of the limits on the integral.

\[
\int_0^\infty (1 + 2x) e^{-x} \, dx = \lim_{t \to \infty} \left[ -(3 + 2x) e^{-x} \right]_0^t = \lim_{t \to \infty} \left( 3 - (3 + 2t) e^{-t} \right)
\]

Step 4
We now need to evaluate the limit in our answer from the previous step and note that, in this case, we really can’t just “evaluate” at infinity! We need to do the limiting process here to make sure we get the correct answer.

We will need to do a quick L’Hôpitals Rule on the second term to properly evaluate it. Here is the limit work.
\[ \int_0^\infty (1 + 2x) \, dx = \lim_{t \to \infty} 3 - \lim_{t \to \infty} \frac{3 + 2t}{e^t} = 3 - \lim_{t \to \infty} \frac{2}{e^t} = 3 - 0 = 3 \]

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (i.e. not an infinity). Therefore, the integral converges and its value is 3.

---

2. Determine if the following integral converges or diverges. If the integral converges determine its value.

\[ \int_{-\infty}^{0} (1 + 2x) e^{-x} \, dx \]

Hint: Don’t forget that we can’t do the integral as long as there is an infinity in one of the limits!

Step 1
First, we need to recall that we can’t do the integral as long as there is an infinity in one of the limits. Therefore, we’ll need to eliminate the infinity first as follows,

\[ \lim_{t \to -\infty} t \int_0^t (1 + 2x) e^{-x} \, dx = \lim_{t \to -\infty} \int_0^t (1 + 2x) e^{-x} \, dx \]

Note that this step really is needed for these integrals! For some integrals we can use basic logic and “evaluate” at infinity to get the answer. However, many of these kinds of improper integrals can’t be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can “evaluate” at infinity we need to be in the habit of doing this for those that can’t be done that way.

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we need to do integration by parts to evaluate this integral. Here is the integration work.

\[ u = 1 + 2x \quad \rightarrow \quad du = 2 \, dx \]
\[ dv = e^{-x} \, dx \quad \rightarrow \quad v = -e^{-x} \]

\[ \int (1 + 2x) e^{-x} \, dx = -(1 + 2x) e^{-x} + 2 \int e^{-x} \, dx = -(1 + 2x) e^{-x} - 2e^{-x} + c = -(3 + 2x) e^{-x} + c \]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and so we held off dealing with them until the next step.
Step 3
Okay, now let’s take care of the limits on the integral.

\[
\int_{-\infty}^{0} (1 + 2x) e^{-x} \, dx = \lim_{t \to -\infty} \left( - (3 + 2x) e^{-x} \right)_{t}^{0} = \lim_{t \to -\infty} \left( (3 + 2t) e^{-t} - 3 \right)
\]

Step 4
We now need to evaluate the limit in our answer from the previous step. In this case we can see that the first term will go to negative infinity since it is just a product of one factor that goes to negative infinity and another factor that goes to infinity. Therefore the full limit will also be negative infinity since the constant second term won’t affect the final value of the limit.

\[
\int_{-\infty}^{0} (1 + 2x) e^{-x} \, dx = \lim_{t \to -\infty} (3 + 2t) e^{-t} - \lim_{t \to -\infty} 3 = (-\infty)(\infty) - 3 = -\infty - 3 = -\infty
\]

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and but was negative infinity. Therefore, the integral diverges.

---

3. Determine if the following integral converges or diverges. If the integral converges determine its value.

\[
\int_{-5}^{1} \frac{1}{10 + 2z} \, dz
\]

Hint : Don’t forget that we can’t do the integral as long as there is a division by zero in the integrand at some point in the interval of integration!

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at \( z = -5 \) and this is the lower limit of integration. We know that as long as that discontinuity is there we can’t do the integral. Therefore, we’ll need to eliminate the discontinuity first as follows,

\[
\int_{-5}^{1} \frac{1}{10 + 2z} \, dz = \lim_{t \to -5} \int_{t}^{1} \frac{1}{10 + 2z} \, dz
\]

Don’t forget that the limits on these kinds of integrals must be one-sided limits. Because the interval of integration is \([-5, 1]\) we are only interested in the values of \( z \) that are greater than -5 and so we must use a right hand limit to reflect that fact.

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.
In this case we can do a simple Calc I substitution. Here is the integration work.

\[\int \frac{1}{10 + 2z} \, dz = \frac{1}{2} \ln|10 + 2z| + c\]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let’s take care of the limits on the integral.

\[\int_{-5}^{1} \frac{1}{10 + 2z} \, dz = \lim_{t \to -5^+} \left[ \frac{1}{2} \ln|10 + 2z| \right]_{t}^{1} = \lim_{t \to -5^+} \left( \frac{1}{2} \ln|12| - \frac{1}{2} \ln|10 + 2t| \right)\]

Step 4
We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

\[\int_{-5}^{1} \frac{1}{10 + 2z} \, dz = \lim_{t \to -5^+} \left( \frac{1}{2} \ln|12| - \frac{1}{2} \ln|10 + 2t| \right) = \frac{1}{2} \ln|12| + \infty = \infty\]

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and but was infinity. Therefore, the integral diverges.

4. Determine if the following integral converges or diverges. If the integral converges determine its value.

\[\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2 - 4}} \, dw\]

Hint: Don’t forget that we can’t do the integral as long as there is a division by zero in the integrand at some point in the interval of integration!

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at \(w = 2\) and this is the upper limit of integration. We know that as long as that discontinuity is there we can’t do the integral. Therefore, we’ll need to eliminate the discontinuity first as follows,

\[\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2 - 4}} \, dw = \lim_{t \to 2^-} \int_{1}^{t} \frac{4w}{\sqrt[3]{w^2 - 4}} \, dw\]
Don’t forget that the limits on these kinds of integrals must be one-sided limits. Because the interval of integration is \([1, 2]\) we are only interested in the values of \(t\) that are less than 2 and so we must use a left hand limit to reflect that fact.

**Step 2**

Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

\[
\int \frac{4w}{\sqrt{w^2 - 4}} \, dw = 3\left(\frac{w^2}{2} - 4\right)^{\frac{3}{2}} + c
\]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and so we held off dealing with them until the next step.

