Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Three Dimensional Space

Introduction

Here are a set of problems for which no solutions are available. The main intent of these problems is to have a set of problems available for any instructors who are looking for some extra problems.

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Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.

Here is a list of topics in this chapter that have problems written for them.

The 3-D Coordinate System – No problems written yet.
Equations of Lines – No problems written yet.
Equations of Planes – No problems written yet.
Quadric Surfaces – No problems written yet.
Functions of Several Variables – No problems written yet.
Vector Functions – No problems written yet.
Calculus with Vector Functions – No problems written yet.
Tangent, Normal and Binormal Vectors – No problems written yet.
Arc Length with Vector Functions – No problems written yet.
Curvature – No problems written yet.
Velocity and Acceleration – No problems written yet.
Cylindrical Coordinates – No problems written yet.
Spherical Coordinates – No problems written yet

The 3-D Coordinate System

1. Give the projection of \( P = (3, -4, 6) \) onto the three coordinate planes.

Solution
There really isn’t a lot to do with this problem. We know that the \( xy \)-plane is given by the equation \( z = 0 \) and so the projection into the \( xy \)-plane for any point is simply found by setting the \( z \) coordinate to zero. Similarly for the other two coordinate planes.

So, the projects are then,

\[
xy \text{- plane : } (3, -4, 0) \\
xz \text{- plane : } (3, 0, 6) \\
yz \text{- plane : } (0, -4, 6)
\]

2. Which of the points \( P = (4, -2, 6) \) and \( Q = (-6, -3, 2) \) is closest to the \( yz \)-plane?

Step 1
The shortest distance between any point and any of the coordinate planes will be the distance between that point and its projection onto that plane.
Let’s call the projections of \( P \) and \( Q \) onto the \( yz \)-plane \( \overline{P} \) and \( \overline{Q} \) respectively. They are,

\[
\overline{P} = (0, -2, 6) \quad \quad \overline{Q} = (0, -3, 2)
\]

Step 2
To determine which of these is closest to the \( yz \)-plane we just need to compute the distance between the point and its projection onto the \( yz \)-plane.

Note as well that because only the \( x \)-coordinate of the two points are different the distance between the two points will just be the absolute value of the difference between two \( x \) coordinates.

Therefore,

\[
d(P, \overline{P}) = 4 \quad \quad d(Q, \overline{Q}) = 6
\]

Based on this is should be pretty clear that \( P = (4, -2, 6) \) is closest to the \( yz \)-plane.

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3. Which of the points \( P = (-1, 4, -7) \) and \( Q = (6, -1, 5) \) is closest to the \( z \)-axis?

Step 1
First, let’s note that the coordinates of any point on the \( z \)-axis will be \((0, 0, z)\).

Also the shortest distance from any point not on the \( z \)-axis to the \( z \)-axis will occur if we draw a line straight from the point to the \( z \)-axis in such a way that it forms a right angle with the \( z \)-axis.

So, if we start with any point not on the \( z \)-axis, say \((x_i, y_i, z_i)\), the point on the \( z \)-axis that will be closest to this point is \((0, 0, z_i)\).

Let’s call the point closest to \( P \) and \( Q \) on the \( z \)-axis closest to be \( \overline{P} \) and \( \overline{Q} \) respectively. They are,

\[
\overline{P} = (0, 0, -7) \quad \quad \overline{Q} = (0, 0, 5)
\]

Step 2
To determine which of these is closest to the \( yz \)-plane we just need to compute the distance between the point and its projection onto the \( yz \)-plane.

The distances are,

\[
d(P, \overline{P}) = \sqrt{(-1-0)^2 + (4-0)^2 + (-7-(-7))^2} = \sqrt{17}
\]
\[
d(Q, \overline{Q}) = \sqrt{(6-0)^2 + (-1-0)^2 + (5-5)^2} = \sqrt{37}
\]
Based on this is should be pretty clear that $P = (-1, 4, -7)$ is closest to the $z$-axis.

4. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$7x^2 - 9y^3 = 3x + 1$$

Solution
First notice that because there are two variables in this equation it cannot have a graph in $\mathbb{R}$ since equations in that coordinate system can only have a single variable.

There are two variables in the equation so we know that it will have a graph in $\mathbb{R}^2$.

Likewise, the fact that the equation has two variables means that it will also have a graph in $\mathbb{R}^3$. Although in this case the third variable, $z$, can have any value.

5. List all of the coordinates systems ($\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$) that the following equation will have a graph in. Do not sketch the graph.

$$x^3 + \sqrt{y^2 + 1} - 6z = 2$$

Solution
This equation has three variables and so it will have a graph in $\mathbb{R}^3$.

On other hand because the equation has three variables in it there will be no graph in $\mathbb{R}^2$ (can have at most two variables) and it will not have a graph in $\mathbb{R}$ (can only have a single variable).

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**Equations of Lines**

1. Give the equation of the line through the points $(2, -4, 1)$ and $(0, 4, -10)$ in vector form, parametric form and symmetric form.

Step 1
Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.
We have two points that are on the line. We can use either point and depending on your choice of points you may have different answers that we get here. We will use the first point listed above for our point for no other reason that it is the first one listed.

The parallel vector is really simple to get as well since we can always form the vector from the first point to the second point and this vector will be on the line and so will also be parallel to the line. The vector is,

$$\vec{v} = \langle -2, 8, -11 \rangle$$

Step 2
The vector form of the line is,

$$\vec{r}(t) = \langle 2, -4, 1 \rangle + t \langle -2, 8, -11 \rangle = \langle 2 - 2t, -4 + 8t, 1 - 11t \rangle$$

Step 3
The parametric form of the line is,

$$x = 2 - 2t, \quad y = -4 + 8t, \quad z = 1 - 11t$$

Step 4
To get the symmetric form all we need to do is solve each of the parametric equations for $t$ and then set them all equal to each other. Doing this gives,

$$\frac{2-x}{2} = \frac{4+y}{8} = \frac{1-z}{11}$$

2. Give the equation of the line through the point $(-7, 2, 4)$ and parallel to the line given by $x = 5 - 8t$, $y = 6 + t$, $z = -12t$ in vector form, parametric form and symmetric form.

Step 1
Okay, regardless of the form of the equation we know that we need a point on the line and a vector that is parallel to the line.

We were given a point on the line so no need to worry about that for this problem.

The parallel vector is really simple to get as well since we were told that the new line must be parallel to the given line. We also know that the coefficients of the $t$’s in the equation of the line forms a vector parallel to the line.

So,

$$\vec{v} = \langle -8, 1, -12 \rangle$$
is a vector that is parallel to the given line.

Now, if \( \vec{v} \) is parallel to the given line and the new line must be parallel to the given line then \( \vec{v} \) must also be parallel to the new line.

Step 2
The vector form of the line is,
\[
\vec{r}(t) = \langle -7, 2, 4 \rangle + t \langle -8, 1, -12 \rangle = \langle -7 - 8t, 2 + t, 4 - 12t \rangle
\]

Step 3
The parametric form of the line is,
\[
x = -7 - 8t \quad y = 2 + t \quad z = 4 - 12t
\]

Step 4
To get the symmetric form all we need to do is solve each of the parametric equations for \( t \) and then set them all equal to each other. Doing this gives,
\[
\frac{-7 - x}{8} = \frac{y - 2}{12} = \frac{z - 4}{12}
\]

3. Is the line through the points \((2, 0, 9)\) and \((-4, 1, -5)\) parallel, orthogonal or neither to the line given by \( \vec{r}(t) = \langle 5, 1 - 9t, -8 - 4t \rangle \)?

Step 1
Let’s start this off simply by getting vectors parallel to each of the lines.

For the line through the points \((2, 0, 9)\) and \((-4, 1, -5)\) we know that the vector between these two points will lie on the line and hence be parallel to the line. This vector is,
\[
\vec{v}_1 = \langle 6, -1, 14 \rangle
\]

For the second line the coefficients of the \( t \)'s are the components of the parallel vector so this vector is,
\[
\vec{v}_2 = \langle 0, -9, -4 \rangle
\]

Step 2
Now, from the first components of these vectors it is hopefully clear that they are not scalar multiples. There is no number we can multiply to zero to get 6.
Likewise, we can only multiply 6 by zero to get 0. However, if we multiply the first vector by zero all the components would be zero and that is clearly not the case.

Therefore they are not scalar multiples and so these two vectors are not parallel. This also in turn means that the two lines can’t possibly be parallel either (since each vector is parallel to its respective line).

Step 3
Next,

\[ \mathbf{v}_1 \cdot \mathbf{v}_2 = -47 \]

The dot product is not zero and so these vectors aren’t orthogonal. Because the two vectors are parallel to their respective lines this also means that the two lines are not orthogonal.

4. Determine the intersection point of the line given by \( x = 8 + t, \ y = 5 + 6t, \ z = 4 - 2t \) and the line given by \( \mathbf{r}(t) = \langle -7 + 12t, 3 - t, 14 + 8t \rangle \) or show that they do not intersect.

Step 1
If the two lines do intersect then they must share a point in common. In other words there must be some value, say \( t = t_1 \), and some (probably) different value, say \( t = t_2 \), so that if we plug \( t_1 \) into the equation of the first line and if we plug \( t_2 \) into the equation of the second line we will get the same \( x, y \) and \( z \) coordinates.

Step 2
This means that we can set up the following system of equations.

\[
\begin{align*}
8 + t_1 &= -7 + 12t_2 \\
5 + 6t_1 &= 3 - t_2 \\
4 - 2t_1 &= 14 + 8t_2
\end{align*}
\]

If this system of equations has a solution then the lines will intersect and if it doesn’t have a solution then the lines will not intersect.

Step 3
Solving a system of equations with more equations than unknowns is probably not something that you’ve run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let’s just take the first two equations. We’ll worry about the third equation eventually.

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we’ll not put in any real explanation to the solution work below.
Calculus II

\[ t_1 = -15 + 12t_2 \quad \rightarrow \quad 5 + 6(-15 + 12t_2) = 3 - t_2 \]
\[ -85 + 72t_2 = 3 - t_2 \]
\[ 73t_2 = 88 \quad \rightarrow \quad t_2 = \frac{88}{73} \]
\[ t_1 = -15 + 12\left(\frac{88}{73}\right) = -\frac{39}{73} \]

Step 4
Okay, this is a somewhat “messy” solution, but they will often be that way so we shouldn’t get too excited about it!

Now, recall that to get this solution we used the first two equations. What this means is that if we use this value of \( t_1 \) and \( t_2 \) in the equations of the first and second lines respectively then the \( x \) and \( y \) coordinates will be the same (remember we used the \( x \) and \( y \) equations to find this solution…).

At this point we need to recall that we did have a third equation that also needs to be satisfied at these values of \( t \). In other words we need to plug \( t_1 = -\frac{39}{73} \) and \( t_2 = \frac{88}{73} \) into the third equation and see what we get. Doing this gives,

\[ \frac{370}{73} = 4 - 2\left(-\frac{39}{73}\right) \neq 14 + 8\left(\frac{88}{73}\right) = \frac{1726}{73} \]

Okay, the two sides are not the same. Just what does this mean? In terms of systems of equations it means that \( t_1 = -\frac{39}{73} \) and \( t_2 = \frac{88}{73} \) are NOT a solution to the system of equations in Step 2. Had they been a solution then we would have gotten the same number from both sides.

In terms of whether or not the lines intersect we need to only recall that the third equation corresponds to the \( z \) coordinates of the lines. We know that at \( t_1 = -\frac{39}{73} \) and \( t_2 = \frac{88}{73} \) the two lines will have the same \( x \) and \( y \) coordinates (since they came from solving the first two equations). However, we’ve just shown that they will not give the same \( z \) coordinate.

