CALCULUS I

Solutions to Practice Problems
Review

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Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Review

Review : Functions

1. Perform the indicated function evaluations for \( f(x) = 3 - 5x - 2x^2 \).
(a) $f(4)$ [Solution]  
\[ f(4) = 3 - 5(4) - 2(4)^2 = -49 \]

(b) $f(0)$ [Solution]  
\[ f(0) = 3 - 5(0) - 2(0)^2 = 3 \]

(c) $f(-3)$ [Solution]  
\[ f(-3) = 3 - 5(-3) - 2(-3)^2 = 0 \]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) $f(6-t)$ [Solution]  
\begin{align*}
   f(6-t) &= 3 - 5(6-t) - 2(6-t)^2 \\
   &= 3 - 5(6-t) - 2(36 - 12t + t^2) \\
   &= 3 - 30 + 5t - 72 + 24t - 2t^2 \\
   &= -99 + 29t - 2t^2
\end{align*}

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) $f(7-4x)$ [Solution]  
\begin{align*}
   f(7-4x) &= 3 - 5(7-4x) - 2(7-4x)^2 \\
   &= 3 - 5(7-4x) - 2(49 - 56x + 16x^2) \\
   &= 3 - 35 + 20x - 98 + 112x - 32x^2 \\
   &= -130 + 132x - 32x^2
\end{align*}

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. Also, don’t get excited about the fact that there is both an $x$ and an $h$ here. This works exactly the same way as the first three it will just have a little more algebra involved.

(f) $f(x+h)$ [Solution]
\[ f(x + h) = 3 - 5(x + h) - 2(x + h)^2 \]
\[ = 3 - 5(x + h) - 2(x^2 + 2xh + h^2) \]
\[ = 3 - 5x - 5h - 2x^2 - 4xh - 2h^2 \]

2. Perform the indicated function evaluations for \( g(t) = \frac{t}{2t + 6} \).

(a) \( g(0) \)  
(b) \( g(-3) \)  
(c) \( g(10) \)  
(d) \( g(x^2) \)  
(e) \( g(t + h) \)  
(f) \( g(t^2 - 3t + 1) \)

(a) \( g(0) \) [Solution]
\[ g(0) = \frac{0}{2(0) + 6} = \frac{0}{6} = 0 \]

(b) \( g(-3) \) [Solution]
\[ g(-3) = \frac{-3}{2(-3) + 6} = \frac{-3}{0} \times \]

The minute we see the division by zero we know that \( g(-3) \) does not exist.

(c) \( g(10) \) [Solution]
\[ g(10) = \frac{10}{2(10) + 6} = \frac{10}{26} = \frac{5}{13} \]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) \( g(x^2) \) [Solution]
\[ g(x^2) = \frac{x^2}{2x^2 + 6} \]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. Also, don’t get excited about the fact that there is both a \( t \) and an \( h \) here. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) \( g(t + h) \) [Solution]
\[
g(t) = \frac{t + h}{2(t + h) + 6} = \frac{t + h}{2t + 2h + 6}
\]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

\[
g(t^2 - 3t + 1) \quad [\text{Solution}]
\]

\[
g(t^2 - 3t + 1) = \frac{t^2 - 3t + 1}{2(t^2 - 3t + 1) + 6} = \frac{t^2 - 3t + 1}{2t^2 - 6t + 8}
\]

3. Perform the indicated function evaluations for \( h(z) = \sqrt{1 - z^2} \).

(a) \( h(0) \) 
(b) \( h(-\frac{1}{2}) \) 
(c) \( h(\frac{1}{2}) \) 
(d) \( h(9z) \) 
(e) \( h(z^2 - 2z) \) 
(f) \( h(z + k) \)

(a) \( h(0) \) \ [Solution]

\[ h(0) = \sqrt{1 - 0^2} = \sqrt{1} = 1 \]

(b) \( h(-\frac{1}{2}) \) \ [Solution]

\[ h\left(-\frac{1}{2}\right) = \sqrt{1 - \left(-\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \]

(c) \( h\left(\frac{1}{2}\right) \) \ [Solution]

\[ h\left(\frac{1}{2}\right) = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2} \]

Hint: Don’t let the fact that there are new variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) \( h(9z) \) \ [Solution]

\[ h(9z) = \sqrt{1 - (9z)^2} = \sqrt{1 - 81z^2} \]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) \( h(z^2 - 2z) \) \ [Solution]
\[
h(z^2 - 2z) = \sqrt{1 - (z^2 - 2z)^2} = \sqrt{1 - (z^2 + 4z^2 - 4z^3)} = \sqrt{1 - 4z^2 + 4z^3 - z^4}
\]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. Also, don’t get excited about the fact that there is both a \( z \) and a \( k \) here. This works exactly the same way as the first three it will just have a little more algebra involved.

(f) \( h(z + k) \) [Solution]

\[
h(z + k) = \sqrt{1 - (z + k)^2} = \sqrt{1 - (z^2 + 2zk + k^2)} = \sqrt{1 - z^2 - 2zk - k^2}
\]

4. Perform the indicated function evaluations for \( R(x) = \sqrt{3 + x - \frac{4}{x+1}} \).

(a) \( R(0) \) 
(b) \( R(6) \) 
(c) \( R(-9) \)
(d) \( R(x + 1) \) 
(e) \( R(x^3 - 3) \) 
(f) \( R\left(\frac{1}{x} - 1\right) \)

(a) \( R(0) \) [Solution]

\[
R(0) = \sqrt{3 + 0 - \frac{4}{0+1}} = \sqrt{3} - 4
\]

(b) \( R(6) \) [Solution]

\[
R(6) = \sqrt{3 + 6 - \frac{4}{6+1}} = \sqrt{9 - \frac{4}{7}} = 3 - \frac{4}{7} = \frac{17}{7}
\]

(c) \( R(-9) \) [Solution]

\[
R(-9) = \sqrt{3 + (-9) - \frac{4}{-9+1}} = \sqrt{-6 - \frac{4}{-8}}
\]

In this class we only deal with functions that give real values as answers. Therefore, because we have the square root of a negative number in the first term this function is not defined.