**Step 3**

Okay, now let’s take care of the limits on the integral.

\[
\int_{1}^{2} \frac{4w}{\sqrt{w^2 - 4}} \, dw = \lim_{t \to 2} \left(3\left(\frac{w^2}{2} - 4\right)^{\frac{3}{2}}\right) - \lim_{t \to 1} \left(3\left(\frac{w^2}{2} - 4\right)^{\frac{3}{2}} - 3\left(-\frac{3}{2}\right)^{\frac{3}{2}}\right)
\]

**Step 4**

We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

\[
\int_{1}^{2} \frac{4w}{\sqrt{w^2 - 4}} \, dw = \lim_{t \to 2} \left(3\left(t^2 - 4\right)^{\frac{3}{2}} - 3\left(-3\right)^{\frac{3}{2}}\right) = -3\left(-3\right)^{\frac{3}{2}} = \left(-3\right)^{\frac{3}{2}}
\]

**Step 5**

The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (i.e. not an infinity). Therefore, the integral converges and its value is \((-3)^{\frac{3}{2}}\).

5. Determine if the following integral converges or diverges. If the integral converges determine its value.

\[
\int_{-\infty}^{1} \sqrt{6-y} \, dy
\]

Hint: Don’t forget that we can’t do the integral as long as there is an infinity in one of the limits!
First, we need to recall that we can’t do the integral as long as there is an infinity in one of the limits. Therefore, we’ll need to eliminate the infinity first as follows,

\[ \int_{-\infty}^{1} \sqrt{6-y} \, dy = \lim_{t \to -\infty} \int_{t}^{1} \sqrt{6-y} \, dy \]

Note that this step really is needed for these integrals! For some integrals we can use basic logic and “evaluate” at infinity to get the answer. However, many of these kinds of improper integrals can’t be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can “evaluate” at infinity we need to be in the habit of doing this for those that can’t be done that way.

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

\[ \int \sqrt{6-y} \, dy = -\frac{2}{3} (6-y)^{\frac{3}{2}} + c \]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let’s take care of the limits on the integral.

\[ \int_{-\infty}^{1} \sqrt{6-y} \, dy = \lim_{t \to -\infty} \left( -\frac{2}{3} (6-y)^{\frac{3}{2}} \right) \bigg|_{t}^{1} = \lim_{t \to -\infty} \left( -\frac{2}{3} (5)^{\frac{3}{2}} + \frac{2}{3} (6-t)^{\frac{3}{2}} \right) \]

Step 4
We now need to evaluate the limit in our answer from the previous step. Here is the limit work.

\[ \int \sqrt{6-y} \, dy = \lim_{t \to -\infty} \left( -\frac{2}{3} (5)^{\frac{3}{2}} + \frac{2}{3} (6-t)^{\frac{3}{2}} \right) = -\frac{2}{3} (5)^{\frac{3}{2}} + \infty = \infty \]

Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and but was infinity. Therefore, the integral 

\[ \text{diverges} \]

6. Determine if the following integral converges or diverges. If the integral converges determine its value.
Hint: Don’t forget that we can’t do the integral as long as there is an infinity in one of the limits!

Step 1
First, we need to recall that we can’t do the integral as long as there is an infinity in one of the limits. Therefore, we’ll need to eliminate the infinity first as follows,

\[
\int_2^\infty \frac{9}{(1-3z)^4} \, dz = \lim_{t \to \infty} \int_2^t \frac{9}{(1-3z)^4} \, dz
\]

Note that this step really is needed for these integrals! For some integrals we can use basic logic and “evaluate” at infinity to get the answer. However, many of these kinds of improper integrals can’t be done that way! This is the only way to make sure we can deal with the infinite limit in those cases.

So even if this ends up being one of the integrals in which we can “evaluate” at infinity we need to be in the habit of doing this for those that can’t be done that way.

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

\[
\int \frac{9}{(1-3z)^4} \, dz = \frac{1}{(1-3z)^3} + c
\]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and so we held off dealing with them until the next step.

Step 3
Okay, now let’s take care of the limits on the integral.

\[
\int_2^\infty \frac{9}{(1-3z)^4} \, dz = \lim_{t \to \infty} \left[ \frac{1}{(1-3z)^3} \right]_2^t = \lim_{t \to \infty} \left( \frac{1}{(1-3t)^3} - \frac{1}{125} \right)
\]

Step 4
We now need to evaluate the limit in our answer from the previous step. Here is the limit work

\[
\int_2^\infty \frac{9}{(1-3z)^4} \, dz = \lim_{t \to \infty} \left( \frac{1}{(1-3t)^3} + \frac{1}{125} \right) = \frac{1}{125}
\]
Step 5
The final step is to write down the answer!

In this case, the limit we computed in the previous step existed and was finite (i.e. not an infinity). Therefore, the integral converges and its value is $\frac{1}{125}$.

7. Determine if the following integral converges or diverges. If the integral converges determine its value.

$$\int_{0}^{4} \frac{x}{x^2 - 9} \, dx$$

Hint: Don’t forget that we can’t do the integral as long as there is a division by zero in the integrand at some point in the interval of integration! Also, do not just assume the division by zero will be at one of the limits of the integral.

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at $x = 3$ and note that this is between the limits of the integral. We know that as long as that discontinuity is there we can’t do the integral.

However, recall from the notes in this section that we can only deal with discontinuities that if they occur at one of the limits of the integral. So, we’ll need to break up the integral at $x = 3$.

$$\int_{0}^{4} \frac{x}{x^2 - 9} \, dx = \int_{0}^{3} \frac{x}{x^2 - 9} \, dx + \int_{3}^{4} \frac{x}{x^2 - 9} \, dx$$

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn’t have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let’s proceed with the problem.

We can eliminate the discontinuity in each as follows,

$$\int_{0}^{4} \frac{x}{x^2 - 9} \, dx = \lim_{t \to 3^-} \int_{0}^{t} \frac{x}{x^2 - 9} \, dx + \lim_{t \to 3^+} \int_{t}^{4} \frac{x}{x^2 - 9} \, dx$$

Don’t forget that the limits on these kinds of integrals must be one-sided limits.