In other words, there are no values of \( t_1 \) and \( t_2 \) for which the two lines will have the same \( x \), \( y \) and \( z \) coordinates. Hence we can now say that the two lines do not intersect.

Before leaving this problem let’s note that it doesn’t matter which two equations we solve in Step 3. Different sets of equations will lead to different values of \( t_1 \) and \( t_2 \) but they will still not satisfy the remaining equation for this problem and we will get the same result of the lines not intersecting.

5. Determine the intersection point of the line through the points \((1, -2, 13)\) and \((2, 0, -5)\) and the line given by \( \vec{r}(t) = \langle 2 + 4t, -1 - t, 3 \rangle \) or show that they do not intersect.

Step 1
Calculus II

Because we don’t have the equation for the first line that will be the first thing we’ll need to do. The vector between the two points (and hence parallel to the line) is,

\[ \vec{v} = \langle 1, 2, -18 \rangle \]

Using the first point listed the equation of the first line is then,

\[ \vec{r}(t) = \langle 1, -2, 13 \rangle + t \langle 1, 2, -18 \rangle = \langle 1 + t, -2 + 2t, 13 - 18t \rangle \]

Step 2
If the two lines do intersect then they must share a point in common. In other words there must be some value, say \( t = t_1 \), and some (probably) different value, say \( t = t_2 \), so that if we plug \( t_1 \) into the equation of the first line and if we plug \( t_2 \) into the equation of the second line we will get the same \( x \), \( y \) and \( z \) coordinates.

Step 3
This means that we can set up the following system of equations.

\[
1 + t_1 = 2 + 4t_2 \\
-2 + 2t_1 = -1 - t_2 \\
13 - 18t_1 = 3
\]

If this system of equations has a solution then the lines will intersect and if it doesn’t have a solution then the lines will not intersect.

Step 4
Solving a system of equations with more equations than unknowns is probably not something that you’ve run into all that often to this point. The basic process is pretty much the same however with a couple of minor (but very important) differences.

Start off by picking any two of the equations (so we now have two equations and two unknowns) and solve that system. For this problem let’s just take the first and third equation. We’ll worry about the second equation eventually.

Note that for this system the third equation should definitely be used here since we can quickly just solve that for \( t_1 \).

Solving a system of two equations and two unknowns is something everyone should be familiar with at this point so we’ll not put in any real explanation to the solution work below.

\[ t_1 = \frac{5}{9} \quad \rightarrow \quad 1 + \frac{5}{9} = 2 + 4t_2 \quad \rightarrow \quad t_2 = -\frac{1}{9} \]

Step 5
Now, recall that to get this solution we used the first and third equations. What this means is that if we use this value of \( t_1 \) and \( t_2 \) in the equations of the first and second lines respectively then the \( x \) and \( z \) coordinates will be the same (remember we used the \( x \) and \( z \) equations to find this solution….).
At this point we need to recall that we did have another equation that also needs to be satisfied at these values of $t$. In other words we need to plug $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ into the second equation and see what we get. Doing this gives,

$$-2 + 2\left(\frac{5}{9}\right) = -\frac{8}{9} = -1 - \left(-\frac{1}{9}\right)$$

Okay, the two sides are the same. Just what does this mean? In terms of systems of equations it means that $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ is a solution to the system of equations in Step 3.

In terms of whether or not the lines intersect we now know that at $t_1 = \frac{5}{9}$ and $t_2 = -\frac{1}{9}$ the two lines will have the same $x$, $y$ and $z$ coordinates (since they satisfy all three equations).

In other words these two lines do intersect.

Before leaving this problem let’s note that it doesn’t matter which two equations we solve in Step 4. Different sets of equations will lead to the same values of $t_1$ and $t_2$ leading to the two lines intersecting.

6. Does the line given by $x = 9 + 21t$, $y = -7$, $z = 12 - 11t$ intersect the $xy$-plane? If so, give the point.

Step 1
If the line intersects the $xy$-plane there will be a point on the line that is also in the $xy$-plane. Recall as well that any point in the $xy$-plane will have a $z$ coordinate of $z = 0$.

Step 2
So, to determine if the line intersects the $xy$-plane all we need to do is set the equation for the $z$ coordinate equal to zero and solve it for $t$, if that’s possible.

Doing this gives,

$$12 - 11t = 0 \quad \rightarrow \quad t = \frac{12}{11}$$

Step 3
So, we were able to solve for $t$ in this case and so we can now say that the line does intersect the $xy$-plane.

Step 4
All we need to do to finish this off this problem is find the full point of intersection. We can find this simply by plugging $t = \frac{12}{11}$ into the $x$ and $y$ portions of the equation of the line.

Doing this gives,

$$x = 9 + 21\left(\frac{12}{11}\right) = \frac{351}{11} \quad \quad y = -7$$
The point of intersection is then: \( \left( \frac{351}{11}, -7, 0 \right) \).

7. Does the line given by \( x = 9 + 21t, \ y = -7, \ z = 12 - 11t \) intersect the \( xz \)-plane? If so, give the point.

Step 1
If the line intersects the \( xz \)-plane there will be a point on the line that is also in the \( xz \)-plane. Recall as well that any point in the \( xz \)-plane will have a \( y \) coordinate of \( y = 0 \).

Step 2
So, to determine if the line intersects the \( xz \)-plane all we need to do is set the equation for the \( y \) coordinate equal to zero and solve it for \( t \), if that’s possible.

However, in this case we can see that is clearly not possible since the \( y \) equation is simply \( y = -7 \) and this can clearly never be zero.

Step 3
Therefore, the line does not intersect the \( xz \)-plane.

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**Equations of Planes**

1. Write down the equation of the plane containing the points \( (4, -3, 1), \ (-3, -1, 1) \) and \( (4, -2, 8) \).

Step 1
To make the work on this problem a little easier let’s “name” the points as,

\[
P = (4, -3, 1) \quad Q = (-3, -1, 1) \quad R = (4, -2, 8)
\]

Now, we know that in order to write down the equation of a plane we’ll need a point (we have three so that’s not a problem!) and a vector that is normal to the plane.

Step 2
We’ll need to do a little work to get a normal vector.

First, we’ll need two vectors that lie in the plane and we can get those from the three points we’re given. Note that there are lots of possible vectors that we could use here. Here are the two that we’ll use for this problem.
Calculus II

\[ \overrightarrow{PQ} = \langle -7, 2, 0 \rangle \quad \overrightarrow{PR} = \langle 0, 1, 7 \rangle \]

Step 3
Now, these two vectors lie in the plane and we know that the cross product of any two vectors will be orthogonal to both of the vectors. Therefore the cross product of these two vectors will also be orthogonal (and so normal!) to the plane.

So, let’s get the cross product.

\[
\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
7 & 2 & 0 \\
0 & 1 & 7
\end{vmatrix} = -7 \vec{i} - 7 \vec{k} - (14 \vec{j} - 7 \vec{k}) = 14 \vec{i} + 49 \vec{j} - 7 \vec{k}
\]

Note that we used the “trick” discussed in the notes to compute the cross product here.

Step 4
Now all we need to do is write down the equation.

We have three points to choose from here. We’ll use the first point simply because it is the first point listed. Any of the others could also be used.

The equation of the plane is,

\[
14(x - 4) + 49(y + 3) - 7(z - 1) = 0 \quad \rightarrow \quad 14x + 49y - 7z = -98
\]

Note that depending on your choice of vectors in Step 2, the order you chose to use them in the cross product computation in Step 3 and the point chosen here will all affect your answer. However, regardless of your choices the equation you get will be an acceptable answer provided you did all the work correctly.

2. Write down the equation of the plane containing the point \((3, 0, -4)\) and orthogonal to the line given by \(\vec{r}(t) = \langle 12 - t, 1 + 8t, 4 + 6t \rangle\).

Step 1
We know that we need a point on the plane and a vector that is normal to the plane. We’ve were given a point that is in the plane so we’re okay there.

Step 2
The normal vector for the plane is actually quite simple to get.

We are told that the plane is orthogonal to the line given in the problem statement. This means that the plane will also be orthogonal to any vector that just happens to be parallel to the line.
Calculus II

From the equation of the line we know that the coefficients of the $t$'s are the components of a vector that is parallel to the line. So, a vector parallel to the line is then,

$$
\vec{v} = \langle -1, 8, 6 \rangle
$$

Now, as mentioned above because this vector is parallel to the line then it will also need to be orthogonal to the plane and hence be normal to the plane. So, a normal vector for the plane is,

$$
\vec{n} = \langle -1, 8, 6 \rangle
$$

Step 3
Now all we need to do is write down the equation. The equation of the plane is,

$$
- (x - 3) + 8(y - 0) + 6(z + 4) = 0 \quad \rightarrow \quad - x + 8y + 6z = -27
$$

3. Write down the equation of the plane containing the point $(-8, 3, 7)$ and parallel to the plane given by $4x + 8y - 2z = 45$.

Step 1
We know that we need a point on the plane and a vector that is normal to the plane. We’ve were given a point that is in the plane so we’re okay there.

Step 2
The normal vector for the plane is actually quite simple to get.

We are told that the plane is parallel to the plane given in the problem statement. This means that any vector normal to one plane will be normal to both planes.

From the equation of the plane we were given we know that the coefficients of the $x, y$ and $z$ are the components of a vector that is normal to the plane. So, a vector normal to the given plane is then,

$$
\vec{n} = \langle 4, 8, -2 \rangle
$$

Now, as mentioned above because this vector is normal to the given plane then it will also need to be normal to the plane we want to find the equation for.

Step 3
Now all we need to do is write down the equation. The equation of the plane is,

$$
4(x + 8) + 8(y - 3) - 2(z - 7) = 0 \quad \rightarrow \quad 4x + 8y - 2z = -22
$$
4. Determine if the plane given by \(4x - 9y - z = 2\) and the plane given by \(x + 2y - 14z = -6\) are parallel, orthogonal or neither.

Step 1
Let’s start off this problem by noticing that the vector \(\vec{n}_1 = \langle 4, -9, -1 \rangle\) will be normal to the first plane and the vector \(\vec{n}_2 = \langle 1, 2, -14 \rangle\) will be normal to the second plane.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors were parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2
If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let’s take a quick look at the normal vectors. We can see that the first component of each vector have the same sign and the same can be said for the third component. However, the second component of each vector has opposite signs.

Therefore, there is no number that we can multiply to \(\vec{n}_1\) that will keep the sign on the first and third component the same and simultaneously changing the sign on the second component. This in turn means the two vectors can’t possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can’t be parallel then the two planes are not parallel.

Step 3
Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

\[
\vec{n}_1 \cdot \vec{n}_2 = 0
\]

The dot product is zero and so the two normal vectors are orthogonal. Therefore the two planes are orthogonal.

5. Determine if the plane given by \(-3x + 2y + 7z = 9\) and the plane containing the points \((-2, 6, 1)\), \((-2, 5, 0)\) and \((-1, 4, -3)\).

Step 1
Let’s start off this problem by noticing that the vector \(\vec{n}_1 = \langle -3, 2, 7 \rangle\) will be normal to the first plane and it would be nice to have a normal vector for the second plane.
We know (Problem 1 from this section!) how to determine a normal vector given three points in the plane. Here is that work.

\[
P = (-2, 6, 1) \quad Q = (-2, 5, 0) \quad R = (-1, 4, -3)
\]

\[
\overrightarrow{QP} = (0, 1, 1) \quad \overrightarrow{RQ} = (1, 1, 3)
\]

\[
\vec{n}_2 = \overrightarrow{QP} \times \overrightarrow{RQ} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\
0 & 1 & 1 \\
-1 & 1 & 3 \end{vmatrix} = 3\hat{i} - \hat{j} - (\hat{k}) = 2\hat{i} - \hat{j} + \hat{k}
\]

Note that we used the “trick” discussed in the notes to compute the cross product here.

