CALCULUS III
Assignment Problems
Line Integrals

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Preface

Here are a set of problems for my Calculus II notes. These problems do not have any solutions available on this site. These are intended mostly for instructors who might want a set of problems to assign for turning in. I try to put up both practice problems (with solutions available) and these problems at the same time so that both will be available to anyone who wishes to use them.
Introduction

Here are a set of problems for which no solutions are available. The main intent of these problems is to have a set of problems available for any instructors who are looking for some extra problems.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have problems written for them.

- Vector Fields
- Line Integrals – Part I
- Line Integrals – Part II
- Line Integrals of Vector Fields
- Fundamental Theorem for Line Integrals
- Conservative Vector Fields
- Green’s Theorem
- Curl and Divergence

Vector Fields

1. Sketch the vector field for \( \vec{F} = -y^2 \mathbf{i} + x \mathbf{j} \).

2. Sketch the vector field for \( \vec{F} = \mathbf{i} + xy \mathbf{j} \).

3. Sketch the vector field for \( \vec{F} = 4y \mathbf{i} + (x + 2) \mathbf{j} \).

4. Compute the gradient vector field for \( f(x, y) = 6x^2 - 9y + x^3 \sqrt{y} \).

5. Compute the gradient vector field for \( f(x, y) = \sin(2x) \cos(3x) \).

6. Compute the gradient vector field for \( f(x, y, z) = z e^{xy} + y^3 \tan(4x) \).

7. Compute the gradient vector field for \( f(x, y, z) = x y^2 z^3 + 4x e^{x^2} - \ln(x - z) \).
**Line Integrals – Part I**

For problems 1 – 10 evaluate the given line integral. Follow the direction of \( C \) as given in the problem statement.

1. Evaluate \( \int_C 3y \, ds \) where \( C \) is the portion of \( x = 9 - y^2 \) from \( y = -1 \) and \( y = 2 \).

2. Evaluate \( \int_C \sqrt{x} + 2xy \, ds \) where \( C \) is the line segment from \((7,3)\) to \((0,6)\).

3. Evaluate \( \int_C y^2 - 10xy \, ds \) where \( C \) is the left half of the circle centered at the origin of radius 6 with counter clockwise rotation.

4. Evaluate \( \int_C x^3 - 2y \, ds \) where \( C \) is given by \( \vec{r}(t) = \left< 4t^4, t^4 \right> \) for \(-1 \leq t \leq 0\).

5. Evaluate \( \int_C z^3 - 4x + 2y \, ds \) where \( C \) is the line segment from \((2,4,-1)\) to \((1,-1,0)\).

6. Evaluate \( \int_C x + 12xz \, ds \) where \( C \) is given by \( \vec{r}(t) = \left< t, \frac{1}{2}t^2, \frac{1}{4}t^4 \right> \) for \(-2 \leq t \leq 1\).

7. Evaluate \( \int_C z^3(x+7) - 2y \, ds \) where \( C \) is the circle centered at the origin of radius 1 centered on the \( x \)-axis at \( x = -3 \). See the sketches below for the direction.

8. Evaluate \( \int_C 6x \, ds \) where \( C \) is the portion of \( y = 3 + x^2 \) from \( x = -2 \) to \( x = 0 \) followed by the portion of \( y = 3 - x^2 \) from \( x = 0 \) to \( x = 2 \) which in turn is followed by the line segment from \((2,-1)\) to \((-1,-2)\). See the sketch below for the direction.
9. Evaluate $\int_{C} 2 - xy \, ds$ where $C$ is the upper half of the circle centered at the origin of radius 1 with the clockwise rotation followed by the line segment from $(1,0)$ to $(3,0)$ which in turn is followed by the lower half of the circle centered at the origin of radius 3 with the clockwise rotation. See the sketch below for the direction.

10. Evaluate $\int_{C} 3xy + (x-1)^2 \, ds$ where $C$ is the triangle with vertices $(0,3)$, $(6,0)$ and $(0,0)$ with the clockwise rotation.

11. Evaluate $\int_{C} x^5 \, ds$ for each of the following curves.
(a) $C$ is the line segment from $(-1,3)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(0,4)$.
(b) $C$ is the portion of $y = 4 - x^4$ from $x = -1$ to $x = 0$.