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.
\[ \int \frac{x}{x^2 - 9} \, dx = \frac{1}{2} \ln \left| x^2 - 9 \right| + c \]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3
Okay, now let’s take care of the limits on the integral.

\[
\int_0^4 \frac{x}{x^2 - 9} \, dx = \lim_{t \to 3^-} \left( \frac{1}{2} \ln \left| t^2 - 9 \right| \right) + \lim_{s \to 3^+} \left( \frac{1}{2} \ln \left| s^2 - 9 \right| \right)
\]

\[
= \lim_{t \to 3^-} \left( \frac{1}{2} \ln \left| t^2 - 9 \right| - \frac{1}{2} \ln (9) \right) + \lim_{s \to 3^+} \left( \frac{1}{2} \ln (7) - \frac{1}{2} \ln \left| s^2 - 9 \right| \right)
\]

Step 4
We now need to evaluate the limits in our answer from the previous step. Here is the limit work.

\[
\int_0^4 \frac{x}{x^2 - 9} \, dx = \lim_{t \to 3^-} \left( \frac{1}{2} \ln \left| t^2 - 9 \right| - \frac{1}{2} \ln (9) \right) + \lim_{s \to 3^+} \left( \frac{1}{2} \ln (7) - \frac{1}{2} \ln \left| s^2 - 9 \right| \right)
\]

\[
= \left[ -\infty - \frac{1}{2} \ln (9) \right] \quad + \quad \left[ \frac{1}{2} \ln (7) + \infty \right]
\]

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5
The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

\[
\int_0^3 \frac{x}{x^2 - 9} \, dx = \lim_{t \to 3^-} \left( \frac{1}{2} \ln \left| t^2 - 9 \right| - \frac{1}{2} \ln (9) \right) = \left[ -\infty - \frac{1}{2} \ln (9) \right] = -\infty
\]

\[
\int_3^4 \frac{x}{x^2 - 9} \, dx = \lim_{s \to 3^+} \left( \frac{1}{2} \ln (7) - \frac{1}{2} \ln \left| s^2 - 9 \right| \right) = \left[ \frac{1}{2} \ln (7) + \infty \right] = \infty
\]

Therefore each of these integrals are divergent. This means that we were, in fact, not able to break up the integral as we did back in Step 1.

This in turn means that the integral **diverges**.
8. Determine if the following integral converges or diverges. If the integral converges determine its value.

\[ \int_{-\infty}^{\infty} \frac{6w^3}{(w^4 + 1)^2} \, dw \]

Hint: Don’t forget that we can’t do the integral as long as there is an infinity in one of the limits! Also, don’t forget that infinities in both limits need an extra step to set up.

Step 1
First, we need to recall that we can’t do the integral as long as there is an infinity in one of the limits. Note as well that in this case we have infinities in both limits and so we’ll need to split up the integral.

The integral can be split up at any point in this case and \( w = 0 \) seems like a good point to use for the split point. Splitting up the integral gives,

\[ \int_{-\infty}^{\infty} \frac{6w^3}{(w^4 + 1)^2} \, dw = \int_{-\infty}^{0} \frac{6w^3}{(w^4 + 1)^2} \, dw + \int_{0}^{\infty} \frac{6w^3}{(w^4 + 1)^2} \, dw \]

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn’t have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let’s proceed with the problem.

Now, we can eliminate the infinities as follows,

\[ \int_{-\infty}^{\infty} \frac{6w^3}{(w^4 + 1)^2} \, dw = \lim_{t \to -\infty} \int_{t}^{0} \frac{6w^3}{(w^4 + 1)^2} \, dw + \lim_{s \to \infty} \int_{0}^{s} \frac{6w^3}{(w^4 + 1)^2} \, dw \]

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

\[ \int \frac{6w^3}{(w^4 + 1)^2} \, dw = -\frac{3}{2} \frac{1}{w^4 + 1} + c \]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.
Step 3
Okay, now let’s take care of the limits on the integral.

\[
\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2} \, dw = \lim_{t \to -\infty} \left[ \frac{-3}{2} \frac{1}{w^4+1} \right]_t^0 + \lim_{s \to \infty} \left[ \frac{-3}{2} \frac{1}{s^4+1} \right]_0^s
\]

\[
= \lim_{t \to -\infty} \left( -\frac{3}{2} + \frac{3}{2} \frac{1}{t^4+1} \right) + \lim_{s \to \infty} \left( -\frac{3}{2} \frac{1}{s^4+1} + \frac{3}{2} \right)
\]

Step 4
We now need to evaluate the limits in our answer from the previous step. Here is the limit work

\[
\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2} \, dw = \lim_{t \to -\infty} \left( -\frac{3}{2} + \frac{3}{2} \frac{1}{t^4+1} \right) + \lim_{s \to \infty} \left( -\frac{3}{2} \frac{1}{s^4+1} + \frac{3}{2} \right)
\]

\[
= \left[ -\frac{3}{2} \right] + \left[ \frac{3}{2} \right]
\]

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5
The final step is to write down the answer!

Now, from the limit work in the previous step we see that,

\[
\int_{-\infty}^{0} \frac{6w^3}{(w^4+1)^2} \, dw = \lim_{t \to -\infty} \left( -\frac{3}{2} + \frac{3}{2} \frac{1}{t^4+1} \right) = -\frac{3}{2}
\]

\[
\int_{0}^{\infty} \frac{6w^3}{(w^4+1)^2} \, dw = \lim_{s \to \infty} \left( -\frac{3}{2} \frac{1}{s^4+1} + \frac{3}{2} \right) = \frac{3}{2}
\]

Therefore each of the integrals are convergent and have the values shown above. This means that we could in fact break up the integral as we did in Step 1. Also, the original integral is now,

\[
\int_{-\infty}^{\infty} \frac{6w^3}{(w^4+1)^2} \, dw = \int_{-\infty}^{0} \frac{6w^3}{(w^4+1)^2} \, dw + \int_{0}^{\infty} \frac{6w^3}{(w^4+1)^2} \, dw
\]

\[
= -\frac{3}{2} + \frac{3}{2}
\]

\[
= 0
\]

Therefore, the integral **converges** and its value is **0**.
9. Determine if the following integral converges or diverges. If the integral converges determine its value.

\[ \int_{1}^{4} \frac{1}{x^2 + x - 6} \, dx \]

Hint: Don’t forget that we can’t do the integral as long as there is a division by zero in the integrand at some point in the interval of integration! Also, do not just assume the division by zero will be at one of the limits of the integral.