Now try to visualize the two planes and these normal vectors. What would the two planes look like if the two normal vectors where parallel to each other? What would the two planes look like if the two normal vectors were orthogonal to each other?

Step 2
If the two normal vectors are parallel to each other the two planes would also need to be parallel.

So, let’s take a quick look at the normal vectors. We can see that the third component of each vector have the same sign while the first and second components each have opposite signs.

Therefore, there is no number that we can multiply to \( \vec{n}_1 \) that will keep the sign on the third component the same and simultaneously changing the sign on the first and second components. This in turn means the two vectors can’t possibly be scalar multiples and this further means they cannot be parallel.

If the two normal vectors can’t be parallel then the two planes are not parallel.

Step 3
Now, if the two normal vectors are orthogonal the two planes will also be orthogonal.

So, a quick dot product of the two normal vectors gives,

\[
\vec{n}_1 \cdot \vec{n}_2 = -1
\]

The dot product is not zero and so the two normal vectors are not orthogonal. Therefore the two planes are not orthogonal.

6. Determine if the line given by \( \vec{r}(t) = \langle -2t, 2 + 7t, -1 - 4t \rangle \) intersects the plane given by \( 4x + 9y - 2z = -8 \) or show that they do not intersect.

Step 1
If the line and the plane do intersect then there must be a value of \( t \) such that if we plug that \( t \) into the equation of the line we’d get a point that lies on the plane. We also know that if a point \( (x, y, z) \) is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2
If you think about it the coordinates of all the points on the line can be written as,

\[
\left( -2t, 2 + 7t, -1 - 4t \right)
\]

for all values of \( t \).

Step 3
So, let’s plug the “coordinates” of the points on the line into the equation of the plane to get,

\[
4(-2t) + 9(2 + 7t) - 2(-1 - 4t) = -8
\]

Step 4
Let’s solve this for \( t \) as follows,

\[
63t + 20 = -8 \quad \rightarrow \quad t = -\frac{4}{9}
\]

Step 5
We were able to find a \( t \) from this equation. What that means is that this is the value of \( t \) that, once we plug into the equation of the line, gives the point of intersection of the line and plane.

So, **the line and plane do intersect** and they will intersect at the point \( \left( \frac{4}{9}, -\frac{10}{9}, \frac{2}{9} \right) \).

Note that all we did to get the point is plug \( t = -\frac{4}{9} \) into the general form for points on the line we wrote down in Step 2.

7. Determine if the line given by \( \vec{r}(t) = \left\langle 4 + t, -1 + 8t, 3 + 2t \right\rangle \) intersects the plane given by \( 2x - y + 3z = 15 \) or show that they do not intersect.

Step 1
If the line and the plane do intersect then there must be a value of \( t \) such that if we plug that \( t \) into the equation of the line we’d get a point that lies on the plane. We also know that if a point \( (x, y, z) \) is on the plane the then the coordinates will satisfy the equation of the plane.

Step 2
If you think about it the coordinates of all the points on the line can be written as,

\[
\left( 4 + t, -1 + 8t, 3 + 2t \right)
\]
for all values of $t$.

Step 3
So, let’s plug the “coordinates” of the points on the line into the equation of the plane to get,

$$2(4+t) - (-1+8t) + 3(3+2t) = 15$$

Step 4
Let’s solve this for $t$ as follows,

$$18 = 15 ??$$

Step 5
Hmmm… So either we’ve just managed to prove that 18 and 15 are in fact the same number or there is something else going on here.

Clearly 18 and 15 are not the same number and so something else must be going on. In fact, all this means is that there is no $t$ that will satisfy the equation we wrote down in Step 3. This in turn means that the line and plane do not intersect.

8. Find the line of intersection of the plane given by $3x + 6y - 5z = -3$ and the plane given by $-2x + 7y - z = 24$.

Step 1
Okay, we know that we need a point and vector parallel to the line in order to write down the equation of the line. In this case neither has been given to us.

First let’s note that any point on the line of intersection must also therefore be in both planes and it’s actually pretty simple to find such a point. Whatever our line of intersection is it must intersect at least one of the coordinate planes. It doesn’t have to intersect all three of the coordinate planes but it will have to intersect at least one.

So, let’s see if it intersects the $xy$-plane. Because the point on the intersection line must also be in both planes let’s set $z = 0$ (so we’ll be in the $xy$-plane!) in both of the equations of our planes.

Doing this gives,

$$3x + 6y = -3$$
$$-2x + 7y = 24$$

Step 2
This is a simple system to solve so we’ll leave it verify that the solution is,

$$x = -5, \quad y = 2$$
The fact that we were able to find a solution to the system from Step 1 means that the line of intersection does in fact intersect the \( xy \)-plane and it does so at the point \((-5,2,0)\). This is also then a point on the line of intersection.

Note that if the system from Step 1 didn’t have a solution then the line of intersection would not have intersected the \( xy \)-plane and we’d need to try one of the remaining coordinate planes.

Step 3
Okay, now we need a vector that is parallel to the line of intersection. This might be a little hard to visualize, but if you think about it the line of intersection would have to be orthogonal to both of the normal vectors from the two planes. This in turn means that any vector orthogonal to the two normal vectors must then be parallel to the line of intersection.

Nicely enough we know that the cross product of any two vectors will be orthogonal to each of the two vectors. So, here are the two normal vectors for our planes and their cross product.

\[
\vec{n}_1 = \langle 3, 6, -5 \rangle \quad \quad \vec{n}_2 = \langle -2, 7, -1 \rangle
\]

\[
\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
3 & 6 & -5 \\
-2 & 7 & -1
\end{vmatrix} = -6\vec{i} + 10\vec{j} + 21\vec{k}
\]

\[
= -6\vec{i} + 10\vec{j} + 21\vec{k} = -3\vec{j} - 35\vec{i} - 12\vec{k} = 29\vec{i} + 13\vec{j} + 33\vec{k}
\]

Note that we used the “trick” discussed in the notes to compute the cross product here.

Step 4
So, we now have enough information to write down the equation of the line of intersection of the two planes. The equation is,

\[
\vec{r}(t) = \langle -5, 2, 0 \rangle + t \langle 29, 13, 33 \rangle = \langle -5 + 29t, 2 + 13t, 33t \rangle
\]

9. Determine if the line given by \( x = 8 - 15t \), \( y = 9t \), \( z = 5 + 18t \) and the plane given by \( 10x - 6y - 12z = 7 \) are parallel, orthogonal or neither.

Step 1
Let’s start off this problem by noticing that the vector \( \vec{v} = \langle -15, 9, 18 \rangle \) will be parallel to the line and the vector \( \vec{n} = \langle 10, -6, -12 \rangle \) will be normal to the plane.
Now try to visualize the line and plane and their corresponding vectors. What would the line and plane look like if the two vectors where orthogonal to each other? What would the line and plane look like if the two vectors were parallel to each other?

Step 2
If the two vectors are orthogonal to each other the line would be parallel to the plane. If you think about this it does make sense. If \( \vec{v} \) is orthogonal to \( \vec{n} \) then it must be parallel to the plane because \( \vec{n} \) is orthogonal to the plane. Then because the line is parallel to \( \vec{v} \) it must also be parallel to the plane.

So, let’s do a quick dot product here.

\[ \vec{v} \cdot \vec{n} = -420 \]

The dot product is not zero and so the two vectors aren’t orthogonal to each other. Therefore, the line and plane are not parallel.

Step 3
If the two vectors are parallel to each other the line would be orthogonal to the plane. If you think about this it does make sense. The line is parallel to \( \vec{v} \) which we’ve just assumed is parallel to \( \vec{n} \). We also know that \( \vec{n} \) is orthogonal to the plane and so anything that is parallel to \( \vec{n} \) (the line for instance) must also be orthogonal to the plane.

In this case it looks like we have the following relationship between the two vectors.

\[ \vec{v} = -\frac{2}{3} \vec{n} \]

The two vectors are parallel and so the line and plane are orthogonal.

---

**Quadric Surfaces**

1. Sketch the following quadric surface.

\[ \frac{y^2}{9} + z^2 = 1 \]

Solution
This is a cylinder that is centered on the \( x \)-axis. The cross sections of the cylinder will be ellipses.

Make sure that you can “translate” the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.
Here are a couple of sketches of the region. We’ve given them with the more traditional axes as well as “boxed” axes to help visualize the surface.
2. Sketch the following quadric surface.

\[ \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{6} = 1 \]

Solution
This is an ellipsoid and because the numbers in the denominators of each of the terms are not the same we know that it won’t be a sphere.

Here are a couple of sketches of the region. We’ve given them with the more traditional axes as well as “boxed” axes to help visualize the surface.
3. Sketch the following quadric surface.

\[ z = \frac{x^2}{4} + \frac{y^2}{4} - 6 \]

Solution
This is an elliptic paraboloid that is centered on the \( z \)-axis. Because the \( x \) and \( y \) terms are positive we know that it will open upwards. The “-6” tells us that the surface will start at \( z = -6 \). We can also say that because the coefficients of the \( x \) and \( y \) terms are identical the cross sections of the surface will be circles.

Here are a couple of sketches of the region. We’ve given them with the more traditional axes as well as “boxed” axes to help visualize the surface.
4. Sketch the following quadric surface.

\[ y^2 = 4x^2 + 16z^2 \]

Solution
This is a cone that is centered on the $y$-axis and because the coefficients of the $x$ and $z$ terms are different the cross sections of the surface will be ellipses.

Make sure that you can “translate” the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We’ve given them with the more traditional axes as well as “boxed” axes to help visualize the surface.
5. Sketch the following quadric surface.

\[ x = 4 - 5y^2 - 9z^2 \]

Solution
This is an elliptic paraboloid that is centered on the \( x \)-axis. Because the \( y \) and \( z \) terms are negative we know that it will open in the negative \( x \) direction. The “4” tells us that the surface will start at \( x = 4 \). We can also say that because the coefficients of the \( y \) and \( z \) terms are different the cross sections of the surface will be ellipses.

Make sure that you can “translate” the equations given in the notes to the other coordinate axes. Once you know what they look like when centered on one of the coordinates axes then a simple and predictable variable change will center them on the other coordinate axes.

Here are a couple of sketches of the region. We’ve given them with the more traditional axes as well as “boxed” axes to help visualize the surface.
1. Find the domain of the following function.

\[ f(x, y) = \sqrt{x^2 - 2y} \]

**Solution**

There really isn’t all that much to this problem. We know that we can’t have negative numbers under the square root and so the we’ll need to require that whatever \((x, y)\) is it will need to satisfy,

\[ x^2 - 2y \geq 0 \]

Let’s do a little rewriting on this so we can attempt to sketch the domain.

\[ x^2 \geq 2y \quad \Rightarrow \quad y \leq \frac{1}{2}x^2 \]

So, it looks like we need to be on or below the parabola above. The domain is illustrated by the green area and red line in the sketch below.
2. Find the domain of the following function.

\[ f(x, y) = \ln (2x - 3y + 1) \]

Solution

There really isn’t all that much to this problem. We know that we can’t have negative numbers or zero in a logarithm so we’ll need to require that whatever \((x, y)\) is it will need to satisfy,

\[ 2x - 3y + 1 > 0 \]

Since this is the only condition we need to meet this is also the domain of the function.

Let’s do a little rewriting on this so we can attempt to sketch the domain.

\[ 2x + 1 > 3y \quad \Rightarrow \quad y < \frac{2}{3}x + \frac{1}{3} \]

So, it looks like we need to be below the line above. The domain is illustrated by the green area in the sketch below.
Calculus II

Note that we dashed the graph of the “bounding” line to illustrate that we don’t take points from the line itself.

