CALCULUS III
Solutions to Practice Problems
Applications of Partial Derivatives

Paul Dawkins
# Table of Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Preface</strong></td>
<td>ii</td>
</tr>
<tr>
<td><strong>Applications of Partial Derivatives</strong></td>
<td>3</td>
</tr>
<tr>
<td>Tangent Planes and Linear Approximations</td>
<td>3</td>
</tr>
<tr>
<td>Gradient Vector, Tangent Planes and Normal Lines</td>
<td>4</td>
</tr>
<tr>
<td>Relative Minimums and Maximums</td>
<td>6</td>
</tr>
<tr>
<td>Absolute Minimums and Maximums</td>
<td>12</td>
</tr>
<tr>
<td>Lagrange Multipliers</td>
<td>20</td>
</tr>
</tbody>
</table>
Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
Applications of Partial Derivatives

Tangent Planes and Linear Approximations

1. Find the equation of the tangent plane to \( z = x^2 \cos(\pi y) - \frac{6}{xy^2} \) at \((2, -1)\).

Step 1
First we know we’ll need the two 1st order partial derivatives. Here they are,

\[
\begin{align*}
  f_x &= 2x \cos(\pi y) + \frac{6}{x^2 y^2} \\
  f_y &= -\pi x^2 \sin(\pi y) + \frac{12}{xy^3}
\end{align*}
\]

Step 2
Now we also need the two derivatives from the first step and the function evaluated at \((2, -1)\). Here are those evaluations,

\[
\begin{align*}
  f(2, -1) &= -7 \\
  f_x(2, -1) &= -\frac{5}{2} \\
  f_y(2, -1) &= -6
\end{align*}
\]

Step 3
The tangent plane is then,

\[
z = -7 - \frac{5}{2} (x - 2) - 6(y + 1) = -\frac{5}{2} x - 6y - 8
\]

2. Find the equation of the tangent plane to \( z = x\sqrt{x^2 + y^2} + y^3 \) at \((-4, 3)\).

Step 1
First we know we’ll need the two 1st order partial derivatives. Here they are,

\[
\begin{align*}
  f_x &= \sqrt{x^2 + y^2} + \frac{x^2}{\sqrt{x^2 + y^2}} \\
  f_y &= \frac{xy}{\sqrt{x^2 + y^2}} + 3y^2
\end{align*}
\]

Step 2
Now we also need the two derivatives from the first step and the function evaluated at \((-4, 3)\). Here are those evaluations,
Step 3
The tangent plane is then,

\[ z = 7 + \frac{41}{5}(x + 4) + \frac{123}{5}(y - 3) = \frac{41}{5}x + \frac{123}{5}y - 34 \]

3. Find the linear approximation to \( z = 4x^2 - ye^{2x+y} \) at \((-2, 4)\).

Step 1
Recall that the linear approximation to a function at a point is really nothing more than the tangent plane to that function at the point.

So, we know that we’ll first need the two 1st order partial derivatives. Here they are,

\[ f_x = 8x - 2ye^{2x+y} \quad f_y = -e^{2x+y} - ye^{2x+y} \]

Step 2
Now we also need the two derivatives from the first step and the function evaluated at \((-2, 4)\). Here are those evaluations,

\[ f(-2, 4) = 12 \quad f_x(-2, 4) = -24 \quad f_y(-2, 4) = -5 \]

Step 3
The linear approximation is then,

\[ L(x, y) = 12 - 24(x + 2) - 5(y - 4) = -24x - 5y - 16 \]

**Gradient Vector, Tangent Planes and Normal Lines**

1. Find the tangent plane and normal line to \( x^2y = 4ze^{x+y} - 35 \) at \((3, -3, 2)\).

Step 1
First we need to do a quick rewrite of the equation as,

\[ x^2y - 4ze^{x+y} = -35 \]
Step 2
Now we need the gradient of the function on the left side of the equation from Step 1 and its value at 
\((3, -3, 2)\). Here are those quantities.

\[
\nabla f = \left\langle 2xy - 4ze^{xy}, x^2 - 4ze^{xy}, -4e^{xy} \right\rangle \quad \nabla f (3, -3, 2) = \left\langle -26, 1, -4 \right\rangle
\]

Step 3
The tangent plane is then,

\[
-26(x - 3) + (1)(y + 3) - 4(z - 2) = 0 \quad \Rightarrow \quad -26x + y - 4z = -89
\]

The normal line is,

\[
\vec{r}(t) = (3, -3, 2) + t(-26, 1, -4) = (3 - 26t, -3 + t, 2 - 4t)
\]

2. Find the tangent plane and normal line to \(\ln \left( \frac{x}{2y} \right) = z^2(x - 2y) + 3z + 3\) at \((4, 2, -1)\).

Step 1
First we need to do a quick rewrite of the equation as,

\[
\ln \left( \frac{x}{2y} \right) - z^2(x - 2y) - 3z = 3
\]

Step 2
Now we need the gradient of the function on the left side of the equation from Step 1 and its value at 
\((4, 2, -1)\). Here are those quantities.

\[
\nabla f = \left\langle \frac{1}{x} - z^2, -\frac{1}{y} + 2z^2, -2z(x - 2y) - 3 \right\rangle \quad \nabla f (4, 2, -1) = \left\langle -\frac{3}{4}, \frac{3}{2}, -3 \right\rangle
\]

Step 3
The tangent plane is then,

\[
-\frac{1}{4}(x - 4) + \frac{3}{2}(y - 2) - 3(z + 1) = 0 \quad \Rightarrow \quad -\frac{1}{4}x + \frac{3}{2}y - 3z = 3
\]

The normal line is,

\[
\vec{r}(t) = (4, 2, -1) + t\left(-\frac{1}{4}, \frac{3}{2}, -3\right) = \left(4 - \frac{1}{4}t, 2 + \frac{3}{2}t, -1 - 3t\right)
\]
Relative Minimums and Maximums

1. Find and classify all the critical points of the following function.

\[ f(x, y) = (y - 2)x^2 - y^2 \]

Step 1
We’re going to need a bunch of derivatives for this problem so let’s get those taken care of first.

\[
\begin{align*}
  f_x &= 2(y - 2)x \\
  f_y &= x^2 - 2y \\
  f_{xx} &= 2(y - 2) \\
  f_{xy} &= 2x \\
  f_{yy} &= -2
\end{align*}
\]

Step 2
Now, let’s find the critical points for this problem. That means solving the following system.

\[
\begin{align*}
  f_x &= 0 : 2(y - 2)x = 0 & \Rightarrow & y = 2 \text{ or } x = 0 \\
  f_y &= 0 : x^2 - 2y = 0
\end{align*}
\]

As shown above we have two possible options from the first equation. We can plug each into the second equation to get the critical points for the equation.

\[
\begin{align*}
  y &= 2 : x^2 - 4 = 0 & \Rightarrow & x = \pm 2 & \Rightarrow & (2, 2) \text{ and } (-2, 2) \\
  x &= 0 : -2y = 0 & \Rightarrow & y = 0 & \Rightarrow & (0, 0)
\end{align*}
\]

Be careful in writing down the solution to this system of equations. One of the biggest mistakes students make here is to just write down all possible combinations of x and y values they get. That is not how these types of systems are solved!