Step 1
First, notice that there is a division by zero issue (and hence a discontinuity) in the integrand at \( x = 2 \) and note that this is between the limits of the integral. We know that as long as that discontinuity is there we can’t do the integral.

However, recall from the notes in this section that we can only deal with discontinuities that if they occur at one of the limits of the integral. So, we’ll need to break up the integral at \( x = 2 \).

\[ \int_{1}^{4} \frac{1}{x^2 + x - 6} \, dx = \int_{1}^{2} \frac{1}{(x + 3)(x - 2)} \, dx + \int_{2}^{4} \frac{1}{(x + 3)(x - 2)} \, dx \]

Remember as well, that we can only break up the integral like this provided both of the new integrals are convergent! If it turns out that even one of them is divergent then it will turn out that we couldn’t have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not, let’s proceed with the problem.

We can eliminate the discontinuity in each as follows,

\[ \int_{1}^{4} \frac{1}{x^2 + x - 6} \, dx = \lim_{t \to 2} \int_{1}^{t} \frac{1}{(x + 3)(x - 2)} \, dx + \lim_{s \to 2} \int_{s}^{4} \frac{1}{(x + 3)(x - 2)} \, dx \]

Don’t forget that the limits on these kinds of integrals must be one-sided limits.

Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we will need to do some partial fractions in order to the integral. Here is the partial fraction work.

\[ \frac{1}{(x + 3)(x - 2)} = \frac{A}{x + 3} + \frac{B}{x - 2} \quad \Rightarrow \quad 1 = A(x - 2) + B(x + 3) \]
\[ x = 2 : \quad 1 = 5B \quad \Rightarrow \quad A = -\frac{1}{5} \]
\[ x = -3 : \quad 1 = -5A \quad \Rightarrow \quad B = \frac{1}{5} \]

The integration work is then,
\[ \int \frac{1}{(x+3)(x-2)} \, dx = \int \frac{\frac{1}{5}}{x-2} - \frac{\frac{1}{5}}{x+3} \, dx = \frac{1}{5} \ln|x-2| - \frac{1}{5} \ln|x+3| + c \]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3
Okay, now let’s take care of the limits on the integral.
\[ \int_{1}^{4} \frac{1}{x^2 + x - 6} \, dx = \lim_{t \to 2^+} \left( \frac{1}{5} \ln|t-2| - \frac{1}{5} \ln|t+3| \right) + \lim_{s \to 2^+} \left( \frac{1}{5} \ln|s-2| - \frac{1}{5} \ln|s+3| \right) \]
\[ = \lim_{t \to 2^+} \left( \frac{1}{5} \ln|t-2| - \frac{1}{5} \ln|t+3| - \left( \frac{1}{5} \ln(1) - \frac{1}{5} \ln(4) \right) \right) \]
\[ + \lim_{s \to 2^+} \left( \frac{1}{5} \ln(2) - \frac{1}{5} \ln(7) - \left( \frac{1}{5} \ln|s-2| - \frac{1}{5} \ln|s+3| \right) \right) \]

Step 4
We now need to evaluate the limits in our answer from the previous step. Here is the limit work.
\[ \int_{1}^{4} \frac{1}{x^2 + x - 6} \, dx = \lim_{t \to 2^+} \left( \frac{1}{5} \ln|t-2| - \frac{1}{5} \ln|t+3| - \left( \frac{1}{5} \ln(1) - \frac{1}{5} \ln(4) \right) \right) \]
\[ + \lim_{s \to 2^+} \left( \frac{1}{5} \ln(2) - \frac{1}{5} \ln(7) - \left( \frac{1}{5} \ln|s-2| - \frac{1}{5} \ln|s+3| \right) \right) \]
\[ = \left[ -\infty - \frac{1}{5} \ln(5) + \frac{1}{5} \ln(4) \right] + \left[ \frac{1}{5} \ln(2) - \frac{1}{5} \ln(7) + \frac{1}{5} \ln(5) + \infty \right] \]

Note that we put the answers for each limit in brackets to make it clear what each limit was. This will be important for the next step.

Step 5
The final step is to write down the answer!

Now, from the limit work in the previous step we see that,
\[ \int_1^2 \frac{1}{(x+3)(x-2)} \, dx = \lim_{s \to 2^+} \left( \frac{1}{5} \ln|s-2| - \frac{1}{5} \ln|s+3| - \left( \frac{1}{5} \ln(1) - \frac{1}{5} \ln(4) \right) \right) = -\infty \]

\[ \int_2^4 \frac{1}{(x+3)(x-2)} \, dx = \lim_{t \to 2^-} \left( \frac{1}{5} \ln(2) - \frac{1}{5} \ln(7) - \left( \frac{1}{5} \ln|s-2| - \frac{1}{5} \ln|s+3| \right) \right) = \infty \]

Therefore each of these integrals are divergent. This means that we were, in fact, not able to break up the integral as we did back in Step 1.

This in turn means that the integral \textbf{diverges}.

10. Determine if the following integral converges or diverges. If the integral converges determine its value.

\[ \int_{-\infty}^0 \frac{1}{e^x - x^2} \, dx \]

Hint: Be very careful with this problem as it is nothing like what we did in the notes. However, you should be able to take the material from the notes and use that to figure out how to do this problem.

Step 1
Now there is clearly an infinite limit here, but also notice that there is a discontinuity at \( x = 0 \) that we’ll need to deal with.

Based on the material in the notes it should make sense that, provided both integrals converge, we should be able to split up the integral at any point. In this case let’s split the integral up at \( x = -1 \). Doing this gives,

\[ \int_{-\infty}^0 \frac{1}{e^x - x^2} \, dx = \int_{-\infty}^{-1} \frac{1}{e^x - x^2} \, dx + \int_{-1}^0 \frac{1}{e^x - x^2} \, dx \]

Keep in mind that splitting up the integral like this can only be done if both of the integrals converge! If it turns out that even one of them is divergent then it will turn out that we couldn’t have done this and the original integral will be divergent.