3. Find the domain of the following function.

\[ f(x, y, z) = \frac{1}{x^2 + y^2 + 4z} \]

Solution

There really isn’t all that much to this problem. We know that we can’t have division by zero so we’ll need to require that whatever \((x, y, z)\) is it will need to satisfy,

\[ x^2 + y^2 + 4z \neq 0 \]

Since this is the only condition we need to meet this is also the domain of the function.

Let’s do a little rewriting on this so we can attempt to identify the domain a little better.

\[ 4z \neq -x^2 - y^2 \quad \Rightarrow \quad z \neq \frac{-x^2 - y^2}{4} \]

So, it looks like we need to avoid points, \((x, y, z)\), that are on the elliptic paraboloid given by the equation above.

4. Find the domain of the following function.

\[ f(x, y) = \frac{1}{x} + \sqrt{y + 4} - \sqrt{x + 1} \]
Solution

There really isn’t all that much to this problem. We know that we can’t have division by zero and we can’t take square roots of negative numbers and so we’ll need to require that whatever \((x, y)\) is it will need to satisfy the following three conditions.

\[ x \geq -1, \; x \neq 0 \quad \text{and} \quad y \geq -4 \]

This is also our domain since these are the only conditions require in order for the function to exist.

A sketch of the domain is shown below. We can take any point in the green area or on the red lines with the exception of the \(y\)-axis (i.e. \(x \neq 0\)) as indicated by the black dashes on the \(y\)-axis.

5. Identify and sketch the level curves (or contours) for the following function.

\[ 2x - 3y + z^2 = 1 \]

Step 1

We know that level curves or contours are given by setting \(z = k\). Doing this in our equation gives,

\[ 2x - 3y + k^2 = 1 \]

Step 2

A quick rewrite of the equation from the previous step gives us,

\[ y = \frac{2}{3}x + \frac{k^2 - 1}{3} \]
So, the level curves for this function will be lines with slope \( \frac{2}{3} \) and a y-intercept of \( \left( 0, \frac{k^2 - 1}{3} \right) \).

Note as well that there will be no restrictions on the values of \( k \) that we can use, as there sometimes are. Also note that the sign of \( k \) will not matter so, with the exception of the level curve for \( k = 0 \), each level curve will in fact arise from two different values of \( k \).

Step 3
Below is a sketch of some level curves for some values of \( k \) for this function.

6. Identify and sketch the level curves (or contours) for the following function.

\[
4z + 2y^2 - x = 0
\]

Step 1
We know that level curves or contours are given by setting \( z = k \). Doing this in our equation gives,

\[
4k + 2y^2 - x = 0
\]

Step 2
A quick rewrite of the equation from the previous step gives us,

\[
x = 2y^2 + 4k
\]
So, the level curves for this function will be parabolas opening to the right and starting at $4k$.

Note as well that there will be no restrictions on the values of $k$ that we can use, as there sometimes are.

Step 3
Below is a sketch of some level curves for some values of $k$ for this function.

7. Identify and sketch the level curves (or contours) for the following function.

$$y^2 = 2x^2 + z$$

Step 1
We know that level curves or contours are given by setting $z = k$. Doing this in our equation gives,

$$y^2 = 2x^2 + k$$

Step 2
For this problem the value of $k$ will affect the type of graph of the level curve.

First, if $k = 0$ the equation will be,

$$y^2 = 2x^2 \quad \Rightarrow \quad y = \pm \sqrt{2} \, x$$
So, in this case the level curve (actually curves if you think about it) will be two lines through the origin. One is increasing and the other is decreasing.

Next, let’s take a look at what we get if \( k > 0 \). In this case a quick rewrite of the equation from Step 2 gives,

\[
\frac{y^2}{k} - \frac{2x^2}{k} = 1
\]

Because we know that \( k \) is positive we see that we have a hyperbola with the \( y \) term the positive term and the \( x \) term the negative term. This means that the hyperbola will be symmetric about the \( y \)-axis and opens up and down.

Finally, what do we get if \( k < 0 \). In this case the equation is,

\[
-\frac{2x^2}{k} + \frac{y^2}{k} = 1
\]

Now, be careful with this equation. In this case we have negative values of \( k \). This means that the \( x \) term is in fact positive (the minus sign will cancel against the minus sign in the \( k \)). Likewise, the \( y \) term will be negative (it’s got a negative \( k \) in the denominator). Therefore, we’ll have a hyperbola that is symmetric about the \( x \)-axis and opens right and left.

Step 3
Below is a sketch of some level curves for some values of \( k \) for this function.
8. Identify and sketch the traces for the following function.

\[ 2x - 3y + z^2 = 1 \]

**Step 1**
We have two traces. One we get by plugging \( x = a \) into the equation and the other we get by plugging \( y = b \) into the equation. Here is what we get for each of these.

\[
\begin{align*}
  x = a & : \\
  y = b & : \\
\end{align*}
\]

\[
\begin{align*}
  2a - 3y + z^2 &= 1 \\
  2x - 3b + z^2 &= 1 \\
\end{align*}
\]

\[
\begin{align*}
  y &= \frac{1}{3} z^2 + \frac{2a - 1}{3} \\
  x &= -\frac{1}{2} z^2 + \frac{3b - 1}{2} \\
\end{align*}
\]

**Step 2**
Okay, we’re now into a realm that many students have issues with initially. We no longer have equations in terms of \( x \) and \( y \). Instead we have one equation in terms of \( x \) and \( z \) and another in terms of \( y \) and \( z \).

Do not get excited about this! They work the same way that equations in terms of \( x \) and \( y \) work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.
Just because we have an \( x \) doesn’t mean that it must be the horizontal axis and just because we have a \( y \) doesn’t mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

In this case since both equation have a \( z \) in them and it is squared we’ll let \( z \) be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the \( x = a \) trace we’ll have a parabola that opens upwards with vertex at \( \left( 0, \frac{2a-1}{3} \right) \) and for the \( y = b \) trace we’ll have a parabola that opens downwards with vertex at \( \left( 0, \frac{3b+1}{2} \right) \).

Step 3
Below is a sketch for each of the traces.
9. Identify and sketch the traces for the following function.

\[ 4z + 2y^2 - x = 0 \]

Step 1
We have two traces. One we get by plugging \( x = a \) into the equation and the other we get by plugging \( y = b \) into the equation. Here is what we get for each of these.

\[
\begin{align*}
  x = a & : \quad 4z + 2y^2 - a = 0 \quad \rightarrow \quad z = -\frac{1}{2}y^2 + \frac{a}{4} \\
  y = b & : \quad 4z + 2b^2 - x = 0 \quad \rightarrow \quad x = 4z + 2b^2
\end{align*}
\]

Step 2
Okay, we’re now into a realm that many students have issues with initially. We no longer have equations in terms of \( x \) and \( y \). Instead we have one equation in terms of \( x \) and \( z \) and another in terms of \( y \) and \( z \).

Do not get excited about this! They work the same way that equations in terms of \( x \) and \( y \) work! The only difference is that we need to make a decision on which variable will be the horizontal axis variable and which variable will be the vertical axis variable.
Just because we have an $x$ doesn’t mean that it must be the horizontal axis and just because we have a $y$ doesn’t mean that it must be the vertical axis! We set up the axis variables in a way that will be convenient for us.

In this case since both equation have a $z$ in them we’ll let $z$ be the horizontal axis variable for both of the equations.

So, given that convention for the axis variables this means that for the $x = a$ trace we’ll have a parabola that opens to the left with vertex at $\left(\frac{a}{4},0\right)$ and for the $y = b$ trace we’ll have a line with slope of 4 and an $x$-intercept at $\left(0, 2b^2\right)$.

Step 3
Below is a sketch for each of the traces.
Vector Functions

1. Find the domain for the vector function: \( \mathbf{r}(t) = \left( t^2 + 1, \frac{1}{t+2}, \sqrt{t+4} \right) \)

Step 1
The domain of the vector function is simply the largest possible set of \( t \)'s that we can use in all the components of the vector function.

The first component will exist for all values of \( t \) and so we won’t exclude any values of \( t \) from that component.

The second component clearly requires us to avoid \( t = -2 \) so we don’t have division by zero in that component.

We’ll also need to require that \( t \geq -4 \) so avoid taking the square root of negative numbers in the third component.
Step 2
Putting all of the information from the first step together we can see that the domain of this function is,

\[ t \geq -4, \ t \neq -2 \]

Note that we can’t forget to add the \( t \neq -2 \) onto this since -2 is larger than -4 and would be included otherwise!

2. Find the domain for the vector function :

\[ \mathbf{r}(t) = \left\langle \ln(4-t^2), \sqrt{t+1} \right\rangle \]

Step 1
The domain of the vector function is simply the largest possible set of \( t \)'s that we can use in all the components of the vector function.

We know that we can’t take logarithms of negative values or zero and so from the first term we know that we’ll need to require that,

\[ 4-t^2 > 0 \quad \Rightarrow \quad -2 < t < 2 \]

We’ll also need to require that \( t \geq -1 \) so avoid taking the square root of negative numbers in the second component.

Step 2
Putting all of the information from the first step together we can see that the domain of this function is,

\[ -1 \leq t < 2 \]

Remember that we want the largest possible set of \( t \)'s for which all the components will exist.  So we can’t take values of \(-2 < t < -1\) because even though those are okay in the first component but they aren’t in the second component.  Likewise even though we can include \( t \geq 2 \) in the second component we can’t plug them into the first component and so we can’t include them in the domain of the function.

3. Sketch the graph of the vector function : \( \mathbf{r}(t) = \langle 4t, 10-2t \rangle \)

Step 1
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the “points” we get out of them.

This will work provided we pick the “correct” values of \( t \) that gives us good points that we can use to actually determine what the graph is.
So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

\[
\begin{align*}
  x &= 4t \\
  y &= 10 - 2t
\end{align*}
\]

Step 2
Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only \( x \) and \( y \) that will have the same graph as the vector function.

We can do this as follows,

\[
  x = 4t \quad \rightarrow \quad t = \frac{1}{4} x \quad \rightarrow \quad y = 10 - 2\left(\frac{1}{4} t\right) = 10 - \frac{1}{2} t
\]

So, it looks like the graph of the vector function will be a line with slope \(-\frac{1}{2}\) and \( y \)-intercept of \((0, 10)\).

Step 3
A sketch of the graph is below.

For illustration purposes we also put in a set of vectors for variety of \( t \)'s just to show that with enough of them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (i.e. the equation involving only \( x \) and \( y \)).

4. Sketch the graph of the vector function : \( \vec{r}(t) = \langle t + 1, \frac{1}{3}t^2 + 3 \rangle \)

Step 1
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the “points” we get out of them.
This will work provided we pick the “correct” values of $t$ that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

\[
x = t + 1 \\
y = \frac{1}{4} t^2 + 3
\]

Step 2
Now, recall from when we looked at parametric equations we eliminated the parameter from the parametric equations to get an equation involving only $x$ and $y$ that will have the same graph as the vector function.

We can do this as follows,

\[
x = t + 1 \quad \rightarrow \quad t = x - 1 \quad \rightarrow \quad y = \frac{1}{4} (x - 1)^2 + 3
\]

So, it looks like the graph of the vector function will be a parabola with vertex $(1, 3)$ and opening upwards.

Step 3
A sketch of the graph is below.

For illustration purposes we also put in a set of vectors for variety of $t$’s just to show that with enough of them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (i.e. the equation involving only $x$ and $y$).
5. Sketch the graph of the vector function: \( \vec{r}(t) = (4\sin(t), 8\cos(t)) \)

**Step 1**
One way to sketch the graph of vector functions of course is to just compute a bunch of vectors and then recall that we consider them to be position vectors and plot the “points” we get out of them.

This will work provided we pick the “correct” values of \( t \) that gives us good points that we can use to actually determine what the graph is.

