CALCULUS II
Assignment Problems
Parametric Equations and Polar Coordinates

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Preface

Here are a set of problems for my Calculus II notes. These problems do not have any solutions available on this site. These are intended mostly for instructors who might want a set of problems to assign for turning in. I try to put up both practice problems (with solutions available) and these problems at the same time so that both will be available to anyone who wishes to use them.

Parametric Equations and Polar Coordinates

Introduction

Here are a set of problems for which no solutions are available. The main intent of these problems is to have a set of problems available for any instructors who are looking for some extra problems.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have problems written for them.

- Parametric Equations and Curves
- Tangents with Parametric Equations
- Area with Parametric Equations
- Arc Length with Parametric Equations
- Surface Area with Parametric Equations

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**Parametric Equations and Curves**

For problems 1 – 9 eliminate the parameter for the given set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on $x$ and $y$.

1. $x = 2 + t \quad y = t^2 - 4t + 7$
2. $x = 2 + t \quad y = t^2 - 4t + 7 \quad -3 \leq t \leq 1$
3. $x = 1 - t^2 \quad y = 3 + 2t$
4. $x = 1 - t^2 \quad y = 3 + 2t \quad -2 \leq t \leq 3$
5. $x = \frac{1}{2} \sqrt{t} \quad y = t - \sqrt{t} - 6 \quad t \geq 0$
6. $x = \frac{1}{2} \sqrt{t} \quad y = t - \sqrt{t} - 6 \quad 8 \leq t \leq 20$
7. $x = -6 \cos(4t) \quad y = 2 \sin(4t) \quad -\frac{\pi}{4} \leq t \leq \frac{\pi}{8}$
8. $x = 1 - 3 \sin\left(\frac{1}{2}t\right) \quad y = 4 \cos\left(\frac{1}{2}t\right)$
9. $x = 6 - 7e^{-2t} \quad y = 4 + 3e^{-2t}$

10. Answer each of the questions about the following set of parametric equations
    
    $x = 3 \cos(at) \quad y = 3 \sin(at) \quad 0 \leq t \leq 2\pi$
    
    (a) Sketch the graph of the parametric curve for $a = 1$.
    
    (b) Sketch the graph of the parametric curve for $a = 6$.
    
    (c) Sketch the graph of the parametric curve for $a = \frac{1}{3}$.
    
    (d) In general, for $a > 0$, how does the value of $a$ affect the graph of the parametric curve?

For problems 11 – 21 the path of a particle is given by the set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.
(i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(ii) Limits on x and y.

(iii) A range of t’s for a single trace of the parametric curve.

(iv) The number of traces of the curve the particle makes if an overall range of t’s is provided in the problem.

11. \( x = 6 \cos \left( \frac{1}{3} t \right) \quad y = 2 + \sin \left( \frac{1}{3} t \right) \quad 0 \leq t \leq 75\pi \\
12. \( x = 7 - 3 \sin (2t) \quad y = 4 + 2 \cos (2t) \\
13. \( x = 6 \cos^2 (3t) \quad y = 2 - 3 \sin^2 (3t) \quad -\frac{5}{6}\pi \leq t \leq 3\pi \\
14. \( x = \sqrt{2 + \cos^2 \left( \frac{2}{7} t \right)} \quad y = \frac{1}{2} \sin \left( \frac{2}{7} t \right) \\
15. \( x = 6 - \sin^3 (4t) \quad y = 2 \sin (4t) \quad -127\pi \leq t \leq 201\pi \\
16. \( x = 3 + \cos \left( \frac{1}{6} t \right) \quad y = 4 + \cos^2 \left( \frac{1}{6} t \right) \quad -90\pi \leq t \leq 216\pi \\
17. \( x = e^{-3t} \quad y = 2e^{12t} \\
18. \( x = 1 + e^{3t} \quad y = e^{6t} \quad -1 \leq t \leq 6 \\
19. \( x = 1 - \ln (t) \quad y = \left[ \ln (t) \right]^2 \quad t > 0 \\
20. \( x = \cos \left( \frac{1}{2} t \right) \quad y = \sec \left( \frac{1}{2} t \right) \quad -\pi < t < \pi \\
21. \( x = \sin (2t) \quad y = \sin^2 (2t) - 4 \sin (2t) \quad -\frac{21}{4}\pi \leq t \leq \frac{17}{4}\pi \\

For problems 22 – 27 write down a set of parametric equations for the given equation that meets the given extra conditions (if any).