We got \( x = \pm 2 \) above only because we assumed first that \( y = 2 \) and so that leads to the two solutions listed in that first line above. Likewise we only got \( y = 0 \) because we first assumed that \( x = 0 \) which leads to the third solution listed above in the second line. The points \((0, 2), (-2, 0)\) and \((2, 0)\) are NOT solutions to this system as can be easily checked by plugging them into the second equation in the system.

So, do not just “mix and match” all possible values of x and y into points and call them all solutions. This will often lead to points that are not solutions to the system of equations. You need to always keep in
mind what assumptions you had to make in order to get certain \( x \) or \( y \) values in the solution process and only match those values up with the assumption you had to make.

So, in summary, this function has three critical points: \((0,0)\), \((-2,2)\), \((2,2)\).

Also, before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3
Next, we’ll need the following,

\[
D(x, y) = f_{xx}f_{yy} - [f_{xy}]^2 = [2(y - 2)][-2] - [2x]^2 = -4(y - 2) - 4x^2
\]

Step 4
With \(D(x, y)\) we can now classify each of the critical points as follows.

\[
\begin{align*}
(0,0) \quad & : \quad D(0,0) = 8 > 0 \quad \Rightarrow f_{xx}(0,0) = -4 < 0 \quad \text{Relative Maximum} \\
(-2,2) \quad & : \quad D(-2,2) = -16 < 0 \quad \text{Saddle Point} \\
(2,2) \quad & : \quad D(2,2) = -16 < 0 \quad \text{Saddle Point}
\end{align*}
\]

Don’t forget to check the value of \(f_{xx}\) when \(D\) is positive so we can get the correct classification (i.e. maximum or minimum) and also recall that for negative \(D\) we don’t need the second check as we know the critical point will be a saddle point.

2. Find and classify all the critical points of the following function.

\[
f(x, y) = 7x - 8y + 2xy - x^2 + y^3
\]

Step 1
We’re going to need a bunch of derivatives for this problem so let’s get those taken care of first.

\[
\begin{align*}
f_x & = 7 + 2y - 2x \\
f_{xx} & = -2 \\
f_y & = -8 + 2x + 3y^2 \\
f_{xy} & = 2 \\
f_{yy} & = 6y
\end{align*}
\]

Step 2
Now, let’s find the critical points for this problem. That means solving the following system.

\[
\begin{align*}
f_x & = 0 \quad \Rightarrow \quad 7 + 2y - 2x = 0 \\
f_y & = 0 \quad \Rightarrow \quad -8 + 2x + 3y^2 = 0 \quad \Rightarrow \quad x = 4 - \frac{3}{2}y^2
\end{align*}
\]
As shown above we solved the second equation for \( x \) and we can now plug this into the first equation as follows,

\[
0 = 7 + 2y - 2\left(4 - \frac{1}{2}y^2\right) = 3y^2 + 2y - 1 = (3y - 1)(y + 1) \quad \rightarrow \quad y = -1, \quad y = \frac{1}{3}
\]

This gives two values of \( y \) which we can now plug back into either of our equations to find corresponding \( x \) values. Here is that work.

\[
y = -1 : \quad x = 4 - \frac{3}{2}(-1)^2 = \frac{5}{2} \quad \Rightarrow \quad \left(\frac{5}{2}, -1\right)
\]

\[
y = \frac{1}{3} : \quad x = 4 - \frac{3}{2}\left(\frac{1}{3}\right)^2 = \frac{23}{9} \quad \Rightarrow \quad \left(\frac{23}{9}, \frac{1}{3}\right)
\]

Be careful in writing down the solution to this system of equations. One of the biggest mistakes students make here is to just write down all possible combinations of \( x \) and \( y \) values they get. That is not how these types of systems are solved!

We got \( x = \frac{5}{2} \) above only because we assumed first that \( y = -1 \) and so that leads to the solution listed in first line above. Likewise we only got \( x = \frac{23}{9} \) because we first assumed that \( y = \frac{1}{3} \) which leads to the second solution listed in the second line above. The points \( \left(\frac{5}{2}, -1\right) \) and \( \left(\frac{23}{9}, -1\right) \) are NOT a solutions to this system as can be easily checked by plugging these points into the either of the equations in the system.

So, do not just “mix and match” all possible values of \( x \) and \( y \) into points and call them all solutions. This will often lead to points that are not solutions to the system of equations. You need to always keep in mind what assumptions you had to make in order to get certain \( x \) or \( y \) values in the solution process and only match those values up with the assumption you had to make.

So, in summary, this function has two critical points: \( \left(\frac{5}{2}, -1\right), \left(\frac{23}{9}, \frac{1}{3}\right) \).

Before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3

Next, we’ll need the following,

\[
D(x, y) = f_{xx}f_{yy} - \left[f_{xy}\right]^2 = -2\left[6y\right] - \left[2\right]^2 = -12y - 4
\]

Step 4

With \( D(x, y) \) we can now classify each of the critical points as follows.

\[
\begin{array}{c|c|c|c}
\text{(Point)} & D(x, y) & f_{xx}(x, y) & \text{Type} \\
\hline
\left(\frac{5}{2}, -1\right) & 8 > 0 & -2 < 0 & \text{Relative Maximum} \\
\left(\frac{23}{9}, \frac{1}{3}\right) & -8 < 0 & & \text{Saddle Point}
\end{array}
\]
Don’t forget to check the value of $f_{xx}$ when $D$ is positive so we can get the correct classification (i.e. maximum or minimum) and also recall that for negative $D$ we don’t need the second check as we know the critical point will be a saddle point.

3. Find and classify all the critical points of the following function.

$$f(x,y) = (3x + 4x^3)(y^2 + 2y)$$

Step 1

We’re going to need a bunch of derivatives for this problem so let’s get those taken care of first.

Do not make these derivatives harder than really are! Do not multiply the function out! We just have a function of $x$’s times a function of $y$’s. Take advantage of that when doing the derivatives.

$$f_x = (3 + 12x^2)(y^2 + 2y) \quad f_y = (3x + 4x^3)(2y + 2)$$

$$f_{xx} = 24x(y^2 + 2y) \quad f_{xy} = (3 + 12x^2)(2y + 2) \quad f_{yy} = 2(3x + 4x^3)$$

Step 2

Now, let’s find the critical points for this problem. That means solving the following system.

$$f_x = 0 : (3 + 12x^2)(y^2 + 2y) = 0$$

$$f_y = 0 : (3x + 4x^3)(2y + 2) = 0$$

We could start the solution process with either of these equations as both are pretty simple to solve. Let’s start with the first equation.

$$\left(3 + 12x^2\right)(y^2 + 2y) = \left(3 + 12x^2\right)(y)(y + 2) = 0 \quad \rightarrow \quad y = 0, y = -2, x = \pm \frac{1}{2}i$$

Okay, we’ve got something to deal with at this point. We clearly get four different values to work with here. Two of them, however, are complex. One of the unspoken rules here is that we are only going to work with real values and so we will ignore any complex answers and work with only the real values.