So, to avoid doing that, recall that because we consider these to be position vectors we can write down a corresponding set of parametric equations that we can use to sketch the graph. The parametric equations for this vector function are,

\[
\begin{align*}
x &= 4\sin(t) \\
y &= 8\cos(t)
\end{align*}
\]

**Step 2**
Now, recall from our look at parametric equations we now know that this will be the graph of an ellipse centered at the origin with right/left points a distance of 4 from the origin and top/bottom points a distance of 8 from the origin.

**Step 3**
A sketch of the graph is below.
For illustration purposes we also put in a set of vectors for variety of \( t \)'s just to show that with enough them we would have also gotten the graph. Of course, it was easier to eliminate the parameter and just graph the algebraic equations (\( i.e. \) the equation involving only \( x \) and \( y \)).

6. Identify the graph of the vector function without sketching the graph.

\[
\vec{r}(t) = \left< 3 \cos(6t), -4, \sin(6t) \right>
\]

Step 1
To identify the graph of this vector function (without graphing) let’s first write down the set of parametric equations we get from this vector function. They are,

\[
x = 3 \cos(6t)
\]
\[
y = -4
\]
\[
z = \sin(6t)
\]

Step 2
Now, from the \( x \) and \( z \) equations we can see that we have an ellipse in the \( xz \)-plane that is given by the following equation.
\[
\frac{x^2}{9} + z^2 = 1
\]

From the \( y \) equation we know that this ellipse will not actually be in the \( xz \)-plane but parallel to the \( xz \)-plane at \( y = -4 \).

---

7. Identify the graph of the vector function without sketching the graph.

\[
\vec{r}(t) = \langle 2-t, 4+7t, -1-3t \rangle
\]

**Solution**

There really isn’t a lot to do with this problem. The equation should look very familiar to you. We saw quite a few of these types of equations in the Equations of Lines and Equations of Planes sections.

From those sections we know that the graph of this equation is a line in \( \mathbb{R}^3 \) that goes through the point \( (2, 4, -1) \) and parallel to the vector \( \vec{v} = \langle -1, 7, -3 \rangle \).

---

8. Write down the equation of the line segment starting at \( (1, 3) \) and ending at \( (-4, 6) \).

**Solution**

There really isn’t a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,

\[
\vec{r}(t) = (1-t)\langle 1, 3 \rangle + t\langle -4, 6 \rangle \quad 0 \leq t \leq 1
\]

Don’t forget the limits on \( t \)! Without that you have the full line that goes through those two points instead of the line segment from \( (1, 3) \) to \( (-4, 6) \).

---

9. Write down the equation of the line segment starting at \( (0, 2, -1) \) and ending at \( (7, -9, 2) \).

**Solution**

There really isn’t a lot to do with this problem. All we need to do is use the formula we derived in the notes for this section.

The line segment is,
\[ \mathbf{r}(t) = (1-t)(0,2,-1) + t(7,-9,2) \quad 0 \leq t \leq 1 \]

Don’t forget the limits on \( t \)! Without that you have the full line that goes through those two points instead of the line segment from \((0,2,-1)\) to \((7,-9,2)\).

---

**Calculus with Vector Functions**

1. Evaluate the following limit.

\[ \lim_{t \to 1} \left( e^{t-1}, 4t, \frac{t-1}{t^2 - 1} \right) \]

**Solution**

There really isn’t a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

\[
\lim_{t \to 1} \left( e^{t-1}, 4t, \frac{t-1}{t^2 - 1} \right) = \left( \lim_{t \to 1} e^{t-1}, \lim_{t \to 1} 4t, \lim_{t \to 1} \frac{t-1}{t^2 - 1} \right) = \left( e^0, 4, \frac{1}{2} \right) = \left( 1, 4, \frac{1}{2} \right)
\]

Don’t forget L’Hospital’s Rule! We needed that for the third term.

---

2. Evaluate the following limit.

\[ \lim_{t \to 2} \left( \frac{1-e^{t^2}}{t^2 + t - 2} \mathbf{i} + \mathbf{j} + \left( t^2 + 6t \right) \mathbf{k} \right) \]

**Solution**

There really isn’t a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

\[
\lim_{t \to 2} \left( \frac{1-e^{t^2}}{t^2 + t - 2} \mathbf{i} + \mathbf{j} + \left( t^2 + 6t \right) \mathbf{k} \right) = \lim_{t \to 2} \frac{1-e^{t^2}}{t^2 + t - 2} \mathbf{i} + \lim_{t \to 2} \mathbf{j} + \lim_{t \to 2} \left( t^2 + 6t \right) \mathbf{k} = \frac{1}{3} \mathbf{i} + \mathbf{j} - 8 \mathbf{k}
\]
Calculus II

Don’t forget L’Hospital’s Rule! We needed that for the first term.

3. Evaluate the following limit.

\[
\lim_{t \to \infty} \left( \frac{1}{t^3} , \frac{2t^2}{1-t-t^2} , e^{-t} \right)
\]

Solution

There really isn’t a lot to do here with this problem. All we need to do is take the limit of all the components of the vector.

\[
\lim_{t \to \infty} \left( \frac{1}{t^3} , \frac{2t^2}{1-t-t^2} , e^{-t} \right) = \lim_{t \to \infty} \left( \frac{1}{t^3} , \frac{2t^2}{1-t-t^2} , e^{-t} \right) = \left( \lim_{t \to \infty} \frac{1}{t^3} , \lim_{t \to \infty} \frac{2t^2}{1-t-t^2} , \lim_{t \to \infty} e^{-t} \right) = (0, -2, 0)
\]

Don’t forget your basic limit at infinity processes/facts.

4. Compute the derivative of the following limit.

\[
\vec{r} (t) = (t^3 - 1) \vec{i} + e^{2t} \vec{j} + \cos(t) \vec{k}
\]

Solution

There really isn’t a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

\[
\vec{r}' (t) = 3t^2 \vec{i} + 2e^{2t} \vec{j} - \sin(t) \vec{k}
\]

5. Compute the derivative of the following limit.

\[
\vec{r} (t) = \left( \ln (t^2 + 1) , te^{-t} , 4 \right)
\]

Solution

There really isn’t a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

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Calculus II

\[
\vec{r}''(t) = \left\langle \frac{2t}{t^2+1} e^{-t} - t e^{-t}, 0 \right\rangle
\]

Make sure you haven’t forgotten your basic differentiation formulas such as the chain rule (the first term) and the product rule (the second term).

6. Compute the derivative of the following limit.

\[
\vec{r}(t) = \left\langle \frac{t+1}{t-1}, \tan(4t), \sin^2(t) \right\rangle
\]

Solution
There really isn’t a lot to do here with this problem. All we need to do is take the derivative of all the components of the vector.

\[
\vec{r}''(t) = \left\langle \frac{(1)(t-1)-(t+1)(1)}{(t-1)^2}, 4 \sec^2(4t), 2 \sin(t) \cos(t) \right\rangle
\]

\[
= \left\langle \frac{-2}{(t-1)^2}, 4 \sec^2(4t), 2 \sin(t) \cos(t) \right\rangle
\]

Make sure you haven’t forgotten your basic differentiation formulas such as the quotient rule (the first term) and the chain rule (the third term).

7. Evaluate \(\int \vec{r}(t) \, dt\), where \(\vec{r}(t) = t^3 \hat{i} - \frac{2t}{t^2+1} \hat{j} + \cos^2(3t) \hat{k}\).

Solution
There really isn’t a lot to do here with this problem. All we need to do is integrate of all the components of the vector.

\[
\int \vec{r}(t) \, dt = \int t^3 \, dt \hat{i} - \int \frac{2t}{t^2+1} \, dt \hat{j} + \int \cos^2(3t) \, dt \hat{k}
\]

\[
= \left[ \frac{1}{4} t^4 - \ln|t^2+1| \right] \hat{i} + \left[ \frac{1}{2} \left( 1 + \cos(6t) \right) \right] \hat{j} + \left[ \frac{1}{2} \sin(6t) \right] \hat{k}
\]

Don’t forget the basic integration rules such as the substitution rule (second term) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (third term).
We didn’t put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you’ll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

8. Evaluate \( \int_{-1}^{2} \mathbf{r}(t) \, dt \) where \( \mathbf{r}(t) = \left( 6, 6t^2 - 4t, te^{2t} \right) \)

Solution

There really isn’t a lot to do here with this problem. All we need to do is integrate all the components of the vector.

\[
\int \mathbf{r}(t) \, dt = \left\langle \int 6 \, dt, \int 6t^2 - 4t \, dt, \int te^{2t} \, dt \right\rangle \\
= \left\langle \int 6 \, dt, \int 6t^2 - 4t \, dt, \frac{1}{2}te^{2t} - \frac{1}{2}e^{2t} \right\rangle = \left\langle 6t, 2t^3 - 2t^2, \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} \right\rangle
\]

Don’t forget the basic integration rules such integration by parts (third term). Also recall that one way to do definite integral is to first do the indefinite integral and then do the evaluation.

The answer for this problem is then,

\[
\int_{-1}^{2} \mathbf{r}(t) \, dt = \left\langle 6t, 2t^3 - 2t^2, \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} \right\rangle \bigg|_{-1}^{2} \\
= \left\langle -6, -4, -\frac{3}{4}e^4 + \frac{3}{4} \right\rangle - \left\langle -6, -4, -\frac{3}{4}e^{-2} + \frac{3}{4} \right\rangle = \left\langle 12, 8, \frac{3}{4}e^4 - \frac{3}{4}e^{-2} \right\rangle
\]

We didn’t put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you’ll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

9. Evaluate \( \int \mathbf{\ddot{r}}(t) \, dt \), where \( \mathbf{\ddot{r}}(t) = \left( (1-t)\cos(t^2 - 2t), \cos(t)\sin(t), \sec^2(4t) \right) \)

Solution

There really isn’t a lot to do here with this problem. All we need to do is integrate all the components of the vector.
\[
\int \vec{r}(t) \, dt = \left( \int (1-t) \cos(t^2-2t) \, dt, \int \cos(t) \sin(t) \, dt, \int \sec^2(4t) \, dt \right) \\
= \left( \int (1-t) \cos(t^2-2t) \, dt, \int \frac{1}{2} \sin(2t) \, dt, \int \sec^2(4t) \, dt \right) \\
= \left( -\frac{1}{2} \sin(t^2-2t), -\frac{1}{4} \cos(2t), \frac{1}{4} \tan(4t) \right) + \vec{c}
\]

Don’t forget the basic integration rules such as the substitution rule (all terms) and some of the basic trig formulas (half angle and double angle formulas) you need to do some of the integrals (second term).

We didn’t put a lot of the integration details into the solution on the assumption that you do know your integration skills well enough to follow what is going on. If you are somewhat rusty with your integration skills then you’ll need to go back to the integration material from both Calculus I and Calculus II to refresh your integration skills.