22. \( x = \sin \left( 3 - y^2 \right) + \cos^2 \left( y \right) \\
23. \( y = \frac{6 \cos (x) - 8}{x^2 + 9x} \\
24. \( x^2 + y^2 = 100 \) and the parametric curve resulting from the parametric equations should be at \( (0,10) \) when \( t = 0 \) and the curve should have a clockwise rotation.
25. \( x^2 + y^2 = 100 \) and the parametric curve resulting from the parametric equations should be at \((0,10)\) when \( t = 0 \) and the curve should have a counter clockwise rotation.

26. \( \frac{x^2}{25} + y^2 = 1 \) and the parametric curve resulting from the parametric equations should be at \((-5,0)\) when \( t = 0 \) and the curve should have a counter clockwise rotation.

27. \( \frac{x^2}{25} + y^2 = 1 \) and the parametric curve resulting from the parametric equations should be at \((-5,0)\) when \( t = 0 \) and the curve should have a clockwise rotation.

28. Eliminate the parameter for the following set of parametric equations and identify the resulting equation.

\[
x = h + a \cos(\omega t) \quad x = k + b \sin(\omega t)
\]

**Tangents with Parametric Equations**

For problems 1 – 3 compute \( \frac{dy}{dx} \) and \( \frac{d^2y}{dx^2} \) for the given set of parametric equations.

1. \( x = 7t^2 - 9t \) \quad \( y = t^6 + 2t^2 \)

2. \( x = \tan(2t) - 12 \) \quad \( y = 3 \sin(2t) + \sec(2t) + 4t \)

3. \( x = \ln(3t^3) + 8t \) \quad \( y = \ln(t^4) - 6 \ln(t^2) \)

For problems 4 – 7 find the equation of the tangent line(s) to the given set of parametric equations at the given point.

4. \( x = t^3 + \cos(\pi t) \) \quad \( y = 4t + \sin(2t + 6) \) at \( t = -3 \)

5. \( x = t^2 + 2t - 1 \) \quad \( y = t^3 + 7t^2 + 8t \) at \( t = 1 \)

6. \( x = 6 - e^{3-9t} \) \quad \( y = t^3 + 3t^2 - 18t + 2 \) at \((5,2)\)

7. \( x = t^2 + 5t - 6 \) \quad \( y = t^2 + 2t - 8 \) at \((-6,7)\)

For problems 8 and 9 find the values of \( t \) that will have horizontal or vertical tangent lines for the given set of parametric equations.

8. \( x = t^3 - 5t^2 + t + 1 \) \quad \( y = t^4 + 8t^3 + 3t^2 \)
9. \( x = 7t^2 + e^{2-t^2} \quad y = 10 \sin \left( \frac{1}{2} t \right) - 1 \)

**Area with Parametric Equations**

For problems 1 – 3 determine the area of the region below the parametric curve given by the set of parametric equations. For each problem you may assume that each curve traces out exactly once from right to left for the given range of \( t \). For these problems you should only use the given parametric equations to determine the answer.

1. \( x = t^2 + 5t - 1 \quad y = 40 - t^2 \quad -2 \leq t \leq 5 \)
2. \( x = 3 \cos^2(t) - \sin^2(t) \quad y = 6 + \cos(t) \quad -\frac{\pi}{2} \leq t \leq 0 \)
3. \( x = e^{4t} - 2 \quad y = 4 + e^{4t} - e^{2t} \quad -6 \leq t \leq 1 \)

**Arc Length with Parametric Equations**

For all the problems in this section you should only use the given parametric equations to determine the answer.

For problems 1 – 5 determine the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of \( t \)’s.