So, not worrying about if this was really possible to do or not let’s proceed with the problem.

Now, we can eliminate the problems as follows,

\[ \int_{-\infty}^0 \frac{1}{e^x - x^2} \, dx = \lim_{t \to -\infty} \int_t^{-1} \frac{1}{e^x - x^2} \, dx + \lim_{s \to 0} \int_{-1}^s \frac{1}{e^x - x^2} \, dx \]
Step 2
Next, let’s do the integral. We’ll not be putting a lot of explanation/detail into the integration process. By this point it is assumed that your integration skills are getting pretty good. If you find your integration skills are a little rusty you should go back and do some practice problems from the appropriate earlier sections.

In this case we can do a simple Calc I substitution. Here is the integration work.

\[
\int \frac{e^x}{x^2} \, dx = -\frac{1}{e^x} + c
\]

Note that we didn’t do the definite integral here. The limits don’t really affect how we do the integral and the integral for each was the same with only the limits being different so no reason to do the integral twice.

Step 3
Okay, now let’s take care of the limits on the integral.

\[
\lim_{s \to -\infty} \int_{s}^{1} \frac{1}{e^x} \, dx = \lim_{t \to -\infty} \left( -e^t \right)_{-1}^{1} + \lim_{t \to -\infty} \left( -e^t \right)_{0}^{1} = \lim_{s \to 0^+} \left( -e^{-s} + e^{s} \right) + \lim_{s \to 0^+} \left( -e^{-s} + e^{-1} \right)
\]

Step 4
We now need to evaluate the limits in our answer from the previous step. Here is the limit work

\[
\lim_{t \to -\infty} \left( -e^{-t} + e^{t} \right) + \lim_{s \to 0^+} \left( -e^{-s} + e^{-1} \right)
\]

Note that,

\[
\lim_{s \to 0^+} \left( -e^{-s} \right) = -\infty
\]

since we are doing a left hand limit and so s will be negative. This in turn means that,

\[
\lim_{s \to 0^+} \left( -e^s \right) = 0
\]

Step 5
The final step is to write down the answer!

Now, from the limit work in the previous step we see that,
Therefore each of the integrals are convergent and have the values shown above. This means that we could in fact break up the integral as we did in Step 1. Also, the original integral is now,

\[
\int_{-\infty}^{0} \frac{1}{e^x} \, dx = \int_{-\infty}^{-1} \frac{1}{e^x} \, dx + \int_{-1}^{0} \frac{1}{e^x} \, dx
\]

\[= -e^{-1} + 1 + e^{-1} = 1\]

Therefore, the integral **converges** and its value is 1.

---

**Comparison Test for Improper Integrals**

1. Use the Comparison Test to determine if the following integral converges or diverges.

\[
\int_{1}^{\infty} \frac{1}{x^3 + 1} \, dx
\]

Hint: Start off with a guess. Do you think this will converge or diverge?

Step 1

The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The “+1” in the denominator does not really change the size of the denominator as \( x \) gets really large and so hopefully it makes sense that we can guess that this integral should behave like,

\[
\int_{1}^{\infty} \frac{1}{x^3} \, dx
\]

Then, by the fact from the previous section, we know that this integral converges since \( p = 3 > 1 \).

Therefore, we can guess that the integral,

\[
\int_{1}^{\infty} \frac{1}{x^3 + 1} \, dx
\]
will converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we’ve guessed the integral converges we now know that it converges and that’s all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we’ve guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn’t correct.

All we’ve done is make a guess. Now we need to prove that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don’t always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let’s continue on with the problem.

Hint : Now that we’ve guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (i.e. not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3
Okay, now that we know we need to find a larger function that we know converges.
So, let's start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making the denominator smaller. Also note that for \( x > 1 \) (which we can assume from the limits on the integral) we have,

\[
x^3 + 1 > x^3
\]

Therefore, we have,

\[
\frac{1}{x^3 + 1} < \frac{1}{x^3}
\]

since we replaced the denominator with something that we know is smaller.

Step 4
Finally, we know that,

\[
\int_{1}^{\infty} \frac{1}{x^3} \, dx
\]

converges. Then because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

\[
\int_{1}^{\infty} \frac{1}{x^3 + 1} \, dx
\]

must also converge.

2. Use the Comparison Test to determine if the following integral converges or diverges.

\[
\int_{3}^{\infty} \frac{z^2}{z^3 - 1} \, dz
\]

Hint: Start off with a guess. Do you think this will converge or diverge?

Step 1
The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The “-1” in the denominator does not really change the size of the denominator as \( z \) gets really large and so hopefully it makes sense that we can guess that this integral should behave like,

\[
\int_{3}^{\infty} \frac{z^2}{z^3} \, dz = \int_{3}^{\infty} \frac{1}{z} \, dz
\]

Then, by the fact from the previous section, we know that this integral diverges since \( p = 1 \leq 1 \).
Therefore, we can guess that the integral,

\[ \int_{3}^{\infty} \frac{z^2}{z^3 - 1} \, dz \]

will diverge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we’ve guessed the integral diverges we now know that it diverges and that’s all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we’ve guessed that the integral diverges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn’t correct.

All we’ve done is make a guess. Now we need to prove that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don’t always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let’s continue on with the problem.

Hint: Now that we’ve guessed the integral diverges do we want a larger or smaller function that we know diverges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral diverges so if we find a smaller function that we know diverges the area analogy tells us that there would be an infinite amount of area under the smaller function.

Our function, which would be larger, would then also have an infinite amount of area under it. There is no way we can have a finite amount of area covering an infinite amount of area!

Note that the opposite situation does us no good. If we find a larger function that we know diverges (and hence will have a infinite amount of area under it) our function (which is now smaller) can have either a finite amount of area or an infinite area under it.

In other words, if we find a larger function that we know diverges this will tell us nothing about our function. However, if we find a smaller function that we know diverges this will force our function to also diverge.
Therefore we need to find a smaller function that we know diverges.

Step 3
Okay, now that we know we need to find a smaller function that we know diverges.