---

**Tangent, Normal and Binormal Vectors**

1. Find the unit tangent vector for the vector function: \( \vec{r}(t) = \left( t^2 + 1, 3 - t, t^3 \right) \)

Step 1
From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

\[
\vec{r}'(t) = \left( 2t, -1, 3t^2 \right)
\]

\[
\left\| \vec{r}'(t) \right\| = \sqrt{(2t)^2 + (-1)^2 + (3t^2)^2} = \sqrt{1 + 4t^2 + 9t^4}
\]

Step 2
The unit tangent vector for this vector function is then,

\[
\vec{T}(t) = \frac{1}{\sqrt{1 + 4t^2 + 9t^4}} \left( 2t, -1, 3t^2 \right) = \left( \frac{2t}{\sqrt{1 + 4t^2 + 9t^4}}, -\frac{1}{\sqrt{1 + 4t^2 + 9t^4}}, \frac{3t^2}{\sqrt{1 + 4t^2 + 9t^4}} \right)
\]

---

2. Find the unit tangent vector for the vector function: \( \vec{r}(t) = t e^{2t} \hat{i} + \left( 2 - t^2 \right) \hat{j} - e^{2t} \hat{k} \)
Calculus II

Step 1
From the notes in this section we know that to get the unit tangent vector all we need is the derivative of the vector function and its magnitude. Here are those quantities.

\[ \vec{r}'(t) = (e^{2t} + 2te^{2t}) \hat{i} - 2t \hat{j} - 2e^{2t} \hat{k} \]

\[ \| \vec{r}'(t) \| = \sqrt{(e^{2t} + 2te^{2t})^2 + (-2t)^2 + (-2e^{2t})^2} = \sqrt{(e^{2t} + 2te^{2t})^2 + 4t^2 + 4e^{4t}} \]

Step 2
The unit tangent vector for this vector function is then,

\[ \vec{T}(t) = \frac{1}{\sqrt{(e^{2t} + 2te^{2t})^2 + 4t^2 + 4e^{4t}}} \left( (e^{2t} + 2te^{2t}) \hat{i} - 2t \hat{j} - 2e^{2t} \hat{k} \right) \]

Note that because of the "mess" with this one we didn’t distribute the magnitude through to each term as we usually do with these. This problem is a good example of just how "messy" these can get.

3. Find the tangent line to \( \vec{r}(t) = \cos(4t) \hat{i} + 3 \sin(4t) \hat{j} + t^3 \hat{k} \) at \( t = \pi \).

Step 1
First we’ll need to get the tangent vector to the vector function. The tangent vector is,

\[ \vec{r}'(t) = -4 \sin(4t) \hat{i} + 12 \cos(4t) \hat{j} + 3t^2 \hat{k} \]

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we’ll just go with this.

Step 2
Now we actually need the tangent vector at the value of \( t \) given in the problem statement and not the "full" tangent vector. We’ll also need the point on the vector function at the value of \( t \) from the problem statement. These are,

\[ \vec{r}(\pi) = \cos(4\pi) \hat{i} + 3 \sin(4\pi) \hat{j} + \pi^3 \hat{k} = \hat{i} + \pi^3 \hat{k} \]

\[ \vec{r}'(\pi) = -4 \sin(4\pi) \hat{i} + 12 \cos(4\pi) \hat{j} + 3\pi^2 \hat{k} = 12 \hat{j} + 3\pi^2 \hat{k} \]

Step 3
To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course these are just the quantities we found in the previous step.

The tangent line is then,
Calculus II

\[ \vec{r}(t) = \vec{i} + \pi^3 \vec{k} + t(12 \vec{j} + 3\pi^2 \vec{k}) = \vec{i} + 12t \vec{j} + (\pi^3 + 3\pi^2 t) \vec{k} \]

4. Find the tangent line to \( \vec{r}(t) = \left< 7e^{2-t}, \frac{16}{t^4}, 5-t \right> \) at \( t = 2 \).

Step 1
First we’ll need to get the tangent vector to the vector function. The tangent vector is,

\[ \vec{r}'(t) = \left< -7e^{2-t}, -\frac{48}{t^4}, -1 \right> \]

Note that we could use the unit tangent vector here if we wanted to but given how messy those tend to be we’ll just go with this.

Step 2
Now we actually need the tangent vector at the value of \( t \) given in the problem statement and not the “full” tangent vector. We’ll also need the point on the vector function at the value of \( t \) from the problem statement. These are,

\[ \vec{r}(2) = \left< 7, 2, 3 \right> \]
\[ \vec{r}'(2) = \left< -7, -3, -1 \right> \]

Step 3
To write down the equation of the tangent line we need a point on the line and a vector parallel to the line. Of course these are just the quantities we found in the previous step.

The tangent line is then,

\[ \vec{r}(t) = \left< 7, 2, 3 \right> + t \left< -7, -3, -1 \right> = \left< 7 - 7t, 2 - 3t, 3 - t \right> \]

5. Find the unit normal and the binormal vectors for the following vector function.

\[ \vec{r}(t) = \left< \cos(2t), \sin(2t), 3 \right> \]

Step 1
We first need the unit tangent vector so let’s get that.
\[
\vec{r}'(t) = \langle -2 \sin(2t), 2 \cos(2t), 0 \rangle \quad \|\vec{r}'(t)\| = \sqrt{4 \sin^2(2t) + 4 \cos^2(2t)} = 2
\]
\[
\vec{T}(t) = \frac{1}{2} \langle -2 \sin(2t), 2 \cos(2t), 0 \rangle = \langle -\sin(2t), \cos(2t), 0 \rangle
\]

Step 2
The unit normal vector is then,
\[
\vec{T}'(t) = \langle -2 \cos(2t), -2 \sin(2t), 0 \rangle \quad \|\vec{T}'(t)\| = \sqrt{4 \cos^2(2t) + 4 \sin^2(2t)} = 2
\]
\[
\vec{N}(t) = \frac{1}{2} \langle -2 \cos(2t), -2 \sin(2t), 0 \rangle = \langle -\cos(2t), -\sin(2t), 0 \rangle
\]

Step 3
Finally, the binormal vector is,
\[
\vec{B}(t) = \vec{T}(t) \times \vec{N}(t) = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
-\sin(2t) & \cos(2t) & 0 \\
-\cos(2t) & -\sin(2t) & 0
\end{vmatrix}
= \sin^2(2t) \vec{k} - (-\cos^2(2t) \vec{k}) = (\sin^2(2t) + \cos^2(2t)) \vec{k} = \vec{k} = \langle 0, 0, 1 \rangle = \vec{B}(t)
\]

---

**Arc Length with Vector Functions**

1. Determine the length of \( \vec{r}(t) = (3 - 4t) \vec{i} + 6t \vec{j} - (9 + 2t) \vec{k} \) from \(-6 \leq t \leq 8\).

Step 1
We first need the magnitude of the derivative of the vector function. This is,
\[
\vec{r}'(t) = -4 \vec{i} + 6 \vec{j} - 2 \vec{k}
\]
\[
\|\vec{r}'(t)\| = \sqrt{16 + 36 + 4} = \sqrt{56} = 2\sqrt{14}
\]

Step 2
The length of the curve is then,
\[
L = \int_{-6}^{8} 2\sqrt{14} \, dt = 2\sqrt{14} \left|_{-6}^{8} \right| = \frac{28\sqrt{14}}{}
2. Determine the length of $\mathbf{r}(t) = \left\langle \frac{1}{2} t^3, 4t, \sqrt{2} t^2 \right\rangle$ from $0 \leq t \leq 2$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\mathbf{r}'(t) = \left\langle t^2, 4, 2\sqrt{2}t \right\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{t^4 + 16 + 8t^2} = \sqrt{t^4 + 8t^2 + 16} = \sqrt{(t^2 + 4)^2} = t^2 + 4$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Step 2
The length of the curve is then,

$$L = \int_0^2 t^2 + 4 \, dt = \left[ \frac{3}{2} t^3 \right]_0^2 = \frac{3}{2}$$

Note that if we’d not simplified the magnitude this would have been a very difficult integral to compute!

3. Find the arc length function for $\mathbf{r}(t) = \left\langle t^2, 2t^3, 1-t^3 \right\rangle$.

Step 1
We first need the magnitude of the derivative of the vector function. This is,

$$\mathbf{r}'(t) = \left\langle 2t, 6t^2, -3t^2 \right\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{4t^2 + 36t^4 + 9t^4} = \sqrt{t^2 \left( 4 + 45t^2 \right)} = t \sqrt{4 + 45t^2} = t \sqrt{4 + 45t^2}$$

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Note as well that because we are assuming that we are starting at $t = 0$ for this kind of problem it is safe to assume that $t \geq 0$ and so $\sqrt{t^2} = |t| = t$.

Step 2
The arc length function is then,
Calculus II

\[ s(t) = \int_0^t u\sqrt{4 + 45u^2} \, du = \frac{1}{135} \left( 4 + 45u^2 \right)^{3/2} \left[ \left( 4 + 45t^2 \right)^{3/2} - 8 \right] \]

4. Find the arc length function for \( \vec{r}(t) = \langle 4t, -2t, \sqrt{5} t^2 \rangle \).

**Step 1**
We first need the magnitude of the derivative of the vector function. This is,

\[ \vec{r}'(t) = \langle 4, -2, 2\sqrt{5} t \rangle \]

\[ \left\| \vec{r}'(t) \right\| = \sqrt{16 + 4 + 20t^2} = \sqrt{20 + 20t^2} = \sqrt{20(1 + t^2)} = 2\sqrt{5}\sqrt{1 + t^2} \]

**Step 2**
The arc length function is then,

\[ s(t) = \int_0^t 2\sqrt{5}\sqrt{1 + u^2} \, du \]

Do not always expect these integrals to be “simple” integrals. They will often require techniques more involved than just a standard Calculus I substitution. In this case we need the following trig substitution.

\[ u = \tan \theta \quad du = \sec^2 \theta \, d\theta \quad \sqrt{1 + u^2} = \sqrt{1 + \tan^2 \theta} = \sqrt{\sec^2 \theta} = |\sec \theta| \]

The limits of the integral become,

\[ u = 0 : 0 = \tan \theta \rightarrow \theta = 0 \quad u = t > 0 : t = \tan \theta \rightarrow \theta = \tan^{-1}(t) \]

Now, as noted we know that \( t > 0 \) and so we can safely assume that from the \( u = t \) limit we will get \( 0 < \theta < \frac{\pi}{2} \). This in turn means that we will always be in the first quadrant and we know that secant is positive in the first quadrant. Therefore we can remove the absolute values bars on the secant above.

The arc length function is now,

\[ s(t) = \int_0^{\tan^{-1}(t)} 2\sqrt{5} \sec \theta \, d\theta = \sqrt{5} \left[ \sec \theta \tan \theta + \ln|\sec \theta + \tan \theta| \right]_0^{\tan^{-1}(t)} \]

\[ = \sqrt{5} \left[ \sec(\tan^{-1}(t))\tan(\tan^{-1}(t)) + \ln|\sec(\tan^{-1}(t)) + \tan(\tan^{-1}(t))| \right] \]
Now we know that $\tan(\tan^{-1}(t)) = t$ so that will simplify our answer a little. Let’s take a look at the
secant term to see if we can simplify that as well. First, from our limit work recall that $\theta = \tan^{-1}(t)$. Or
with a little rewrite we have,
\[
\tan \theta = t = \frac{t}{1} = \frac{\text{opposite}}{\text{adjacent}}
\]
Construct a right triangle with opposite side being $t$ and the adjacent side being 1. The hypotenuse is then
$\sqrt{t^2 + 1}$. This in turn means that $\sec \theta = \sqrt{t^2 + 1}$. So,
\[
\sec(\tan^{-1}(t)) = \sec(\theta) = \sqrt{t^2 + 1}
\]
With this simplification our arc length function is then,
\[
s(t) = \sqrt{5 \left[ t\sqrt{t^2 + 1} + \ln(\sqrt{t^2 + 1} + t) \right]}
\]
There was some slightly unpleasant simplification here but once we did that we got a much nicer arc
length function.

5. Determine where on the curve given by $\mathbf{r}(t) = \langle t^2, 2t^3, 1 - t^3 \rangle$ we are after traveling a distance of 20.

Step 1
From Problem 3 above we know that the arc length function for this vector function is,
\[
s(t) = \frac{1}{135} \left[ (4 + 45t^2)^{3/2} - 8 \right]
\]
We need to solve this for $t$. Doing this gives,
\[
(4 + 45t^2)^{3/2} - 8 = 135s
\]
\[
(4 + 45t^2)^{3/2} = 135s + 8
\]
\[
4 + 45t^2 = (135s + 8)^{2/3}
\]
\[
t^2 = \frac{1}{45} \left[ (135s + 8)^{2/3} - 4 \right] \quad \rightarrow \quad t = \sqrt[3/2]{\frac{1}{45} \left[ (135s + 8)^{2/3} - 4 \right]}
\]
Note that we only used the positive $t$ after taking the root because the implicit assumption from the arc
length function is that $t$ is positive.