So, we now have two possible values of $y$ so let’s plug each of them into the second equation as follows,

$$y = 0 : 2\left(3x + 4x^3\right) = 2x \left(3 + 4x^2\right) = 0 \quad \rightarrow \quad x = 0, x = \pm \frac{\sqrt{2}}{2}i \quad \Rightarrow \quad (0, 0)$$

$$y = -2 : -2\left(3x + 4x^3\right) = 2x \left(3 + 4x^2\right) = 0 \quad \rightarrow \quad x = 0, x = \pm \frac{\sqrt{2}}{2}i \quad \Rightarrow \quad (0, -2)$$

As with the first part of the solution process we only take the real values and so ignore the complex portions from this part as well.
In the previous two problems we made mention at this point to be careful and not just from up points for all possible combinations of the $x$ and $y$ values we have at this point.

One of the reasons that students often do that is because of problems like this one where it appears that we are doing just that. However, we haven’t just randomly formed all combinations here. It just so happened that when we assumed $y = 0$ and $y = -2$ that we just happened to get the same value of $x$, $x = 0$. In general, this won’t happen and so do not read into this problem that we always just form all possible combinations of the $x$ and $y$ values to get the critical points for a function. We must always pay attention to the assumptions made at the start of each step.

So, in summary, this function has two critical points: $(0, -2), (0, 0)$.

Before proceeding with the next step we should note that there are multiple ways to solve this system. The process you used may not be the same as the one we used here. However, regardless of the process used to solve the system, the solutions should always be the same.

Step 3
Next, we’ll need the following,

$$D(x, y) = f_{xx}f_{yy} - [f_{xy}]^2$$

$$= [24x(y^2 + 2y)][2(3x + 4x^3)] - [(3+12x^2)(2y+2)]^2$$

$$= 48x(3x + 4x^3)(y^2 + 2y) - (3+12x^2)(2y+2)^2$$

Step 4
With $D(x, y)$ we can now classify each of the critical points as follows.

$$(0, -2) : D(0, -2) = -36 < 0 \quad \text{Saddle Point}$$

$$(0, 0) : D(0, 0) = -36 < 0 \quad \text{Saddle Point}$$

Don’t always expect every problem to have at least one relative extrema. As this example has shown it is completely possible to have only saddle points.

4. Find and classify all the critical points of the following function.

$$f(x, y) = 3y^3 - x^2y^2 + 8y^2 + 4x^2 - 20y$$

Step 1
We’re going to need a bunch of derivatives for this problem so let’s get those taken care of first.

$$f_x = -2xy^2 + 8x \quad f_y = 9y^2 - 2x^2y + 16y - 20$$

$$f_{xx} = -2y^2 + 8 \quad f_{xy} = -4xy \quad f_{yy} = 18y - 2x^2 + 16$$
Step 2  
Now, let’s find the critical points for this problem. That means solving the following system.

\[ f_x = 0 : \quad -2xy^2 + 8x = 2x(4 - y^2) = 0 \quad \Rightarrow \quad y = \pm 2 \quad \text{or} \quad x = 0 \]
\[ f_y = 0 : \quad 9y^2 - 2x^2y + 16y - 20 = 0 \]

As shown above we have three possible options from the first equation. We can plug each into the second equation to get the critical points for the equation.

\[ y = -2 : \quad 4x^2 - 16 = 0 \quad \Rightarrow \quad x = \pm 2 \quad \Rightarrow \quad (2, -2) \quad \text{and} \quad (-2, -2) \]
\[ y = 2 : \quad -4x^2 + 48 = 0 \quad \Rightarrow \quad x = \pm 2\sqrt{3} \quad \Rightarrow \quad (2\sqrt{3}, 2) \quad \text{and} \quad (-2\sqrt{3}, 2) \]
\[ x = 0 : \quad 9y^2 + 16y - 20 = 0 \quad \Rightarrow \quad y = \frac{-16\pm\sqrt{76}}{18} \quad \Rightarrow \quad \left(0, \frac{-16\pm\sqrt{76}}{18}\right) \quad \text{and} \quad \left(0, \frac{-16\pm\sqrt{76}}{18}\right) \]

As we noted in the first two problems in this section be careful to only write down the actual solutions as found in the above work. Do not just write down all possible combinations of \(x\) and \(y\) from each of the three lines above. If you do that for this problem you will end up with a large number of points that are not critical points.

Also, do not get excited about the “mess” (i.e. roots) involved in some of the critical points. They will be a fact of life with these problems on occasion.

So, in summary, this function has the following six critical points.

\((-2, -2), (2, -2), (2\sqrt{3}, 2), (-2\sqrt{3}, 2), \left(0, \frac{-16\pm\sqrt{76}}{18}\right), \left(0, \frac{-16\pm\sqrt{76}}{18}\right)\).

Step 3  
Next, we’ll need the following,

\[ D(x, y) = f_{xx}f_{yy} - \left[f_{xy}\right]^2 \]
\[ = \left[-2y^2 + 8\right]\left[18y - 2x^2 + 16\right] - \left[-4xy\right]^2 \]
\[ = \left[-2y^2 + 8\right]\left[18y - 2x^2 + 16\right] - 16x^2y^2 \]

Step 4  
With \(D(x, y)\) we can now classify each of the critical points as follows.
\((-2,-2)\) : \(D(-2,-2) = -256 < 0\) \hspace{1cm} \text{Saddle Point} \\
\((2,-2)\) : \(D(2,-2) = -256 < 0\) \hspace{1cm} \text{Saddle Point} \\
\((-2\sqrt{3},2)\) : \(D(-2\sqrt{3},2) = -768 < 0\) \hspace{1cm} \text{Saddle Point} \\
\((2\sqrt{3},2)\) : \(D(2\sqrt{3},2) = -768 < 0\) \hspace{1cm} \text{Saddle Point} \\
\(\left(0,\frac{-16\sqrt{3}}{18}\right)\) : \(D\left(0,\frac{-16\sqrt{3}}{18}\right) = 180.4 > 0\) \hspace{1cm} \(f_{xx}\left(0,\frac{-16\sqrt{3}}{18}\right) = -5.8 < 0\) \hspace{1cm} \text{Relative Maximum} \\
\(\left(0,\frac{-16\sqrt{3}}{18}\right)\) : \(D\left(0,\frac{-16\sqrt{3}}{18}\right) = 205.1 > 0\) \hspace{1cm} \(f_{xx}\left(0,\frac{-16\sqrt{3}}{18}\right) = 6.6 > 0\) \hspace{1cm} \text{Relative Minimum} \\

Don’t forget to check the value of \(f_{xx}\) when \(D\) is positive so we can get the correct classification (\(i.e.\) maximum or minimum) and also recall that for negative \(D\) we don’t need the second check as we know the critical point will be a saddle point.

---

**Absolute Minimums and Maximums**

1. Find the absolute minimum and absolute maximum of \(f(x,y) = 192x^3 + y^2 - 4xy^2\) on the triangle with vertices \((0,0)\), \((4,2)\) and \((-2,2)\).