So, let’s start with the function from the integral. It is a fraction and we know that we can make a fraction smaller by making the denominator larger. Also note that for \( z > 3 \) (which we can assume from the limits on the integral) we have,

\[
z^3 - 1 < z^3
\]

Therefore, we have,

\[
\frac{z^2}{z^3 - 1} > \frac{z^2}{z^3} = \frac{1}{z}
\]

since we replaced the denominator with something that we know is larger.

Step 4
Finally, we know that,

\[
\int_3^\infty \frac{1}{z} \, dz
\]

diverges. Then because the function in this integral is smaller than the function in the original integral the Comparison Test tells us that,

\[
\int_3^\infty \frac{z^2}{z^3 - 1} \, dz
\]

must also diverge.

3. Use the Comparison Test to determine if the following integral converges or diverges.

\[
\int_4^\infty \frac{e^{-y}}{y} \, dy
\]

Hint : Start off with a guess. Do you think this will converge or diverge?

Step 1
The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

We need to be a little careful with the guess for this problem. We might be tempted to use the fact from the previous section to guess diverge since the exponent on the \( y \) in the denominator is \( p = 1 \leq 1 \).
That would be incorrect however. Recall that the fact requires a constant in the numerator and we clearly do not have that in this case. In fact what we have in the numerator is $e^{-y}$ and this goes to zero very fast as $y \to \infty$ and so there is a pretty good chance that this integral will in fact converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we’ve guessed the integral converges we now know that it converges and that’s all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we’ve guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn’t correct.

All we’ve done is make a guess. Now we need to prove that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don’t always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let’s continue on with the problem.

Hint : Now that we’ve guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (i.e. not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.
Step 3
Okay, now that we know we need to find a larger function that we know converges.

So, let’s start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making the denominator smaller. From the limits on the integral we can see that,

\[ y > 4 \]

Therefore, we have,

\[ \frac{e^{-y}}{y} < \frac{e^{-y}}{4} \]

since we replaced the denominator with something that we know is smaller.

Step 4
Finally, we will need to prove that,

\[ \int_{4}^{\infty} \frac{e^{-y}}{y} \, dy \]

converges. However, after the previous section that shouldn’t be too difficult. Here is that work.

\[
\lim_{t \to \infty} \left. \frac{e^{-y}}{y} \right|_{4}^{t} = \lim_{t \to \infty} \left( -\frac{1}{4} e^{-t} + \frac{1}{4} e^{-4} \right) = \frac{1}{4} e^{-4}
\]

The limit existed and was finite and so we know that,

\[ \int_{4}^{\infty} \frac{1}{4} e^{-y} \, dy \]

converges.

Therefore, because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

\[ \int_{4}^{\infty} \frac{e^{-y}}{y} \, dy \]

must also converge.

4. Use the Comparison Test to determine if the following integral converges or diverges.

\[ \int_{1}^{\infty} \frac{z - 1}{z^4 + 2z^2} \, dz \]
Hint: Start off with a guess. Do you think this will converge or diverge?

Step 1
The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

Both the numerator and denominator of this function are polynomials and we know that as $z \to \infty$ the behavior of each of the polynomials will be the same as the behavior of the largest power of $z$. Therefore, it looks like this integral should behave like,

$$\int_1^\infty \frac{z}{z^4} \, dz = \int_1^\infty \frac{1}{z^3} \, dz$$

Then, by the fact from the previous section, we know that this integral converges since $p = 3 > 1$.

Therefore, we can guess that the integral,

$$\int_1^\infty \frac{z-1}{z^4 + 2z^2} \, dz$$

will converge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we’ve guessed the integral converges we now know that it converges and that’s all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we’ve guessed that the integral converges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn’t correct.

All we’ve done is make a guess. Now we need to prove that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don’t always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let’s continue on with the problem.

Hint: Now that we’ve guessed the integral converges do we want a larger or smaller function that we know converges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.
We want to prove that the integral converges so if we find a larger function that we know converges the area analogy tells us that there would be a finite (i.e. not infinite) amount of area under the larger function.

Our function, which would be smaller, would then also have a finite amount of area under it. There is no way we can have an infinite amount of area inside of a finite amount of area!

Note that the opposite situation does us no good. If we find a smaller function that we know converges (and hence will have a finite amount of area under it) our function (which is now larger) can have either a larger finite amount of area or an infinite area under it.

In other words, if we find a smaller function that we know converges this will tell us nothing about our function. However, if we find a larger function that we know converges this will force our function to also converge.

Therefore we need to find a larger function that we know converges.

Step 3
Okay, now that we know we need to find a larger function that we know converges.

So, let’s start with the function from the integral. It is a fraction and we know that we can make a fraction larger by making numerator larger or the denominator smaller.

Note that for $z > 1$ (which we can assume from the limits on the integral) we have,

$$z - 1 < z$$

Therefore, we have,

$$\frac{z - 1}{z^4 + 2z^2} < \frac{z}{z^4 + 2z^2} = \frac{1}{z^3 + 2z}$$

since we replaced the numerator with something that we know is larger.

Step 4
It is at this point that students again often make mistakes with this kind of problem. After doing one manipulation of the numerator or denominator they stop the manipulation and declare that the new function must converge (since that is what we want after all) and move on to the next problem.

Recall however that we must know that the new function converges and we’ve not gotten to a function yet that we know converges. To get to a function that we know converges we need to do one more manipulation of the function.

Again, note that for $z > 1$ we have,

$$z^3 + 2z > z^3$$

Therefore, we have,
Calculus II

\[ \frac{1}{z^3 + 2z} < \frac{1}{z^3} \]

since we replaced the denominator with something that we know is smaller.

Step 5
Finally, putting the results of Steps 3 & 4 together we have,

\[ \frac{z - 1}{z^4 + 2z^2} < \frac{1}{z^3} \]

and we know that,

\[ \int_1^\infty \frac{1}{z^3} \, dz \]

converges. Then because the function in this integral is larger than the function in the original integral the Comparison Test tells us that,

\[ \int_1^\infty \frac{z - 1}{z^4 + 2z^2} \, dz \]

must also converge.