Step 2
We could use this to reparameterize the vector function however that would lead to a particularly unpleasant function in this case.

The key here is to simply realize that what we are being asked to compute is the value of the reparameterized vector function, \( \vec{r}(t(s)) \) when \( s = 20 \). Or, in other words, we want to compute \( \vec{r}(t(20)) \).

So, first,

\[
t(20) = \sqrt{\frac{1}{15} \left[ (135(20) + 8)^{\frac{2}{3}} - 4 \right]} = 2.05633
\]

Our position after traveling a distance of 20 is then,

\[
\vec{r}(t(20)) = \vec{r}(2.05633) = \left< 4.22849, 17.39035, -7.69518 \right>
\]

---

**Curvature**

1. Find the curvature of \( \vec{r}(t) = \langle \cos(2t), -\sin(2t), 4t \rangle \).

**Step 1**
We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector won’t be that bad to work with so let’s go with that formula. Here’s the unit tangent vector work.

\[
\vec{T}(t) = \frac{1}{\sqrt{ \vec{r}'(t) \cdot \vec{r}'(t) }} = \left\{ \begin{array}{c}
\frac{\sin(2t)}{\sqrt{5}}, -\frac{\cos(2t)}{\sqrt{5}}, \frac{2}{\sqrt{5}}
\end{array} \right\}
\]

**Step 2**
Now, what we really need is the magnitude of the derivative of the unit tangent vector so here is that work,
Calculus II

\( \vec{T}'(t) = \left( -\frac{1}{\sqrt{5}} \cos(2t), \frac{1}{\sqrt{5}} \sin(2t), 0 \right) \)

\[ \| \vec{T}'(t) \| = \sqrt{\frac{1}{5}\cos^2(2t) + \frac{1}{5}\sin^2(2t)} = \frac{1}{\sqrt{5}} \]

Step 3
The curvature is then,

\[ \kappa = \frac{\| \vec{T}'(t) \|}{\| \vec{T}''(t) \|} = \frac{2/\sqrt{5}}{2/\sqrt{5}} = \frac{1}{2} \]

So, in this case, the curvature will be independent of \( t \). That won’t always be the case so don’t expect this to happen every time.

2. Find the curvature of \( \vec{r}(t) = \langle 4t, -t^2, 2t^3 \rangle \).

Step 1
We have two formulas we can use here to compute the curvature. One requires us to take the derivative of the unit tangent vector and the other requires a cross product.

Either will give the same result. The real question is which will be easier to use. Cross products can be a pain to compute but then some of the unit tangent vectors can be quite messy to take the derivative of. So, basically, the one we use will be the one that will probably be the easiest to use.

In this case it looks like the unit tangent vector will involve lots of quotients that would probably be unpleasant to take the derivative of. So, let’s go with the cross product formula this time.

We’ll need the first and second derivative of the vector function. Here are those.

\[ \vec{r}'(t) = \langle 4, -2t, 6t^2 \rangle \quad \vec{r}''(t) = \langle 0, -2, 12t \rangle \]

Step 2
Next, we need the cross product of these two derivatives. Here is that work.

\[ \vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2t & 6t^2 \\ 0 & -2 & 12t \end{vmatrix} = -24t^2 \hat{i} - 8\hat{k} - 48t \hat{j} + 12t^2 \hat{i} - 12t^2 \hat{i} - 48t \hat{j} - 8\hat{k} \]

Step 3
We now need a couple of magnitudes.

\[ \| \vec{r}'(t) \times \vec{r}''(t) \| = \sqrt{144t^4 + 2304t^2 + 64} \quad \| \vec{r}'(t) \| = \sqrt{16 + 4t^2 + 36t^4} \]

The curvature is then,
Calculus II

\[ \kappa = \frac{\| \vec{r}'(t) \times \vec{r}''(t) \|}{\| \vec{r}'(t) \|} = \frac{\sqrt{144t^4 + 2304t^2 + 64}}{\sqrt{16 + 4t^2 + 36t^4}} = \sqrt{\frac{36t^4 + 576t^2 + 16}{9t^4 + t^2 + 4}} \]

A fairly messy formula here, but these will often be that way.

\[ \kappa = \frac{\| \vec{r}'(t) \times \vec{r}''(t) \|}{\| \vec{r}'(t) \|} = \frac{\sqrt{144t^4 + 2304t^2 + 64}}{\sqrt{16 + 4t^2 + 36t^4}} = \sqrt{\frac{36t^4 + 576t^2 + 16}{9t^4 + t^2 + 4}} \]

\[ \kappa = \frac{\| \vec{r}'(t) \times \vec{r}''(t) \|}{\| \vec{r}'(t) \|} = \frac{\sqrt{144t^4 + 2304t^2 + 64}}{\sqrt{16 + 4t^2 + 36t^4}} = \sqrt{\frac{36t^4 + 576t^2 + 16}{9t^4 + t^2 + 4}} \]

\[ \kappa = \frac{\| \vec{r}'(t) \times \vec{r}''(t) \|}{\| \vec{r}'(t) \|} = \frac{\sqrt{144t^4 + 2304t^2 + 64}}{\sqrt{16 + 4t^2 + 36t^4}} = \sqrt{\frac{36t^4 + 576t^2 + 16}{9t^4 + t^2 + 4}} \]

A fairly messy formula here, but these will often be that way.

Velocity and Acceleration

1. An object’s acceleration is given by \( \vec{a} = 3t \vec{i} - 4e^{-t} \vec{j} + 12t^2 \vec{k} \). The object’s initial velocity is \( \vec{v}(0) = \vec{j} - 3\vec{k} \) and the object’s initial position is \( \vec{r}(0) = -5\vec{i} + 2\vec{j} - 3\vec{k} \). Determine the object’s velocity and position functions.

Step 1
To determine the velocity function all we need to do is integrate the acceleration function.

\[ \vec{v}(t) = \int (3t \vec{i} - 4e^{-t} \vec{j} + 12t^2 \vec{k}) \, dt = \frac{3}{2} t^2 \vec{i} + 4e^{-t} \vec{j} + 4t^3 \vec{k} + \vec{c} \]

Don’t forget the “constant” of integration, which in this case is actually the vector \( \vec{c} = c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k} \).
To determine the constant of integration all we need is to use the value \( \vec{v}(0) \) that we were given in the problem statement.

\[ \vec{j} - 3\vec{k} = \vec{v}(0) = 4\vec{j} + c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k} \]

To determine the values of \( c_1, c_2, \) and \( c_3 \) all we need to do is set the various components equal.

\[ \vec{i} : 0 = c_1 \]
\[ \vec{j} : 1 = 4 + c_2 \quad \Rightarrow \quad c_1 = 0, \ c_2 = -3, \ c_3 = -3 \]
\[ \vec{k} : -3 = c_3 \]

The velocity is then,

\[ \vec{v}(t) = \frac{3}{2} t^2 \vec{i} + \left(4e^{-t} - 3\right) \vec{j} + \left(4t^3 - 3\right) \vec{k} \]

Step 2
The position function is simply the integral of the velocity function we found in the previous step.
\[ \vec{r}(t) = \int \frac{1}{2}t^2 \, \vec{i} + \left( 4e^{-t} - 3 \right) \vec{j} + \left( 4t^3 - 3 \right) \vec{k} \, dt = \frac{1}{2}t^2 \, \vec{i} + \left( 4e^{-t} - 3t \right) \vec{j} + \left( t^4 - 3t \right) \vec{k} + \vec{c} \]

We’ll use the value of \( \vec{r}(0) \) from the problem statement to determine the value of the constant of integration.

\[-5\vec{i} + 2\vec{j} - 3\vec{k} = \vec{r}(0) = -4 \vec{j} + c_1 \vec{i} + c_2 \vec{j} + c_3 \vec{k} \]

\[ \vec{i} : -5 = c_1 \]
\[ \vec{j} : 2 = -4 + c_2 \quad \Rightarrow \quad c_1 = -5, \quad c_2 = 6, \quad c_3 = -3 \]

The position function is then,

\[ \vec{r}(t) = \left( \frac{1}{2}t^2 - 5 \right) \vec{i} + \left( 4e^{-t} - 3t + 6 \right) \vec{j} + \left( t^4 - 3t - 3 \right) \vec{k} \]

2. Determine the tangential and normal components of acceleration for the object whose position is given by \( \vec{r}(t) = \langle \cos(2t), -\sin(2t), 4t \rangle \).

Step 1
First we need the first and second derivatives of the position function.

\[ \vec{r}'(t) = \langle -2 \sin(2t), -2 \cos(2t), 4 \rangle \quad \vec{r}''(t) = \langle -4 \cos(2t), 4 \sin(2t), 0 \rangle \]

Step 2
Next we’ll need the following quantities.

\[ \| \vec{r}'(t) \| = \sqrt{4 \sin^2(2t) + 4 \cos^2(2t) + 16} = \sqrt{20} = 2\sqrt{5} \]

\[ \vec{r}'(t) \cdot \vec{r}''(t) = 8 \sin(2t) \cos(2t) - 8 \sin(2t) \cos(2t) + 0 = 0 \]

\[ \vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 \sin(2t) & -2 \cos(2t) & 4 \\ -4 \cos(2t) & 4 \sin(2t) & 0 \end{vmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -16 \cos(2t) & -8 \sin^2(2t) & -8 \cos^2(2t) \end{vmatrix} = -16 \sin(2t) \vec{i} - 16 \cos(2t) \vec{j} - 8 \vec{k} \]
\[ \|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{256 \sin^2(2t) + 256 \cos^2(2t) + 64} = \sqrt{320} = 8\sqrt{5} \]

Step 3
The tangential component of the acceleration is,
\[ a_t = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{\|\vec{r}''(t)\|} = 0 \]

The normal component of the acceleration is,
\[ a_n = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}''(t)\|} = \frac{8\sqrt{5}}{2\sqrt{5}} = 4 \]

**Cylindrical Coordinates**

1. Convert the Cartesian coordinates for \((4, -5, 2)\) into Cylindrical coordinates.

Step 1
From the point we’re given we have,
\[ x = 4 \quad y = -5 \quad z = 2 \]

So, we already have the \(z\) coordinate for the Cylindrical coordinates.

Step 2
Remember as well that for \(r\) and \(\theta\) we’re going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting \(r\) is easy.
\[ r = \sqrt{(4)^2 + (-5)^2} = \sqrt{41} \]

Step 3
Finally we need to get \(\theta\).
\[ \theta_1 = \tan^{-1}\left(\frac{-5}{4}\right) = -0.8961 \quad \theta_2 = -0.8961 + \pi = 2.2455 \]
If we look at the three dimensional coordinate system from above we can see that $\theta_1$ is in the fourth quadrant and $\theta_2$ is in the second quadrant. Likewise from our $x$ and $y$ coordinates the point is in the fourth quadrant (as we look at the point from above).

This in turn means that we need to use $\theta_1$ for our point.

The Cylindrical coordinates are then,

$$\left(\sqrt{41}, -0.8961, 2\right)$$

2. Convert the Cartesian coordinates for $(-4, -1, 8)$ into Cylindrical coordinates.

Step 1
From the point we’re given we have,

$$x = -4 \quad y = -1 \quad z = 8$$

So, we already have the $z$ coordinate for the Cylindrical coordinates.

Step 2
Remember as well that for $r$ and $\theta$ we’re going to do the same conversion work as we did in converting a Cartesian point into Polar coordinates.