1. \( x = 3 + 9t \quad y = 10 - 15t \quad -5 \leq t \leq 8 \)
2. \( x = 6(3 + t)^{\frac{3}{2}} \quad y = -3t^{\frac{3}{2}} \quad -2 \leq t \leq 1 \)
3. \( x = 4t^2 - 3 \quad y = 3t \quad 0 \leq t \leq 5 \)
4. \( x = 3 + t \quad y = 6 + (t - 1)^2 \quad 1 \leq t \leq 3 \)
5. \( x = t^2 - 1 \quad y = t^4 + 5 \quad 0 \leq t \leq 1 \)

For problems 6 and 7 a particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

6. \( x = 6 \cos^2(3t) \quad y = 2 - 3 \sin^2(3t) \quad -\frac{5}{6} \pi \leq t \leq 3 \pi \)
7. \( x = 3 + \cos \left( \frac{1}{6} t \right) \quad y = 4 + \cos^2 \left( \frac{1}{6} t \right) \quad -90 \pi \leq t \leq 216 \pi \)
For problems 8 – 10 set up, but do not evaluate, an integral that gives the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of t's.

8. \( x = t \cos(2t) \) \quad \( y = \sin(3t) \) \quad \( 2 \leq t \leq 3 \)

9. \( x = 1 - \sin\left(1 + \sqrt{t}\right) \) \quad \( y = \sin\left(e^{-t}\right) \) \quad \( 1 \leq t \leq 4 \)

10. \( x = \ln(t + 2) \) \quad \( y = \frac{1}{t + 7} \) \quad \( -1 \leq t \leq 2 \)

**Surface Area with Parametric Equations**

For all the problems in this section you should only use the given parametric equations to determine the answer.

For problems 1 – 4 determine the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

1. Rotate \( x = t^2 - 3 \) \quad \( y = 2 + t^2 \) \quad \( 0 \leq t \leq 5 \) about the x-axis.

2. Rotate \( x = -8t \) \quad \( y = 6 + t^2 \) \quad \( -3 \leq t \leq 0 \) about the y-axis.

3. Rotate \( x = t^2 \) \quad \( y = t^4 - 2 \) \quad \( 0 \leq t \leq 2 \) about the y-axis.

4. Rotate \( x = 2 + t \) \quad \( y = 4e^{-\frac{t}{4}} \) \quad \( -1 \leq t \leq 2 \) about the x-axis.

For problems 5 – 7 set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

5. Rotate \( x = 2 + e^{\cos(t)} \) \quad \( y = 1 + t^2 \) \quad \( -2 \leq t \leq 0 \) about the x-axis.

6. Rotate \( x = \cos^2(t) \) \quad \( y = 2 \cos(2t) - \sin(t) \) \quad \( 0 \leq t \leq 1 \) about the y-axis.

7. Rotate \( x = t^2 \) \quad \( y = \ln\left(3 + e^{-t}\right) \) \quad \( 0 \leq t \leq 2 \) about the x-axis.

**Polar Coordinates**
1. For the point with polar coordinates \((9, \frac{3\pi}{7})\) determine three different sets of coordinates for the same point all of which have angles different from \(-\frac{2\pi}{3}\) and are in the range \(-2\pi \leq \theta \leq 2\pi\).

2. For the point with polar coordinates \((7, -\frac{4\pi}{7})\) determine three different sets of coordinates for the same point all of which have angles different from \(\frac{3\pi}{7}\) and are in the range \(-2\pi \leq \theta \leq 2\pi\).

3. The polar coordinates of a point are \((14, 2.48)\). Determine the Cartesian coordinates for the point.

4. The polar coordinates of a point are \((-\frac{1}{10}, -5.29)\). Determine the Cartesian coordinates for the point.

5. The Cartesian coordinate of a point are \((-3, 5)\). Determine a set of polar coordinates for the point.

6. The Cartesian coordinate of a point are \((4, -7)\). Determine a set of polar coordinates for the point.

7. The Cartesian coordinate of a point are \((-3, -12)\). Determine a set of polar coordinates for the point.

For problems 8 and 9 convert the given equation into an equation in terms of polar coordinates.

8. \(7x^2y + 8y = 3 - 6x^2 - 6y^2\)

9. \(\frac{7y}{x^2 + y^2 - 8x} = 9 + y^2\)

For problems 10 – 13 convert the given equation into an equation in terms of Cartesian coordinates.

10. \(r - \frac{8\sin \theta}{r} = 2\cos \theta\)

11. \(r^3 \csc \theta = 5\cos \theta - 6\)

12. \(8 - r = r^2 \sin (2\theta)\)

13. \(r = 2a \cos \theta + 2b \sin \theta\)

For problems 14 – 27 sketch the graph of the given polar equation.