Step 1

We’ll need the first order derivatives to start the problem off. Here they are,

\[ f_x = 576x^2 - 4y^2 \quad \quad \quad f_y = 2y - 8xy \]

Step 2

We need to find the critical points for this problem. That means solving the following system.

\[
\begin{align*}
    f_x &= 0 : \quad 576x^2 - 4y^2 = 0 \\
    f_y &= 0 : \quad 2y(1-4x) = 0 \quad \Rightarrow \quad y = 0 \text{ or } x = \frac{1}{4}
\end{align*}
\]

So, we have two possible options from the second equation. We can plug each into the first equation to get the critical points for the equation.

\[
\begin{align*}
    y = 0 : \quad 576x^2 = 0 \quad \Rightarrow \quad x = 0 \quad \Rightarrow \quad (0,0) \\
    x = \frac{1}{4} : \quad 36 - 4y^2 = 0 \quad \Rightarrow \quad y = \pm 3 \quad \Rightarrow \quad \left(\frac{1}{4},3\right) \text{ and } \left(\frac{1}{4},-3\right)
\end{align*}
\]
Okay, we have the three critical points listed above. Also recall that we only use critical points that are actually in the region we are working with. In this case, the last two have $y$ values that clearly are out of the region (we’ve sketched the region in the next step if you aren’t sure you believe this!) and so we can ignore them.

Therefore, the only critical point from this list that we need to use is the first. Note as well that, in this case, this also happens to be one of the points that define the boundary of the region. This will happen on occasion but won’t always.

So, we’ll need the function value for the only critical point that is actually in our region. Here is that value,

$$f(0,0) = 0$$

Step 3
Now, we know that absolute extrema can occur on the boundary. So, let’s start off with a quick sketch of the region we’re working on.

Each of the sides of the triangle can then be defined as follows.

Top : \( y = 2, \ -2 \leq x \leq 4 \)

Right : \( y = \frac{1}{2}x, \ 0 \leq x \leq 4 \)

Left : \( y = -x, \ -2 \leq x \leq 0 \)

Now we need to analyze each of these sides to get potential absolute extrema for \( f(x,y) \) that might occur on the boundary.

Step 4
Let’s first check out the top : \( y = 2, \ -2 \leq x \leq 4 \).

We’ll need to identify the points along the top that could be potential absolute extrema for \( f(x,y) \). This, in essence, requires us to find the potential absolute extrema of the following equation on the interval \(-2 \leq x \leq 4\).
\[ g(x) = f(x, 2) = 192x^3 - 16x + 4 \]

This is really nothing more than a Calculus I absolute extrema problem so we’ll be doing the work here without a lot of explanation. If you don’t recall how to do these kinds of problems you should read through that section in the Calculus I material.

The critical point(s) for \( g(x) \) are,

\[ g'(x) = 576x^2 - 16 = 0 \quad \rightarrow \quad x = \pm \frac{1}{6} \]

So, these two points as well as the \( x \) limits for the top give the following four points that are potential absolute extrema for \( f(x, y) \).

\[ \left( \frac{1}{6}, 2 \right), \left( -\frac{1}{6}, 2 \right), (-2, 2), (4, 2) \]

Recall that, in this step, we are assuming that \( y = 2 \) ! So, the next set of potential absolute extrema for \( f(x, y) \) are then,

\[ f\left( \frac{1}{6}, 2 \right) = \frac{20}{9}, \quad f\left( -\frac{1}{6}, 2 \right) = \frac{42}{9}, \quad f(-2, 2) = -1,500, \quad f(4, 2) = 12,228 \]

Step 5
Next let’s check out the right side : \( y = \frac{1}{2}x, \quad 0 \leq x \leq 4 \). For this side we’ll need to identify possible absolute extrema of the following function on the interval \( 0 \leq x \leq 4 \).

\[ g(x) = f(x, \frac{1}{2}x) = \frac{1}{2}x^2 + 191x^3 \]

The critical point(s) for the \( g(x) \) from this step are,

\[ g'(x) = \frac{1}{2}x + 573x^2 = x\left( \frac{1}{2} + 573x \right) = 0 \quad \rightarrow \quad x = 0, \quad x = -\frac{1}{1146} \]

Now, recall what we are restricted to the interval \( 0 \leq x \leq 4 \) for this portion of the problem and so the second critical point above will not be used as it lies outside this interval.

So, the single point from above that is in the interval \( 0 \leq x \leq 4 \) as well as the \( x \) limits for the right give the following two points that are potential absolute extrema for \( f(x, y) \).

\[ (0, 0), (4, 2) \]

Recall that, in this step, we are assuming that \( y = \frac{1}{2}x \) ! Also note that, in this case, one of the critical points ended up also being one of the endpoints.
Therefore, the next set of potential absolute extrema for \( f(x,y) \) are then,

\[
f(0,0) = 0 \quad f(4,2) = 12,228
\]

Before proceeding to the next step note that both of these have already appeared in previous steps. This will happen on occasion but we can’t, in many cases, expect this to happen so we do need to go through and do the work for each boundary.

The main exception to this is usually the endpoints of our intervals as they will always be shared in two of the boundary checks and so, once done, don’t really need to be checked again. We just included the endpoints here for completeness.

Step 6

Finally, let’s check out the left side : \( y = -x \), \( -2 \leq x \leq 0 \). For this side we’ll need to identify possible absolute extrema of the following function on the interval \( -2 \leq x \leq 0 \).

\[
g(x) = f(x,-x) = x^2 + 188x^3
\]

The critical point(s) for the \( g(x) \) from this step are,

\[
g'(x) = 2x + 564x^2 = 2x(1 + 282x) = 0 \quad \rightarrow \quad x = 0, \quad x = -\frac{1}{282}
\]

Both of these are in the interval \( -2 \leq x \leq 0 \) that we are restricted to for this portion of the problem.

So the two points from above as well as the \( x \) limits for the right give the following three points that are potential absolute extrema for \( f(x,y) \).

\[
(-\frac{1}{282}, 0), \quad (0,0), \quad (-2, 2)
\]

Recall that, in this step we are assuming that \( y = -x \) ! Also note that, in this case, one of the critical points ended up also being one of the endpoints.

Therefore, the next set of potential absolute extrema for \( f(x,y) \) are then,

\[
f\left(-\frac{1}{282}, \frac{1}{282}\right) = \frac{1}{238,5732} \quad f(0,0) = 0 \quad f(-2,2) = -1,500
\]

As with the previous step we can note that both of the end points above have already occurred previously in the problem and didn’t really need to be checked here. They were just included for completeness.

Step 7

Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.
From this list we can see that the absolute maximum of the function will be 12,228 which occurs at
\((4, 2)\) and the absolute minimum of the function will be -1,500 which occurs at \((-2, 2)\).

2. Find the absolute minimum and absolute maximum of \(f(x, y) = (9x^2 - 1)(1 + 4y)\) on the rectangle
given by \(-2 \leq x \leq 3, -1 \leq y \leq 4\).