12. Evaluate $\int_C 3x - 6y \, ds$ for each of the following curves.
   (a) $C$ is the line segment from $(6,0)$ to $(0,3)$ followed by the line segment from $(0,3)$ to $(6,6)$.
   (b) $C$ is the line segment from $(6,0)$ to $(6,6)$.

13. Evaluate $\int_C y^2 - 3z + 2 \, ds$ for each of the following curves.
   (a) $C$ is the line segment from $(1,0,4)$ to $(2,-1,1)$.
   (b) $C$ is the line segment from $(2,-1,1)$ to $(1,0,4)$.

Line Integrals – Part II

For problems 1 – 7 evaluate the given line integral. Follow the direction of $C$ as given in the problem statement.

1. Evaluate $\int_C xy \, dx + (x - y) \, dy$ where $C$ is the line segment from $(0,-3)$ to $(-4,1)$.

2. Evaluate $\int_C e^{3x} \, dx$ where $C$ is portion of $x = \sin(4y)$ from $y = \frac{\pi}{6}$ to $y = \pi$.

3. Evaluate $\int_C x \, dy - (x^2 + y) \, dx$ where $C$ is portion of the circle centered at the origin of radius 3 in the $2^{nd}$ quadrant with clockwise rotation.

4. Evaluate $\int_C dx - 3y^3 \, dy$ where $C$ is given by $\vec{r} (t) = 4 \sin(\pi t) \hat{i} + (t-1)^2 \hat{j}$ with $0 \leq t \leq 1$.

5. Evaluate $\int_C 4y^2 \, dx + 3x \, dy + 2z \, dz$ where $C$ is the line segment from $(4,-1,2)$ to $(1,7,-1)$.

6. Evaluate $\int_C (yz + x) \, dx + yz \, dy - (y + z) \, dz$ where $C$ is given by $\vec{r} (t) = 3t \hat{i} + 4 \sin(t) \hat{j} + 4 \cos(t) \hat{k}$ with $0 \leq t \leq \pi$.
7. Evaluate $\int_C xy \, dy$ where $C$ is the portion of $y = \sqrt{x^2 + 5}$ from $x = -1$ to $x = 2$ followed by the line segment from $(2,3)$ to $(4,-1)$. See the sketch below for the direction.

![Sketch](image)

8. Evaluate $\int_C (y^2 - x) \, dx - 4y \, dy$ where $C$ is the portion of $y = x^2$ from $x = -2$ to $x = 2$ followed by the line segment from $(2,4)$ to $(0,6)$ which in turn is followed by the line segment from $(0,6)$ to $(-2,4)$. See the sketch below for the direction.

![Sketch](image)

9. Evaluate $\int_C (x^2 - 2) \, dx + 7xy^2 \, dy$ for each of the following curves.

(a) $C$ is the portion of $x = -y^2$ from $y = -1$ to $y = 1$.

(b) $C$ is the line segment from $(-1,-1)$ to $(1,1)$.

10. Evaluate $\int_C x^3 + 9y \, dy$ for each of the following curves.
Calculus II

(a) $C$ is the portion of $y = 1 - x^2$ from $x = -1$ to $x = 1$.

(b) $C$ is the line segment from $(-1, 0)$ to $(0, -1)$ followed by the line segment from $(0, -1)$ to $(1, 0)$.

11. Evaluate $\int_C xy^3 \, dx - 4x \, dy$ for each of the following curves.

(a) $C$ is the portion of the circle centered at the origin of radius 7 in the $1^{st}$ quadrant with counter clockwise rotation.

(b) $C$ is the portion of the circle centered at the origin of radius 7 in the $1^{st}$ quadrant with clockwise rotation.

**Line Integrals of Vector Fields**

1. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = 2x^2 \mathbf{i} + (y^2 - 1) \mathbf{j}$ and $C$ is the portion of $\frac{x^2}{25} + \frac{y^2}{9} = 1$ that is in the $1^{st}$, $4^{th}$ and $3^{rd}$ quadrant with the clockwise orientation.

2. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = xy \mathbf{i} + (4x - 2y) \mathbf{j}$ and $C$ is the line segment from $(4, -3)$ to $(7, 0)$.

3. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (x^3 - y) \mathbf{i} + (x^2 + 7x) \mathbf{j}$ and $C$ is the portion of $y = x^3 + 2$ from $x = -1$ to $x = 2$.

4. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = xy \mathbf{i} + (1 + x^2) \mathbf{j}$ and $C$ is given by $\mathbf{r}(t) = e^{6t} \mathbf{i} + (4 - e^{2t}) \mathbf{j}$ for $-2 \leq t \leq 0$.

5. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (3x - 3y) \mathbf{i} + (y^3 - 10) \mathbf{j} + yz \mathbf{k}$ and $C$ is the line segment from $(1, 4, -2)$ to $(3, 4, 6)$.

6. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y, z) = (x + z) \mathbf{i} + y^3 \mathbf{j} + (1 - x) \mathbf{k}$ and $C$ is the portion of the spiral on the $y$-axis given by $\mathbf{r}(t) = \cos(2t) \mathbf{i} - t \mathbf{j} + \sin(2t) \mathbf{k}$ for $-\pi \leq t \leq 2\pi$.

7. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = x^2 \mathbf{i} + (y^2 - x) \mathbf{j}$ and $C$ is the line segment from $(2, 4)$ to $(0, 4)$ followed by the line segment from $(0, 4)$ to $(3, -1)$.

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8. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = xy \vec{i} - 3 \vec{j}$ and $C$ is the portion of $x^2 + \frac{y^2}{4} = 1$ in the 2nd quadrant with clockwise rotation followed by the line segment from $(0, 4)$ to $(4, -2)$. See the sketch below.

9. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = xy^2 \vec{i} + (2y + 3x) \vec{j}$ and $C$ is the portion of $x = y^2 - 1$ from $y = -2$ to $y = 2$ followed by the line segment from $(3, 2)$ to $(0, 0)$ which in turn is followed by the line segment from $(0, 0)$ to $(3, -2)$. See the sketch below.
10. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (1 - y^2)\hat{i} - x\hat{j}$ for each of the following curves.

(a) $C$ is the top half of the circle centered at the origin of radius 1 with the counter clockwise rotation.

(b) $C$ is the bottom half of $x^2 + \frac{y^2}{36}$ with clockwise rotation.

11. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = (x^2 + y + 2)\hat{i} + xy\hat{j}$ for each of the following curves.

(a) $C$ is the portion of $y = x^2 - 2$ from $x = -3$ to $x = 3$.

(b) $C$ is the line segment from $(3,5)$ to $(3,5)$.

14. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = y^2\hat{i} + (1 - 3x)\hat{j}$ for each of the following curves.

(a) $C$ is the line segment from $(1,4)$ to $(-2,3)$.

(b) $C$ is the line segment from $(-2,3)$ to $(1,4)$.

13. Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = -2x\hat{i} + (x + 2y)\hat{j}$ for each of the following curves.

(a) $C$ is the portion of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ in the 1st quadrant with counter clockwise rotation.

(b) $C$ is the portion of $\frac{x^2}{16} + \frac{y^2}{4} = 1$ in the 1st quadrant with clockwise rotation.

**Fundamental Theorem for Line Integrals**

1. Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f(x,y) = 5x - y^2 + 10xy + 9$ and $C$ is given by $\vec{r}(t) = \left< \frac{2t}{t^2 + 1}, 1 - 8t \right>$ with $-2 \leq t \leq 0$.

2. Evaluate $\int_C \nabla f \cdot d\vec{r}$ where $f(x,y,z) = \frac{3x - 8y}{z - 6}$ and $C$ is given by $\vec{r}(t) = 6t\hat{i} + 4\hat{j} + \left(9 - t^3\right)\hat{k}$ with $-1 \leq t \leq 3$. 

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3. Evaluate \( \int_C \mathbf{f} \cdot d\mathbf{r} \) where \( f(x, y) = 20y \cos(x + 3) - yx^3 \) and \( C \) is right half of the ellipse given by \( \left( x + 3 \right)^2 + \frac{(y-1)^2}{16} = 1 \) with clockwise rotation.

4. Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = 2x \mathbf{i} + 4y \mathbf{j} \) and \( C \) is the circle centered at the origin of radius 5 with the counter clockwise rotation. Is \( \int_C \mathbf{F} \cdot d\mathbf{r} \) independent of path? If it is not possible to determine if \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path clearly explain why not.

5. Compute \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = y \mathbf{i} + x^2 \mathbf{j} \) and \( C \) is the circle centered at the origin of radius 5 with the counter clockwise rotation. Is \( \int_C \mathbf{F} \cdot d\mathbf{r} \) independent of path? If it is not possible to determine if \( \int_C \mathbf{F} \cdot d\mathbf{r} \) is independent of path clearly explain why not.