14. \(-7 = r \sin \theta\)

15. \(\theta = \frac{5\pi}{7}\)
16. $\theta = -\frac{9\pi}{5}$

17. $r \cos \theta = 4$

18. $r = 6 \sin \theta$

19. $r = 100$

20. $r = 24 \cos \theta$

21. $r = -15 \sin \theta$

22. $r = 4 + 12 \cos \theta$

23. $r = 7 - 7 \sin \theta$

24. $r = 1 + 3 \sin \theta$

25. $r = 5 - 4 \cos \theta$

26. $r = 8 + 3 \sin \theta$

27. $r = 1 - \cos \theta$

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**Tangents with Polar Coordinates**

1. Find the tangent line to $r = \theta \sin (3\theta)$ at $\theta = \frac{\pi}{2}$.

2. Find the tangent line to $r = \cos (2\theta) - \sin (\theta)$ at $\theta = -\frac{\pi}{4}$.

3. Find the tangent line to $r = \cos (\theta^2 - \theta)$ at $\theta = \pi$.

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**Area with Polar Coordinates**

1. Find the area inside the inner loop of $r = 3 + 10 \sin \theta$.

2. Find the area inside the inner loop of $r = 5 + 12 \cos \theta$. 
3. Find the area inside the graph of $r = 8 + \cos \theta$ and to the right of the $y$-axis.

4. Find the area inside the graph of $r = 5 - 4 \sin \theta$ and the below the $x$-axis.

5. Find the area that is inside $r = 4$ and outside $r = 4 - 2 \sin \theta$.

6. Find the area that is inside $r = 7 - 3 \cos \theta$ and outside $r = 4$.

7. Find the area that is inside $r = 6 + 6 \cos \theta$ and outside $r = 4 - 3 \cos \theta$.

8. Find the area that is inside $r = 4 + 2 \sin \theta$ and outside $r = 5 - \sin \theta$.

9. Find the area that is inside $r = 5 - \sin \theta$ and outside $r = 4 + 2 \sin \theta$.

10. Find the area that is inside both $r = 6 - 4 \sin \theta$ and $r = 5$.

11. Find the area that is inside both $r = 3 + 2 \cos \theta$ and $r = 3 - \cos \theta$.

**Arc Length with Polar Coordinates**

For problems 1 – 3 determine the length of the given polar curve. For these problems you may assume that the curve traces out exactly once for the given range of $\theta$.

1. $r = \frac{1}{\cos \theta}$, $0 \leq \theta \leq \frac{\pi}{3}$

2. $r = \theta^2$, $0 \leq \theta \leq 3\pi$

3. $r = 6 \cos \theta - 3 \sin \theta$, $0 \leq \theta \leq \pi$

For problems 4 – 6 set up, but do not evaluate, an integral that gives the length of the given polar curve. For these problems you may assume that the curve traces out exactly once for the given range of $\theta$.

4. $r = \sin(\theta^2)$, $0 \leq \theta \leq \pi$

5. $r = \cos(1 + \sin \theta)$, $0 \leq \theta \leq 2\pi$

6. $r = e^{-\frac{1}{4} \theta} \cos \theta$, $0 \leq \theta \leq 3\pi$
Surface Area with Polar Coordinates

For problems 1 – 4 set up, but do not evaluate, an integral that gives the surface area of the curve rotated about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of $\theta$.

1. $r = \cos\left(e^{-\frac{1}{2}\theta}\right)$, $0 \leq \theta \leq \frac{\pi}{2}$ rotated about the $x$-axis.

2. $r = \theta \sin \theta$, $0 \leq \theta \leq \frac{\pi}{2}$ rotated about the $y$-axis.

3. $r = \cos(\theta)\sin(2\theta)$, $0 \leq \theta \leq \frac{\pi}{6}$ rotated about the $x$-axis.

4. $r = \theta + \sin \theta$, $\frac{\pi}{2} \leq \theta \leq \pi$ rotated about the $y$-axis.

Arc Length and Surface Area Revisited

Problems have not yet been written for this section and probably won’t be to be honest since this is just a summary section.