Step 1
We'll need the first order derivatives to start the problem off. Here they are,
\[f_x = 18x(1 + 4y) \quad f_y = 4(9x^2 - 1)\]

Step 2
We need to find the critical points for this problem. That means solving the following system.
\[f_x = 0 : \quad 18x(1 + 4y) = 0 \quad f_y = 0 : \quad 4(9x^2 - 1) = 0 \quad \Rightarrow \quad x = \pm \frac{1}{3}\]

So, we have two possible options from the second equation. We can plug each into the first equation to
get the critical points for the equation.

\[x = \frac{1}{3} : \quad 6(1 + 4y) = 0 \quad \Rightarrow \quad y = -\frac{1}{4} \quad \Rightarrow \quad \left(\frac{1}{3}, -\frac{1}{4}\right)\]
\[x = -\frac{1}{3} : \quad -6(1 + 4y) = 0 \quad \Rightarrow \quad y = -\frac{1}{4} \quad \Rightarrow \quad \left(-\frac{1}{3}, -\frac{1}{4}\right)\]

Both of these critical points are in the region we are interested in and so we’ll need the function evaluated
at both of them. Here are those values,
\[f\left(\frac{1}{3}, -\frac{1}{4}\right) = 0 \quad f\left(-\frac{1}{3}, -\frac{1}{4}\right) = 0\]

Step 3
Now, we know that absolute extrema can occur on the boundary. So, let’s start off with a quick sketch of
the region we’re working on.
Each of the sides of the rectangle can then be defined as follows.

Top : \( y = 4, \ -2 \leq x \leq 3 \)

Bottom : \( y = -1, \ -2 \leq x \leq 3 \)

Right : \( x = 3, \ -1 \leq y \leq 4 \)

Left : \( x = -2, \ -1 \leq y \leq 4 \)

Now we need to analyze each of these sides to get potential absolute extrema for \( f(x,y) \) that might occur on the boundary.

Step 4
Let’s first check out the top : \( y = 4, \ -2 \leq x \leq 3 \).

We’ll need to identify the points along the top that could be potential absolute extrema for \( f(x,y) \). This, in essence, requires us to find the potential absolute extrema of the following equation on the interval \(-2 \leq x \leq 3\).

\[
g(x) = f(x, 4) = 17\left(-1 + 9x^2\right)
\]

This is really nothing more than a Calculus I absolute extrema problem so we’ll be doing the work here without a lot of explanation. If you don’t recall how to do these kinds of problems you should read through that section in the Calculus I material.

The critical point(s) for \( g(x) \) are,

\[
g'(x) = 306x = 0 \quad \rightarrow \quad x = 0
\]

This critical point is in the interval we are working on so, this point as well as the \( x \) limits for the top give the following three points that are potential absolute extrema for \( f(x,y) \).
Recall that, in this step, we are assuming that \( y = 4 \). So, the next set of potential absolute extrema for \( f(x, y) \) are then,

\[

c_1 = f(0, 4) = -17, \quad c_2 = f(-2, 4) = 595, \quad c_3 = f(3, 4) = 1360
\]

Step 5

Next, let’s check out the bottom : \( y = -1, \ -2 \leq x \leq 3 \). For this side we’ll need to identify possible absolute extrema of the following function on the interval \( -2 \leq x \leq 3 \).

\[
g(x) = f(x, -1) = -3(-1 + 9x^2)
\]

The critical point(s) for the \( g(x) \) from this step are,

\[
g'(x) = -54x = 0 \quad \rightarrow \quad x = 0
\]

This critical point is in the interval we are working on so, this point as well as the \( x \) limits for the bottom give the following three points that are potential absolute extrema for \( f(x, y) \).

\[
(0, -1) \quad (-2, -1) \quad (3, -1)
\]

Recall that, in this step, we are assuming that \( y = -1 \). So, the next set of potential absolute extrema for \( f(x, y) \) are then,

\[
f(0, -1) = 3, \quad f(-2, -1) = -105, \quad f(3, -1) = -240
\]

Step 6

Let’s now check out the right side : \( x = 3, \ -1 \leq y \leq 4 \). For this side we’ll need to identify possible absolute extrema of the following function on the interval \( -1 \leq y \leq 4 \).

\[
h(y) = f(3, y) = 80(1 + 4y)
\]

The derivative of the \( h(y) \) from this step is,

\[
h'(y) = 320
\]

In this case there are no critical points of the function along this boundary. So, only the limits for the right side are potential absolute extrema for \( f(x, y) \).
Recall that, in this step, we are assuming that \( x = 3 \) ! Therefore, the next set of potential absolute extrema for \( f(x, y) \) are then,

\[
f(3, -1) = -240 \quad f(3, 4) = 1360
\]

Before proceeding to the next step let’s note that both of these points have already been listed in previous steps and so did not really need to be written down here. This will always happen with boundary points (as these are here). Boundary points will always show up in multiple boundary steps.

Step 7
Finally, let’s check out the left side: \( x = -2, \quad -1 \leq y \leq 4 \). For this side we’ll need to identify possible absolute extrema of the following function on the interval \( -1 \leq y \leq 4 \).

\[
h(y) = f(-2, y) = 35(1 + 4y)
\]

The derivative of the \( h(y) \) from this step is,

\[
h'(y) = 140
\]

In this case there are no critical points of the function along this boundary. So, we only the limits for the right side are potential absolute extrema for \( f(x, y) \).

\[
(-2, -1) \quad (-2, 4)
\]

Recall that, in this step, we are assuming that \( x = -2 \) ! Therefore, the next set of potential absolute extrema for \( f(x, y) \) are then,

\[
f(-2, -1) = -105 \quad f(-2, 4) = 595
\]

As with the previous step both of these are boundary points and have appeared in previous steps. They were simply listed here for completeness.

Step 8
Okay, in summary, here are all the potential absolute extrema and their function values for this function on the region we are working on.

\[
f(0, 4) = -17 \quad f(-2, 4) = 595 \quad f(3, 4) = 1360
\]
\[
f(0, -1) = 3 \quad f(-2, -1) = -105 \quad f(3, -1) = -240
\]
From this list we can see that the absolute maximum of the function will be 1360 which occurs at \((3, 4)\) and the absolute minimum of the function will be -240 which occurs at \((3, -1)\).

---

**Lagrange Multipliers**

1. Find the maximum and minimum values of \(f(x, y) = 81x^2 + y^2\) subject to the constraint \(4x^2 + y^2 = 9\).

Step 1
Before proceeding with the problem let’s note because our constraint is the sum of two terms that are squared (and hence positive) the largest possible range of \(x\) is \(-\frac{3}{2} \leq x \leq \frac{3}{2}\) (the largest values would occur if \(y = 0\)). Likewise the largest possible range of \(y\) is \(-3 \leq y \leq 3\) (with the largest values occurring if \(x = 0\)).

Note that, at this point, we don’t know if \(x\) and/or \(y\) will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first actual step in the solution process is then to write down the system of equations we’ll need to solve for this problem.

\[
\begin{align*}
162x &= 8x\lambda \\
2y &= 2y\lambda \\
4x^2 + y^2 &= 9
\end{align*}
\]

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be “easy” and which will be “hard” until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you’ve ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren’t really as “bad” as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.
In this case, simply because the numbers are a little smaller, let’s start with the second equation. A little rewrite of the equation gives us the following,

\[ 2y\lambda - 2y = 2y(\lambda - 1) = 0 \quad \rightarrow \quad y = 0 \quad \text{or} \quad \lambda = 1 \]

Be careful here to not just divide both sides by \( y \) to “simplify” the equation. Remember that you can’t divide by anything unless you know for a fact that it won’t ever be zero. In this case we can see that \( y \) clearly can be zero and if you divide it out to start the solution process you will miss that solution. This is often one of the biggest mistakes that students make when working these kinds of problems.