6. Evaluate \( \int_C \nabla f \cdot d\mathbf{r} \) where \( f(x, y, z) = 2x^2 + x(y - 2)^2 \) and \( C \) is the line segment from \((1, 2, 0)\) to \((-3,10,9)\) followed by the line segment from \((-3,10,9)\) to \((6,0,2)\).

7. Evaluate \( \int_C \nabla f \cdot d\mathbf{r} \) where \( f(x, y) = 4x + 3xy^2 - \ln(x^2 + y^2) \) and \( C \) is the upper half of \( x^2 + y^2 = 1 \) with clockwise rotation followed by the right half of \( (x-1)^2 + \frac{(y-2)^2}{4} = 1 \) with counter clockwise rotation. See the sketch below.

Conservative Vector Fields

For problems 1 – 4 determine if the vector field is conservative.

1. \( \vec{F} = (2xy^3 + e^x \cos(y))\hat{i} + (e^x \sin(y) - 3x^3y^2)\hat{j} \)

2. \( \vec{F} = (xy^2 - 3y^4 + 2)\hat{i} + (xy^2 + x^2y^2 - x)\hat{j} \)

3. \( \vec{F} = (2 + 12xy^2 - 3x^2\sqrt{y})\hat{i} - \left( \frac{x^3}{2\sqrt{y}} - 12x^2y \right)\hat{j} \)

4. \( \vec{F} = \left( 8 - \frac{3x^2}{y} + 5x^4y^2 \right)\hat{i} + \left( 6 + \frac{x^3}{y^2} - 3y^2 + 2x^5y \right)\hat{j} \)

For problems 5 – 11 find the potential function for the vector field.

5. \( \vec{F} = \left( 4x^3 + 3y + \frac{2y^3}{x^3} \right)\hat{i} + \left( 3x - 3y^2 - \frac{3y^2}{x^2} \right)\hat{j} \)

6. \( \vec{F} = (3x^2e^{2y} + 4ye^{4x})\hat{i} - \left( 7 - 2x^3e^{2y} - e^{4x} \right)\hat{j} \)

7. \( \vec{F} = \left( \cos(x)\cos(x + y) - 2y^2 - \sin(x)\sin(x + y) \right)\hat{i} - \left( 4xy + \sin(x)\sin(x + y) \right)\hat{j} \)

8. \( \vec{F} = \left( \frac{4}{x^2} + \frac{2x}{y} + \frac{2}{x^2y^3} \right)\hat{i} + \left( \frac{6}{xy^4} - \frac{1 + x^2}{y^2} \right)\hat{j} \)

9. \( \vec{F} = \left( 2xe^{x^2 - z}\sin(y^2) - 3y^3 \right)\hat{i} + \left( 2ye^{x^2 - z}\cos(y^2) - 9xy^2 \right)\hat{j} + \left( 12z - e^{x^2 - z}\sin(y^2) \right)\hat{k} \)

10. \( \vec{F} = \left( 12x - 5z^2 \right)\hat{i} + \ln(1 + z^2)\hat{j} - \left( 10xz - \frac{2yz}{1 + z^2} \right)\hat{k} \)

11. \( \vec{F} = \left( xy^2e^{y-x} - xy^2ze^{y-x} \right)\hat{i} + \left( 2xyze^{y-x} + xy^2ze^{y-x} \right)\hat{j} + \left( xy^2e^{y-x} - 24z \right)\hat{k} \)

12. Evaluate \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F}(x, y) = \left( \frac{3x^2}{y-1} - 3x^2y \right)\hat{i} + \left( 8y - x^3 - \frac{x^3}{(y-1)^2} \right)\hat{j} \) and \( C \) is the line segment from \((1, 2)\) to \((4, 3)\).
13. Evaluate \[ \int_C \vec{F} \cdot d\vec{r} \] where \( \vec{F}(x, y) = \left( y^2 - 4y + 5 \right) \hat{i} + \left( 2xy - 4x - 9 \right) \hat{j} \) and \( C \) the upper half of \( \frac{x^2}{36} + \frac{y^2}{16} = 1 \) with clockwise rotation.

14. Evaluate \[ \int_C \vec{F} \cdot d\vec{r} \] where \( \vec{F}(x, y) = -\left( 3 - (1 + 2y)e^{x-1} \right) \hat{i} + \left( 3y^2 + 2e^{x-1} \right) \hat{j} \) and \( C \) is the portion of \( y = x^3 + 1 \) from \( x = -2 \) to \( x = 1 \).