So, getting $r$ is easy.

$$r = \sqrt{(-4)^2 + (-1)^2} = \sqrt{17}$$

Step 3
Finally we need to get $\theta$.

$$\theta_1 = \tan^{-1}\left(\frac{-1}{-4}\right) = 0.2450 \quad \theta_2 = 0.2450 + \pi = 3.3866$$

If we look at the three dimensional coordinate system from above we can see that $\theta_1$ is in the first quadrant and $\theta_2$ is in the third quadrant. Likewise from our $x$ and $y$ coordinates the point is in the third quadrant (as we look at the point from above).

This in turn means that we need to use $\theta_2$ for our point.

The Cylindrical coordinates are then,
3. Convert the following equation written in Cartesian coordinates into an equation in Cylindrical coordinates.

\[ x^3 + 2x^2 - 6z = 4 - 2y^2 \]

**Step 1**
There really isn’t a whole lot to do here. All we need to do is plug in the following \( x \) and \( y \) polar conversion formulas into the equation where (and if) possible.

\[
x = r \cos \theta \quad \quad y = r \sin \theta \quad \quad r^2 = x^2 + y^2
\]

**Step 2**
However, first we’ll do a little rewrite.

\[
x^3 + 2x^2 + 2y^2 - 6z = 4 \quad \rightarrow \quad x^3 + 2 \left( x^2 + y^2 \right) - 6z = 4
\]

**Step 3**
Now let’s use the formulas from Step 1 to convert the equation into Cylindrical coordinates.

\[
(r \cos \theta)^3 + 2 \left( r^2 \right) - 6z = 4 \quad \rightarrow \quad r^3 \cos^3 \theta + 2r^2 - 6z = 4
\]

4. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

\[ zr = 2 - r^2 \]

**Solution**
There is not really a lot to do here other than plug in \( r = \sqrt{x^2 + y^2} \) into the equation. Doing this is,

\[
z\sqrt{x^2 + y^2} = 2 - (x^2 + y^2)
\]

5. Convert the following equation written in Cylindrical coordinates into an equation in Cartesian coordinates.

\[ 4 \sin(\theta) - 2 \cos(\theta) = \frac{r}{z} \]
Step 1
There really isn’t a whole lot to do here. All we need to do is to use the following \( x \) and \( y \) polar conversion formulas in the equation where (and if) possible.

\[
x = r \cos \theta \quad y = r \sin \theta \quad r^2 = x^2 + y^2
\]

Step 2
To make the conversion a little easier let’s multiply the equation by \( r \) to get,

\[
4r \sin (\theta) - 2r \cos (\theta) = \frac{r^2}{z}
\]

Step 3
Now let’s use the formulas from Step 1 to convert the equation into Cartesian coordinates.

\[
4y - 2x = \frac{x^2 + y^2}{z}
\]

6. Identify the surface generated by the equation : \( r^2 - 4r \cos (\theta) = 14 \)

Step 1
To identify the surface generated by this equation it’s probably best to first convert the equation into Cartesian coordinates. In this case that’s a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

\[
x^2 + y^2 - 4x = 14
\]

Step 2
To identify this equation (and you do know what it is!) let’s complete the square on the \( x \) part of the equation.

\[
x^2 - 4x + y^2 = 14
\]
\[
x^2 - 4x + 4 + y^2 = 14 + 4
\]
\[
(x - 2)^2 + y^2 = 18
\]

So, we can see that this is a circle centered at \((2, 0)\) with radius \(\sqrt{18}\).

7. Identify the surface generated by the equation : \( z = 7 - 4r^2 \)
Step 1
To identify the surface generated by this equation it’s probably best to first convert the equation into Cartesian coordinates. In this case that’s a pretty simple thing to do.

Here is the equation in Cartesian coordinates.

\[ z = 7 - 4\left(x^2 + y^2\right) = 7 - 4x^2 - 4y^2 \]

Step 2
From the Cartesian equation in Step 1 we can see that the surface generated by the equation is an elliptic paraboloid that starts at \( z = 7 \) and opens down.

---

**Spherical Coordinates**

1. Convert the Cartesian coordinates for \((3, -4, 1)\) into Spherical coordinates.

Step 1
From the point we’re given we have,

\[ x = 3 \quad y = -4 \quad z = 1 \]

Step 2
Let’s first determine \( \rho \).

\[ \rho = \sqrt{(3)^2 + (-4)^2 + (1)^2} = \sqrt{26} \]

Step 3
We can now determine \( \varphi \).

\[ \cos \varphi = \frac{z}{\rho} = \frac{1}{\sqrt{26}} \quad \varphi = \cos^{-1}\left(\frac{1}{\sqrt{26}}\right) = 1.3734 \]

Step 4
Let’s use the \( x \) conversion formula to determine \( \theta \).

\[ \cos \theta = \frac{x}{\rho \sin \varphi} = \frac{3}{\sqrt{26}\sin(1.3734)} = 0.6 \quad \Rightarrow \quad \theta_i = \cos^{-1}(0.6) = 0.9273 \]
This angle is in the first quadrant and if we sketch a quick unit circle we see that a second angle in the fourth quadrant is \( \theta_2 = 2\pi - 0.9273 = 5.3559 \).

If we look at the three dimensional coordinate system from above we can see that from our \( x \) and \( y \) coordinates the point is in the fourth quadrant. This in turn means that we need to use \( \theta_2 \) for our point.

The Spherical coordinates are then,

\[
(\sqrt{26}, 5.3559, 1.3734)
\]

2. Convert the Cartesian coordinates for \((-2, -1, -7)\) into Spherical coordinates.

Step 1
From the point we’re given we have,

\[
x = -2 \quad y = -1 \quad z = -7
\]

Step 2
Let’s first determine \( \rho \).

\[
\rho = \sqrt{(-2)^2 + (-1)^2 + (-7)^2} = \sqrt{54}
\]

Step 3
We can now determine \( \phi \).

\[
\cos \phi = \frac{z}{\rho} = \frac{-7}{\sqrt{54}} \quad \phi = \cos^{-1}\left(\frac{-7}{\sqrt{54}}\right) = 2.8324
\]

Step 4
Let’s use the \( y \) conversion formula to determine \( \theta \).

\[
\sin \theta = \frac{-1}{\rho \sin \phi} = \frac{-1}{\sqrt{54} \sin(2.8324)} = -0.4472 \quad \rightarrow \quad \theta_1 = \sin^{-1}(-0.4472) = -0.4636
\]

This angle is in the fourth quadrant and if we sketch a quick unit circle we see that a second angle in the third quadrant is \( \theta_2 = \pi + 0.4636 = 3.6052 \).

If we look at the three dimensional coordinate system from above we can see that from our \( x \) and \( y \) coordinates the point is in the third quadrant. This in turn means that we need to use \( \theta_2 \) for our point.

The Spherical coordinates are then,
3. Convert the Cylindrical coordinates for \((2, 0.345, -3)\) into Spherical coordinates.

Step 1
From the point we’re given we have,

\[ r = 2 \quad \theta = 0.345 \quad z = -3 \]

So, we already have the value of \(\theta\) for the Spherical coordinates.

Step 2
Next we can determine \(\rho\).

\[ \rho = \sqrt{(2)^2 + (-3)^2} = \sqrt{13} \]

Step 3
Finally, we can determine \(\varphi\).

\[ \cos \varphi = \frac{z}{\rho} = \frac{-3}{\sqrt{13}} \quad \varphi = \cos^{-1}\left(\frac{-3}{\sqrt{13}}\right) = 2.5536 \]

The Spherical coordinates are then,

\(\sqrt{13}, 0.345, 2.5536\)

4. Convert the following equation written in Cartesian coordinates into an equation in Spherical coordinates.

\[ x^2 + y^2 = 4x + z - 2 \]

Step 1
All we need to do here is plug in the following conversion formulas into the equation and do a little simplification.

\[ x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi \]

Step 2
Plugging the conversion formula in gives,

\[
(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = 4(\rho \sin \phi \cos \theta) + \rho \cos \phi - 2
\]

The first two terms can be simplified as follows.

\[
\begin{align*}
\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta &= 4\rho \sin \phi \cos \theta + \rho \cos \phi - 2 \\
\rho^2 \sin^2 \phi \left(\cos^2 \theta + \sin^2 \theta\right) &= 4\rho \sin \phi \cos \theta + \rho \cos \phi - 2 \\
\rho^2 \sin^2 \phi &= 4\rho \sin \phi \cos \theta + \rho \cos \phi - 2
\end{align*}
\]

5. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

\[
\rho^2 = 3 - \cos \phi
\]

Step 1

There really isn’t a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi \\
\rho^2 &= x^2 + y^2 + z^2
\end{align*}
\]

Step 2

To make this problem a little easier let’s first multiply the equation by \(\rho\). Doing this gives,

\[
\rho^3 = 3\rho - \rho \cos \phi
\]

Doing this makes recognizing the right most term a little easier.

Step 3

Using the appropriate conversion formulas from Step 1 gives,

\[
\left(x^2 + y^2 + z^2\right)^{\frac{3}{2}} = 3\sqrt{x^2 + y^2 + z^2} - z
\]

6. Convert the equation written in Spherical coordinates into an equation in Cartesian coordinates.

\[
csc \phi = 2 \cos \theta + 4 \sin \theta
\]

Step 1
There really isn’t a whole lot to do here. All we need to do is to use the following conversion formulas in the equation where (and if) possible

\[
x = \rho \sin \varphi \cos \theta \quad y = \rho \sin \varphi \sin \theta \quad z = \rho \cos \varphi \\
\rho^2 = x^2 + y^2 + z^2
\]

Step 2
To make this problem a little easier let’s first do some rewrite on the equation.

First, let’s deal with the cosecant.

\[
\frac{1}{\sin \varphi} = 2 \cos \theta + 4 \sin \theta \quad \rightarrow \quad 1 = 2 \sin \varphi \cos \theta + 4 \sin \varphi \sin \theta
\]

Next, let’s multiply everything by \( \rho \) to get,

\[
\rho = 2 \rho \sin \varphi \cos \theta + 4 \rho \sin \varphi \sin \theta
\]

Doing this makes recognizing the terms on the right a little easier.

Step 3
Using the appropriate conversion formulas from Step 1 gives,

\[
\sqrt{x^2 + y^2 + z^2} = 2x + 4y
\]

7. Identify the surface generated by the given equation : \( \varphi = \frac{4\pi}{5} \)

Solution
Okay, as we discussed this type of equation in the notes for this section. We know that all points on the surface generated must be of the form \( (\rho, \theta, \frac{4\pi}{5}) \).

So, we can rotate around the \( z \)-axis as much as want them to (i.e. \( \theta \) can be anything) and we can move as far as we want from the origin (i.e. \( \rho \) can be anything). All we need to do is make sure that the point will always make an angle of \( \frac{4\pi}{5} \) with the positive \( z \)-axis.

In other words we have a cone. It will open downwards and the “wall” of the cone will form an angle of \( \frac{4\pi}{5} \) with the positive \( z \)-axis and it will form an angle of \( \frac{\pi}{5} \) with the negative \( z \)-axis.

8. Identify the surface generated by the given equation : \( \rho = -2 \sin \varphi \cos \theta \)
Step 1
Let's first multiply each side of the equation by $\rho$ to get,

$$\rho^2 = -2\rho \sin \varphi \cos \theta$$

Step 2
We can now easily convert this to Cartesian coordinates to get,

$$x^2 + y^2 + z^2 = -2x$$
$$x^2 + 2x + y^2 + z^2 = 0$$

Let's complete the square on the $x$ portion to get,

$$x^2 + 2x + 1 + y^2 + z^2 = 0 + 1$$
$$(x+1)^2 + y^2 + z^2 = 1$$

Step 3
So, it looks like we have a sphere with radius 1 that is centered at $(-1,0,0)$.