Step 4
We now have two possibilities from Step 2. Either \( y = 0 \) or \( \lambda = 1 \). We’ll need to go through both of these possibilities and see what we get.

Let’s start by assuming that \( y = 0 \). In this case we can go directly to the constraint to get,

\[ 4x^2 = 9 \quad \rightarrow \quad x = \pm \frac{3}{2} \]

Therefore from this part we get two points that are potential absolute extrema,

\[ \left( -\frac{3}{2}, 0 \right) \quad \left( \frac{3}{2}, 0 \right) \]

Step 5
Next, let’s assume that \( \lambda = 1 \). In this case, we can plug this into the first equation to get,

\[ 162x = 8x \quad \rightarrow \quad 154x = 0 \quad \rightarrow \quad x = 0 \]

So, under this assumption we must have \( x = 0 \). We can now plug this into the constraint to get,

\[ y^2 = 9 \quad \rightarrow \quad y = \pm 3 \]

So, this part gives us two more points that are potential absolute extrema,

\[ (0, -3) \quad (0, 3) \]

Step 6
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

\[ f\left( -\frac{3}{2}, 0 \right) = \frac{729}{4} \quad f\left( \frac{3}{2}, 0 \right) = \frac{729}{4} \quad f(0, -3) = 9 \quad f(0, 3) = 9 \]

The absolute maximum is then \( \frac{729}{4} = 182.25 \) which occurs at \( \left( -\frac{3}{2}, 0 \right) \) and \( \left( \frac{3}{2}, 0 \right) \). The absolute minimum is 9 which occurs at \( (0, -3) \) and \( (0, 3) \). Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.
2. Find the maximum and minimum values of \( f(x, y) = 8x^2 - 2y \) subject to the constraint \( x^2 + y^2 = 1 \).

Step 1
Before proceeding with the problem let’s note because our constraint is the sum of two terms that are squared (and hence positive) the largest possible range of \( x \) is \(-1 \leq x \leq 1\) (the largest values would occur if \( y = 0 \)). Likewise the largest possible range of \( y \) is \(-1 \leq y \leq 1\) (with the largest values occurring if \( x = 0 \)).

Note that, at this point, we don’t know if \( x \) and/or \( y \) will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first actual step in the solution process is then to write down the system of equations we’ll need to solve for this problem.

\[
\begin{align*}
16x & = 2x\lambda \\
-2 & = 2y\lambda \\
x^2 + y^2 & = 1
\end{align*}
\]

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be “easy” and which will be “hard” until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you’ve ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren’t really as “bad” as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

For this system it looks like maybe the first equation will give us some information to start off with so let’s start with that equation. A quick rewrite of the equation gives us the following,

\[16x - 2x\lambda = 2x(8 - \lambda) = 0 \quad \rightarrow \quad x = 0 \quad \text{or} \quad \lambda = 8\]

Be careful here to not just divide both sides by \( x \) to “simplify” the equation. Remember that you can’t divide by anything unless you know for a fact that it won’t ever be zero. In this case we can see that \( x \) clearly can be zero and if you divide it out to start the solution process you will miss that solution. This is often one of the biggest mistakes that students make when working these kinds of problems.
Step 4
We now have two possibilities from Step 2. Either $x = 0$ or $\lambda = 8$. We’ll need to go through both of these possibilities and see what we get.

Let’s start by assuming that $x = 0$. In this case we can go directly to the constraint to get,

$$y^2 = 1 \quad \rightarrow \quad y = \pm 1$$

Therefore from this part we get two points that are potential absolute extrema,

$$\left(0, -1\right) \quad \left(0, 1\right)$$

Step 5
Next, let’s assume that $\lambda = 8$. In this case, we can plug this into the second equation to get,

$$-2 = 16y \quad \rightarrow \quad y = -\frac{1}{8}$$

So, under this assumption we must have $y = -\frac{1}{8}$. We can now plug this into the constraint to get,

$$x^2 + \frac{1}{64} = 1 \quad \rightarrow \quad x^2 = \frac{63}{64} \quad \rightarrow \quad x = \pm \sqrt{\frac{63}{64}} = \pm \frac{3\sqrt{7}}{8}$$

So, this part gives us two more points that are potential absolute extrema,

$$\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) \quad \left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$$

Step 6
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

$$f \left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) = \frac{65}{8} \quad \quad \quad f \left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) = \frac{65}{8} \quad \quad \quad f \left(0, -1\right) = 2 \quad \quad \quad f \left(0, 1\right) = -2$$

The absolute maximum is then $\frac{65}{8} = 8.125$ which occurs at $\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$ and $\left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right)$. The absolute minimum is -2 which occurs at $\left(0, 1\right)$. Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

---

3. Find the maximum and minimum values of $f(x, y, z) = y^2 - 10z$ subject to the constraint $x^2 + y^2 + z^2 = 36$.

Step 1
Before proceeding with the problem let’s note because our constraint is the sum of three terms that are squared (and hence positive) the largest possible range of $x$ is $-6 \leq x \leq 6$ (the largest values would occur if $y = 0$ and $z = 0$). Likewise we’d get the same ranges for both $y$ and $z$.

Note that, at this point, we don’t know if $x$, $y$ or $z$ will actually be the largest possible value. At this point we are simply acknowledging what they are. What this allows us to say is that whatever our answers will be they must occur in these bounded ranges and hence by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first actual step in the solution process is then to write down the system of equations we’ll need to solve for this problem.

\[
\begin{align*}
0 &= 2x\lambda \\
2y &= 2y\lambda \\
-10 &= 2z\lambda \\
x^2 + y^2 + z^2 &= 36
\end{align*}
\]

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be “easy” and which will be “hard” until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you’ve ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren’t really as “bad” as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

For this system let’s start with the third equation and note that because the left side is -10, or more importantly can never by zero, we can see that we must therefore have $z \neq 0$ and $\lambda \neq 0$. The fact that $\lambda$ can’t be zero is really important for this problem.

Step 4
Okay, because we now know that $\lambda \neq 0$ we can see that the only way for the first equation to be true is to have $x = 0$.

Therefore no matter what else is going on with $y$ and $z$ in this problem we must always have $x = 0$ and we’ll need to keep that in mind.

Step 5
Next, let’s take a look at the second equation. A quick rewrite of this equation gives,
\[ 2y - 2y \lambda = 2y(1 - \lambda) = 0 \quad \rightarrow \quad y = 0 \quad \text{or} \quad \lambda = 1 \]

**Step 6**

We now have two possibilities from Step 4. Either \( y = 0 \) or \( \lambda = 1 \). We’ll need to go through both of these possibilities and see what we get.