15. Evaluate \[ \int_C \vec{F} \cdot d\vec{r} \] where \( \vec{F}(x, y, z) = \frac{x}{\sqrt{x^2 + z^2}} \hat{i} + \left( 2yz - 6y \right) \hat{j} + \left( y^2 + \frac{z}{\sqrt{x^2 + z^2}} \right) \hat{k} \) and \( C \) is the line segment from \( (1, 0, -1) \) to \( (2, -4, 3) \).

16. Evaluate \[ \int_C \vec{F} \cdot d\vec{r} \] where \( \vec{F}(x, y) = (12xy - 2x) \hat{i} + \left( 6x^2 - 8xy \right) \hat{j} + \left( 8 - 4y^2 \right) \hat{k} \) and \( C \) is the spiral given by \( \vec{r}(t) = \left( \sin(\pi t), \cos(\pi t), 3t \right) \) for \( 0 \leq t \leq 6 \).

17. Evaluate \[ \int_C \vec{F} \cdot d\vec{r} \] where \( \vec{F}(x, y) = \left( 8 - 14xy^2 + 2ye^{2x} \right) \hat{i} + \left( e^{2x} - 14x^2 y \right) \hat{j} \) and \( C \) is the curve shown below.

![Diagram of curve]

18. Evaluate \[ \int_C \vec{F} \cdot d\vec{r} \] where \( \vec{F}(x, y) = \left( 6x - 5y^2 + 2xy^3 - 10 \right) \hat{i} + \left( 3x^2 y^2 - 10xy \right) \hat{j} \) and \( C \) is the curve shown below.

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1. Use Green’s Theorem to evaluate \( \int_C (y^2 - y) \, dx + (x^3 + 4) \, dy \) where \( C \) is shown below.

2. Use Green’s Theorem to evaluate \( \int_C (7x + y^2) \, dy - (x^2 - 2y) \, dx \) where \( C \) is the two circles as shown below.
3. Use Green’s Theorem to evaluate \[ \int_C \left( y^2 - 6y \right) \, dx + \left( y^3 + 10y^2 \right) \, dy \] where \( C \) is shown below.

4. Use Green’s Theorem to evaluate \[ \int_C xy^2 \, dx + \left( 1 - xy^3 \right) \, dy \] where \( C \) is shown below.
5. Use Green’s Theorem to evaluate \( \oint_C (y^2 - 4x) \, dx - (2 + x^3 y^2) \, dy \) where \( C \) is shown below.

6. Use Green’s Theorem to evaluate \( \oint_C (y^3 - xy^2) \, dx + (2 - x^3) \, dy \) where \( C \) is shown below.
7. Verify Green’s Theorem for \( \int_C (6 + x^2) \, dx + (1 - 2xy) \, dy \) where \( C \) is shown below by (a) computing the line integral directly and (b) using Green’s Theorem to compute the line integral.

8. Verify Green’s Theorem for \( \int_C (6y - 3y^2 + x) \, dx + yx^3 \, dy \) where \( C \) is shown below by (a) computing the line integral directly and (b) using Green’s Theorem to compute the line integral.
Curl and Divergence

For problems 1 – 3 compute \( \text{div } \vec{F} \) and \( \text{curl } \vec{F} \).

1. \( \vec{F} = \left( 2y - \cos(x) \right) \hat{i} - z^2 e^{3x} \hat{j} + \left( x^2 - 7z \right) \hat{k} \)

2. \( \vec{F} = -\left( 4y - 1 \right) \hat{i} + xy^2 \hat{j} + (x - 3y) \hat{k} \)

3. \( \vec{F} = z^2 \left( y - x \right) \hat{i} + \frac{4y^3}{x} \hat{j} + \left( x^2 - 3z \right) \hat{k} \)

For problems 4 – 6 determine if the vector field is conservative.

4. \( \vec{F} = \left( 2xy^2 - 16x \right) \hat{i} + 2y \left( x^2 - 1 \right) \hat{j} + 9 \hat{k} \)

5. \( \vec{F} = \left( y - 3z \right) \hat{i} + \left( x^2 + y^4 \right) \hat{j} - 4z^2 \hat{k} \)

6. \( \vec{F} = \left( 18x^3 + 4z^3 \right) \hat{i} - 12yz \hat{j} - \left( 6y^2 - 12xz^2 \right) \hat{k} \)