5. Use the Comparison Test to determine if the following integral converges or diverges.

\[ \int_6^\infty \frac{w^2 + 1}{w^3 \left( \cos^3(w) + 1 \right)} \, dw \]

Hint : Start off with a guess. Do you think this will converge or diverge?

Step 1
The first thing that we really need to do here is to take a guess on whether we think the integral converges or diverges.

The numerator of this function is a polynomial and we know that as \( w \to \infty \) the behavior of polynomials will be the same as the behavior of the largest power of \( w \). Also the cosine term in the denominator is bounded and never gets too large or small.

Therefore, it looks like this integral should behave like,

\[ \int_6^\infty \frac{w^2}{w^3} \, dw = \int_6^\infty \frac{1}{w} \, dw \]

Then, by the fact from the previous section, we know that this integral diverges since \( p = 1 \leq 1 \).
Therefore, we can guess that the integral,

\[ \int_{6}^{\infty} \frac{w^2 + 1}{w^3(\cos^2(w) + 1)} \, dw \]

will diverge.

Be careful from this point on! One of the biggest mistakes that many students make at this point is to say that because we’ve guessed the integral diverges we now know that it diverges and that’s all that we need to do and they move on to the next problem.

Another mistake that students often make here is to say that because we’ve guessed that the integral diverges they make sure that the remainder of the work in the problem supports that guess even if the work they do isn’t correct.

All we’ve done is make a guess. Now we need to prove that our guess was the correct one. This may seem like a silly thing to go on about, but keep in mind that at this level the problems you are working with tend to be pretty simple (even if they don’t always seem like it). This means that it will often (or at least often once you get comfortable with these kinds of problems) be pretty clear that the integral converges or diverges.

When these kinds of problems arise in other sections/applications it may not always be so clear if our guess is correct or not and it can take some real work to prove the guess. So, we need to be in the habit of actually doing the work to prove the guess so we are capable of doing it when it is required.

The hard part with these problems is often not making the guess but instead proving the guess! So let’s continue on with the problem.

Hint : Now that we’ve guessed the integral diverges do we want a larger or smaller function that we know diverges?

Step 2
Recall that we used an area analogy in the notes of this section to help us determine if we want a larger or smaller function for the comparison test.

We want to prove that the integral diverges so if we find a smaller function that we know diverges the area analogy tells us that there would be an infinite amount of area under the smaller function.

Our function, which would be larger, would then also have an infinite amount of area under it. There is no way we can have an finite amount of area covering an infinite amount of area!

Note that the opposite situation does us no good. If we find a larger function that we know diverges (and hence will have a infinite amount of area under it) our function (which is now smaller) can have either a finite amount of area or an infinite area under it.

In other words, if we find a larger function that we know diverges this will tell us nothing about our function. However, if we find a smaller function that we know diverges this will force our function to also diverge.
Therefore we need to find a smaller function that we know diverges.

Step 3
Okay, now that we know we need to find a smaller function that we know diverges.

So, let’s start with the function from the integral. It is a fraction and we know that we can make a fraction smaller by making the numerator smaller or the denominator larger. Also note that for \( w > 6 \) (which we can assume from the limits on the integral) we have,

\[
\cos^2 w + 1 > w^2
\]

Therefore, we have,

\[
\frac{w^2 + 1}{w^3 \left( \cos^2 w + 1 \right)} > \frac{w^2}{w^3 \left( \cos^2 w + 1 \right)} = \frac{1}{w \left( \cos^2 w + 1 \right)}
\]

since we replaced the numerator with something that we know is smaller.

Step 4
It is at this point that students again often make mistakes with this kind of problem. After doing one manipulation of the numerator or denominator they stop the manipulation and declare that the new function must diverge (since that is what we want after all) and move on to the next problem.

Recall however that we must know that the new function diverges and we’ve not gotten to a function yet that we know diverges. To get to a function that we know diverges we need to do one more manipulation of the function.

For this step we know that \( 0 \leq \cos^2 w \leq 1 \) and so we will have,

\[
\cos^2 w + 1 < 1 + 1 = 2
\]

Therefore, we have,

\[
\frac{1}{w \left( \cos^2 w + 1 \right)} > \frac{1}{w(2)} = \frac{1}{2w}
\]

since we replaced the denominator with something that we know is larger.

Step 5
Finally, putting the results of Steps 3 & 4 together we have,

\[
\frac{w^2 + 1}{w^3 \left( \cos^2 w + 1 \right)} > \frac{1}{2w}
\]

and we know that,
\[ \int_{6}^{\infty} \frac{1}{2w} \, dw = \frac{1}{2} \int_{6}^{\infty} \frac{1}{w} \, dw \]

diverges. Then because the function in this integral is smaller than the function in the original integral the Comparison Test tells us that,

\[ \int_{6}^{\infty} \frac{w^2 + 1}{w^3 \left( \cos^2 (w) + 1 \right)} \, dw \]

must also diverge.

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**Approximating Definite Integrals**

1. Using \( n = 6 \) approximate the value of \( \int_{1}^{7} \frac{1}{x^3 + 1} \, dx \) using

   \( \text{(a) the Midpoint Rule,} \)

   \( \text{(b) the Trapezoid Rule, and} \)

   \( \text{(c) Simpson’s Rule} \)

Use at least 6 decimal places of accuracy for your work.

\( \text{(a) Midpoint Rule} \)

While it’s not really needed to do the problem here is a sketch of the graph.

![Graph of y vs x](image)

We know that we need to divide the interval \([1, 7]\) into 6 subintervals each with width,
\[ \Delta x = \frac{7-1}{6} = 1 \]

The endpoints of each of these subintervals are represented by the dots on the \( x \) axis on the graph above.

The tick marks between each dot represents the midpoint of each of the subintervals. The \( x \)-values of the midpoints for each of the subintervals are then,

\[ \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \frac{11}{2}, \frac{13}{2} \]

So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

\[
\int_1^7 \frac{1}{x^3+1} \, dx \approx \left(1\right) \left[ f\left(\frac{3}{2}\right) + f\left(\frac{5}{2}\right) + f\left(\frac{7}{2}\right) + f\left(\frac{9}{2}\right) + f\left(\frac{11}{2}\right) + f\left(\frac{13}{2}\right) \right] \\
= 0.33197137
\]

(b) Trapezoid Rule

From the Midpoint Rule work we know that the width of each subinterval is \( \Delta x = 1 \) and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.