Let’s start by assuming that \( y = 0 \) and recall from Step 3 that we also know that \( x = 0 \). In this case we can plug these values into the constraint to get,

\[ z^2 = 36 \quad \rightarrow \quad z = \pm 6 \]

Therefore from this part we get two points that are potential absolute extrema,

\[ (0, 0, -6) \quad \quad (0, 0, 6) \]

**Step 7**

Next, let’s assume that \( \lambda = 1 \). If we head back to the third equation we can see that we now have,

\[ -10 = 2z \quad \rightarrow \quad z = -5 \]

So, under this assumption we must have \( z = -5 \) and recalling once more from Step 3 that we have \( x = 0 \) we can now plug these into the constraint to get,

\[ y^2 + 25 = 36 \quad \rightarrow \quad y^2 = 11 \quad \rightarrow \quad y = \pm \sqrt{11} \]

So, this part gives us two more points that are potential absolute extrema,

\[ (0, -\sqrt{11}, -5) \quad \quad (0, \sqrt{11}, -5) \]

**Step 8**

In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

\[ f(0, -\sqrt{11}, -5) = 61 \quad f(0, \sqrt{11}, -5) = 61 \quad f(0, 0, -6) = 60 \quad f(0, 0, 6) = -60 \]

The absolute maximum is then 61 which occurs at \( (0, -\sqrt{11}, -5) \) and \( (0, \sqrt{11}, -5) \). The absolute minimum is -60 which occurs at \( (0, 0, 6) \). Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.
4. Find the maximum and minimum values of \( f(x, y, z) = xyz \) subject to the constraint \( x + 9y^2 + z^2 = 4 \). Assume that \( x \geq 0 \) for this problem. Why is this assumption needed?

Step 1
Before proceeding with the solution to this problem let’s address why the assumption that \( x \geq 0 \) is needed for this problem.

The answer is simple. Without that assumption this function will not have absolute extrema.

If there are no restrictions on \( x \) then we could make \( x \) as large and negative as we wanted to and we could still meet the constraint simply by chose a very large \( y \) and/or \( z \). Note as well that because \( y \) and \( z \) are both squared we could chose them to be either negative or positive.

If we took our choices for \( x, y \) and \( z \) and plugged them into the function then the function would be similarly large. Also, the larger we chose \( x \) the larger we’d need to choose appropriate \( y \) and/or \( z \) and hence the larger our function would become. Finally, as noted above because we could chose \( y \) and \( z \) to be either positive or negative we could force the function to be either positive or negative with appropriate choices of signs for \( y \) and \( z \).

In other words, if we have no restriction on \( x \), we can make the function arbitrarily large in a positive and negative sense and so this function would not have absolute extrema.

On the other hand, if we put on the restriction on \( x \) that we have we now have the sum of three positive terms that must equal four. This in turn leads to the following largest possible values of the three variables in the problem.

\[
0 \leq x \leq 4 \quad -\frac{2}{3} \leq y \leq \frac{2}{3} \quad -2 \leq z \leq 2
\]

The largest value of \( x \) and the extreme values of \( y \) and \( z \) would occur when the other two variables are zero and in general there is no way to know ahead of time if any of the variables will in fact take on their largest possible values. However, what we can say now is that because all of our variables are bounded then by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

Note as well that all we really need here is a lower limit for \( x \). It doesn’t have to be zero that just makes the above analysis a little bit easier. We could have used the restriction that \( x \geq -8 \) if we’d wanted to. With this restriction we’d still have a bounded set of ranges for \( x, y \) and \( z \) and so the function would still have absolute extrema.

This problem shows just why this step is so important for these problems. If this problem did not have a restriction on \( x \) and we neglected to do this step we’d get the (very) wrong answer! We could still go through the process below and we’d get values that would appear to be absolute extrema. However, as we’ve shown above without any restriction on \( x \) the function would not have absolute extrema.

The issue here is that the Lagrange multiplier process itself is not set up to detect if absolute extrema exist or not. Before we even start the process we need to first make sure that the values we get out of the process will in fact be absolute extrema (i.e. we need to verify that absolute extrema exist).

Step 2
The first step here is to write down the system of equations we’ll need to solve for this problem.
Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be “easy” and which will be “hard” until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you’ve ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren’t really as “bad” as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system let’s start out by multiplying the first equation by \( x \), multiplying the second equation by \( y \) and multiplying the third equation by \( z \). Doing this gives the following “new” system of equations.

\[
\begin{align*}
xz &= 18y\lambda \\
xz &= 2z\lambda \\
x + 9y^2 + z^2 &= 4
\end{align*}
\]

Let’s also note that the constraint won’t be true if all three variables are zero simultaneously. One or two of the variables can be zero but we can’t have all three be zero.

Step 4
Now, let’s set the first and second equations from Step 3 equal. Doing this gives,

\[
x\lambda = 18y^2\lambda \quad \rightarrow \quad (x - 18y^2)\lambda = 0 \quad \rightarrow \quad x = 18y^2 \quad \text{or} \quad \lambda = 0
\]

Let’s also set the second and third equation from Step 3 equal. Doing this gives,

\[
18y^2\lambda = 2z^2\lambda \quad \rightarrow \quad (18y^2 - 2z^2)\lambda = 0 \quad \rightarrow \quad z^2 = 9y^2 \quad \text{or} \quad \lambda = 0
\]

Step 5
Okay, from Step 4 we have two possibilities. Either \( \lambda = 0 \) or we have \( x = 18y^2 \) and \( z^2 = 9y^2 \).

Let’s take care of the first possibility, \( \lambda = 0 \). If we go back to the original system this assumption gives us the following system.
Step 6
We have all sorts of possibilities from Step 5. From the first equation we have two possibilities. Let’s start with \( y = 0 \). Since the third equation from Step 5 won’t really tell us anything (after all it is now \( 0 = 0 \)) let’s move to the second equation. In this case we get either \( x = 0 \) or \( z = 0 \).

Recall that at the end of the third step we noticed that we can’t have all three of the variables be zero but we could have two of them be zero. So, this leads to the following two cases that we can plug into the constraint to find the value of the third variable.

\[
\begin{align*}
\text{Case 1: } & \quad y = 0, x = 0 \quad \Rightarrow \quad z = \pm 2 \\
\text{Case 2: } & \quad y = 0, z = 0 \quad \Rightarrow \quad x = 4
\end{align*}
\]

Step 7
Now, way back in Step 5 we had another possibility : \( x = 18y^2 \) and \( z^2 = 9y^2 \). We have to now take a look at this case. In this case we can plug each of these directly into the constraint to get the following,

\[
18y^2 + 9y^2 + 9y^2 = 36y^2 = 4 \quad \Rightarrow \quad y = \pm \frac{1}{3}
\]
Now we can go back to the two assumptions we started this step off with to get,

\[ x = 18 \left( \frac{1}{2} \right) = 2 \quad \text{ and } \quad z^2 = 9 \left( \frac{1}{2} \right) = 1 \quad \rightarrow \quad z = \pm 1 \]

Now, in most cases, we can’t just “mix and match” all the values of \( x, y \) and \( z \) to from points. In this case however, we can do exactly that. The \( x = 2 \) will arise regardless of the sign on \( y \) because of the \( y^2 \) in the \( x \) assumption. Likewise, because of the \( y^2 \) in the \( z \) assumption each of the \( z \)'s can arise for either \( y \) and so we get all combinations of \( x, y \) and \( z \) for points in this case.

Therefore we get the following four possible absolute extrema from this step.

\[ (2, -\frac{1}{3}, -1), (2, -\frac{1}{3}, 1), (2, \frac{1}{3}, -1), (2, \frac{1}{3}, 1) \]

Step 8
In total, it looks like we have nine points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points and with nine points that seems like a lot of work.

However, in this case, it’s actually quite simple. Recall that the function we’re evaluating is \( f(x, y, z) = xyz \). First, this means that if even one of the variables is zero the whole function will be zero. Therefore, the function evaluations for the five points from Step 6 all give,

\[ f(0, 0, \pm 2) = f(0, \pm \frac{2}{3}, 0) = f(4, 0, 0) = 0 \]

Note the usage of the “\( \pm \)” notation to “simplify” the work here as well.

Now, the potential points from Step 7 are all the same values, with the exception of signs changing occasionally on the \( y \) and \( z \). That means that the function value here will be either \(-\frac{2}{3}\) or \(\frac{2}{3}\) depending on the number of minus signs in the point. So again, not a lot of effort to compute these function values. Here are the evaluations for the points from Step 7.

\[ f(2, -\frac{1}{3}, 1) = f(2, \frac{1}{3}, -1) = -\frac{2}{3} \quad \text{ and } \quad f(2, -\frac{1}{3}, -1) = f(2, \frac{1}{3}, 1) = \frac{2}{3} \]

The absolute maximum is then \(\frac{2}{3}\) which occurs at \( (2, -\frac{1}{3}, -1) \) and \( (2, \frac{1}{3}, 1) \). The absolute minimum is \(-\frac{2}{3}\) which occurs at \( (2, -\frac{1}{3}, 1) \) and \( (2, \frac{1}{3}, -1) \). Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that some of the solution processes for the systems that arise with Lagrange multipliers can be quite involved. It can be easy to get lost in the details of the solution process and forget to go back and take care of one or more possibilities. You need to always be very careful and before finishing a problem go back and make sure that you’ve dealt with all the possible solution paths in the problem.
5. Find the maximum and minimum values of \( f(x, y, z) = 3x^2 + y \) subject to the constraints
\[ 4x - 3y = 9 \quad \text{and} \quad x^2 + z^2 = 9. \]

Step 1
Before proceeding with the problem let’s note that the second constraint is the sum of two terms that are squared (and hence positive). Therefore, the largest possible range of \( x \) is \(-3 \leq x \leq 3\) (the largest values would occur if \( z = 0 \)). We’ll get a similar range for \( z \).

Now, the first constraint is not the sum of two (or more) positive numbers. However, we’ve already established that \( x \) is restricted to \(-3 \leq x \leq 3\) and this will give \(-7 \leq y \leq 1\) as the largest possible range of \( y \)’s. Note that we can easily get this range by acknowledging that the first constraint is just a line and so the extreme values of \( y \) will correspond to the extreme values of \( x \).

So, because we now know that our answers must occur in these bounded ranges by the Extreme Value Theorem we know that absolute extrema will occur for this problem.

This step is an important (and often overlooked) step in these problems. It always helps to know that absolute extrema exist prior to actually trying to find them!

Step 2
The first step here is to write down the system of equations we’ll need to solve for this problem.

\[
\begin{align*}
6x &= 4\lambda + 2x\mu \\
1 &= -3\lambda \\
0 &= 2z\mu \\
4x - 3y &= 9 \\
x^2 + z^2 &= 9
\end{align*}
\]

Step 3
For most of these systems there are a multitude of solution methods that we can use to find a solution. Some may be harder than other, but unfortunately, there will often be no way of knowing which will be “easy” and which will be “hard” until you start the solution process.

Do not be afraid of these systems. They are probably unlike anything you’ve ever really been asked to solve up to this point. Most of the systems can be solved using techniques that you already know and aren’t really as “bad” as they may appear at first glance. Some do require some additional techniques and can be quite messy but for the most part still involve techniques that you do know how to use, you just may not have ever seen them done in the context of solving systems of equations.

With this system we get a “freebie” to start off with. Notice that from the second equation we quickly can see that \( \lambda = -\frac{1}{3} \) regardless of any of the values of the other variables in the system.

Step 4
Next, from the third equation we can see that we have either \( z = 0 \) or \( \mu = 0 \), So, we have 2 possibilities to look at. Let’s take a look at \( z = 0 \) first.
In this case we can go straight to the second constraint to get,

\[ x^2 = 9 \rightarrow x = \pm3 \]

We can in turn plug each of these possibilities into the first constraint to get values for \( y \).

\[
\begin{align*}
  x = -3 & : -12 - 3y = 9 \rightarrow y = -7 \\
  x = 3 & : 12 - 3y = 9 \rightarrow y = 1
\end{align*}
\]

Okay, from this step we have two possible absolute extrema.

\(( -3, -7, 0 ) \quad ( 3, 1, 0 )\)

Step 5
Now let’s go back and take a look at what happens if \( \mu = 0 \). If we plug this into the first equation in our system (and recalling that we also know that \( \lambda = -\frac{1}{3} \)) we get,

\[ 6x = -\frac{4}{3} \rightarrow x = -\frac{2}{3} \]

We can plug this into each of our constraints to get values of \( y \) (from the first constraint) and \( z \) (form the second constraint). Here is that work,

\[
\begin{align*}
  4\left( -\frac{2}{9} \right) - 3y &= 9 \rightarrow y = -\frac{8}{27} \\
  \left( -\frac{2}{3} \right)^2 + z^2 &= 9 \rightarrow z = \pm\frac{5\sqrt{39}}{9}
\end{align*}
\]

This leads to two more potential absolute extrema.

\[
\left( -\frac{2}{3}, -\frac{8}{27}, -\frac{5\sqrt{39}}{9} \right) \quad \left( -\frac{2}{3}, -\frac{8}{27}, \frac{5\sqrt{39}}{9} \right)
\]

Step 6
In total, it looks like we have four points that can potentially be absolute extrema. So, to determine the absolute extrema all we need to do is evaluate the function at each of these points. Here are those function evaluations.

\[
\begin{align*}
  f\left( -3, -7, 0 \right) &= 20 \\
  f\left( 3, 1, 0 \right) &= 28 \\
  f\left( -\frac{2}{9}, -\frac{8}{27}, -\frac{5\sqrt{39}}{9} \right) &= -\frac{85}{27} \\
  f\left( -\frac{2}{9}, -\frac{8}{27}, \frac{5\sqrt{39}}{9} \right) &= -\frac{85}{27}
\end{align*}
\]

The absolute maximum is then 28 which occurs at \((3,1,0)\). The absolute minimum is \(-\frac{85}{27}\) which occurs at \(\left( -\frac{2}{9}, -\frac{8}{27}, -\frac{5\sqrt{39}}{9} \right)\) and \(\left( -\frac{2}{9}, -\frac{8}{27}, \frac{5\sqrt{39}}{9} \right)\). Do not get excited about the absolute extrema occurring at multiple points. That will happen on occasion with these problems.

Before leaving this problem we should note that, in this case, the value of the absolute extrema (as opposed to the location) did not actually depend on the value of \( z \) in any way as the function we were...
optimizing in this problem did not depend on \( z \). This will happen sometimes and we shouldn’t get too worried about it when it does.

Note however that we still need the values of \( z \) for the location of the absolute extrema. We need the values of \( z \) for the location because the points that give the absolute extrema are also required to satisfy the constraint and the second constraint in our problem does involve \( z \)'s!