So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

\[
\int_1^7 \frac{1}{x^3+1} \, dx \approx \left(\frac{1}{2}\right) \left[ f\left(1\right) + 2f\left(2\right) + 2f\left(3\right) + 2f\left(4\right) + 2f\left(5\right) + 2f\left(6\right) + f\left(7\right) \right] \\
= 0.42620830
\]

(b) Simpson’s Rule
From the Midpoint Rule work we know that the width of each subinterval is \( \Delta x = 1 \) and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.

As with the first two parts all we need to do is plug into the formula to use Simpson’s Rule to approximate value of the integral. Doing this gives,

\[
\int_{1}^{7} \frac{1}{x^3+1} \, dx \approx \left( \frac{1}{3} \right) \left[ f(1) + 4f(2) + 2f(3) + 4f(4) + 2f(5) + 4f(6) + f(7) \right]
\]

\[
= 0.37154155
\]

2. Using \( n = 6 \) approximate the value of \( \int_{-1}^{2} \sqrt{e^{-x^2} + 1} \, dx \) using
   (a) the Midpoint Rule,
   (b) the Trapezoid Rule, and
   (c) Simpson’s Rule

Use at least 6 decimal places of accuracy for your work.

(a) Midpoint Rule
While it’s not really needed to do the problem here is a sketch of the graph.
We know that we need to divide the interval \([-1, 2]\) into 6 subintervals each with width,

\[ \Delta x = \frac{2 - (-1)}{6} = \frac{1}{2} \]

The endpoints of each of these subintervals are represented by the dots on the \(x\) axis on the graph above.

The tick marks between each dot represents the midpoint of each of the subintervals. The \(x\)-values of the midpoints for each of the subintervals are then,

\[-\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}\]

So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

\[
\int_{-1}^{2} \sqrt{e^{-x^2} + 1} \, dx \approx \left( \frac{1}{2} \right) \left[ f \left( -\frac{3}{4} \right) + f \left( -\frac{1}{4} \right) + f \left( \frac{1}{4} \right) + f \left( \frac{3}{4} \right) + f \left( \frac{5}{4} \right) + f \left( \frac{7}{4} \right) \right] \\
= 3.70700857
\]

**b) Trapezoid Rule**

From the Midpoint Rule work we know that the width of each subinterval is \( \Delta x = \frac{1}{2} \) and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.
So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

\[
\int_{-1}^{2} \sqrt{x^2 + 1} \, dx \approx \left[ f(-1) + 2f\left(-\frac{1}{2}\right) + 2f\left(0\right) + 2f\left(\frac{1}{2}\right) + 2f\left(1\right) + 2f\left(\frac{3}{2}\right) + f\left(2\right) \right]
\]

\[
= 3.69596543
\]

(b) Simpson’s Rule
From the Midpoint Rule work we know that the width of each subinterval is \( \Delta x = \frac{1}{2} \) and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.

As with the first two parts all we need to do is plug into the formula to use Simpson’s Rule to approximate value of the integral. Doing this gives,
\[
\int_{-1}^{2} \sqrt{e^{-x^2} + 1} \, dx \approx \left( \frac{\sqrt{2}}{3} \right) \left[ f \left( -1 \right) + 4 f \left( -\frac{1}{2} \right) + 2 f \left( 0 \right) + 4 f \left( \frac{1}{2} \right) + 2 f \left( 1 \right) + 4 f \left( \frac{3}{2} \right) + f \left( 2 \right) \right] \\
= 3.70358145
\]

3. Using \( n = 8 \) approximate the value of \( \int_{0}^{4} \cos \left( 1 + \sqrt{x} \right) \, dx \) using
   (a) the Midpoint Rule,
   (b) the Trapezoid Rule, and
   (c) Simpson’s Rule

Use at least 6 decimal places of accuracy for your work.

(a) Midpoint Rule
While it’s not really needed to do the problem here is a sketch of the graph.

\[
\Delta x = \frac{4 - 0}{8} = \frac{1}{2}
\]

The endpoints of each of these subintervals are represented by the dots on the \( x \) axis on the graph above.

The tick marks between each dot represents the midpoint of each of the subintervals. The \( x \)-values of the midpoints for each of the subintervals are then,

\[
\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \frac{9}{4}, \frac{11}{4}, \frac{13}{4}, \frac{15}{4}
\]

So, to use the Midpoint Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,
\[
\int_0^4 \cos(1 + \sqrt{x}) \, dx \approx \frac{1}{2} \left[ f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) \right]
\]

\[= -2.51625938 \]

(b) Trapezoid Rule

From the Midpoint Rule work we know that the width of each subinterval is \( \Delta x = \frac{1}{2} \) and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.

So, to use the Trapezoid Rule to approximate the value of the integral all we need to do is plug into the formula. Doing this gives,

\[
\int_0^4 \cos(1 + \sqrt{x}) \, dx \approx \frac{\Delta x}{2} \left[ f(0) + 2 f\left(\frac{1}{2}\right) + 2 f\left(\frac{1}{2}\right) + 2 f\left(\frac{3}{2}\right) + 2 f\left(2\right) + 2 f\left(\frac{5}{2}\right) + 2 f\left(3\right) + 2 f\left(\frac{7}{2}\right) + f(4) \right]
\]

\[= -2.43000475\]

(b) Simpson’s Rule

From the Midpoint Rule work we know that the width of each subinterval is \( \Delta x = \frac{1}{2} \) and for reference purposes the sketch of the graph along with the endpoints of each subinterval marked by the dots is shown below.
As with the first two parts all we need to do is plug into the formula to use Simpson’s Rule to approximate value of the integral. Doing this gives,

\[
\int_0^4 \cos(1 + \sqrt{x}) \, dx \approx \left( \frac{3}{3} \right) \left[ f(0) + 4f\left(1\right) + 2f\left(\frac{3}{2}\right) + 4f\left(2\right) + 4f\left(\frac{5}{2}\right) + 
2f(3) + 4f\left(\frac{7}{2}\right) + f(4) \right]
\]

= \boxed{-2.47160136}