Note that the fact that the second term is perfectly acceptable has no bearing on the fact that the function will not be defined here. If any portion of the function is not defined upon evaluation then the whole function is not defined at that point. Also note that if we allow complex numbers this function will be defined.

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(d) \( R(x + 1) \) [Solution]
Calculus I

\[ R(x+1) = \sqrt{3 + (x+1)} - \frac{4}{(x+1)+1} = \sqrt{4+x} - \frac{4}{x+2} \]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(e) \( R\left(x^4 - 3\right) \) [Solution]

\[ R\left(x^4 - 3\right) = \sqrt{3 + (x^4 - 3)} - \frac{4}{(x^4 - 3)+1} = \sqrt{x^4} - \frac{4}{x^4 - 2} = x^2 - \frac{4}{x^4 - 2} \]

Hint: Don’t let the fact that there are now variables here instead of numbers get you confused. This works exactly the same way as the first three it will just have a little more algebra involved.

(f) \( R\left(\frac{1}{x} - 1\right) \) [Solution]

\[ R\left(\frac{1}{x} - 1\right) = \sqrt{3 + \left(\frac{1}{x} - 1\right)} - \frac{4}{\left(\frac{1}{x} - 1\right)+1} = \sqrt{2 + \frac{1}{x} - \frac{4}{x}} = \sqrt{2 + \frac{1}{x} - 4x} \]

5. The **difference quotient** of a function \( f(x) \) is defined to be,

\[ \frac{f(x + h) - f(x)}{h} \]

compute the difference quotient for \( f(x) = 4x - 9 \).

Hint: Compute \( f(x + h) \), then compute the numerator and finally compute the difference quotient.

Step 1

\[ f(x + h) = 4(x + h) - 9 = 4x + 4h - 9 \]

Step 2

\[ f(x + h) - f(x) = 4x + 4h - 9 - (4x - 9) = 4h \]

Step 3

\[ \frac{f(x + h) - f(x)}{h} = \frac{4h}{h} = 4 \]
6. The **difference quotient** of a function \( f(x) \) is defined to be,
\[
\frac{f(x+h) - f(x)}{h}
\]
compute the difference quotient for \( g(x) = 6 - x^2 \).

**Hint**: Don’t get excited about the fact that the function is now named \( g(x) \), the difference quotient still works in the same manner it just has \( g \)’s instead of \( f \)’s now. So, compute \( g(x+h) \), then compute the numerator and finally compute the difference quotient.

**Step 1**
\[
g(x+h) = 6 - (x+h)^2 = 6 - x^2 - 2xh - h^2
\]

**Step 2**
\[
g(x+h) - g(x) = 6 - x^2 - 2xh - h^2 - (6 - x^2) = -2xh - h^2
\]

**Step 3**
\[
\frac{g(x+h) - g(x)}{h} = \frac{-2xh - h^2}{h} = -2x - h
\]

7. The **difference quotient** of a function \( f(x) \) is defined to be,
\[
\frac{f(x+h) - f(x)}{h}
\]
compute the difference quotient for \( f(t) = 2t^2 - 3t + 9 \).

**Hint**: Don’t get excited about the fact that the function is now \( f(t) \), the difference quotient still works in the same manner it just has \( t \)’s instead of \( x \)’s now. So, compute \( f(t+h) \), then compute the numerator and finally compute the difference quotient.

**Step 1**
\[
f(t+h) = 2(t+h)^2 - 3(t+h) + 9 = 2(t^2 + 2th + h^2) - 3t - 3h + 9
\]
\[
= 2t^2 + 4th + 2h^2 - 3t - 3h + 9
\]

**Step 2**
\[
f(t+h) - f(t) = 2t^2 + 4th + 2h^2 - 3t - 3h + 9 - (2t^2 - 3t + 9) = 4th + 2h^2 - 3h
\]
Step 3

\[
\frac{f(t + h) - f(t)}{h} = \frac{4th + 2h^2 - 3h}{h} = 4t + 2h - 3
\]

8. The **difference quotient** of a function \( f(x) \) is defined to be,

\[
\frac{f(x + h) - f(x)}{h}
\]

compute the difference quotient for \( y(z) = \frac{1}{z + 2} \).

Hint: Don’t get excited about the fact that the function is now named \( y(z) \), the difference quotient still works in the same manner it just has \( y \)'s and \( z \)'s instead of \( f \)'s and \( x \)'s now. So, compute \( y(z + h) \), then compute the numerator and finally compute the difference quotient.

Step 1

\[
y(z + h) = \frac{1}{z + h + 2}
\]

Step 2

\[
y(z + h) - y(z) = \frac{1}{z + h + 2} - \frac{1}{z + 2} = \frac{z + 2 - (z + h + 2)}{(z + h + 2)(z + 2)} = -\frac{h}{(z + h + 2)(z + 2)}
\]

Note that, when dealing with difference quotients, it will almost always be advisable to combine rational expressions into a single term in preparation of the next step.

Step 3

\[
\frac{y(z + h) - y(z)}{h} = \frac{1}{h} \left( h(z + h) - h(z) \right) = \frac{1}{h} \left( -\frac{h}{(z + h + 2)(z + 2)} \right) = -\frac{1}{(z + h + 2)(z + 2)}
\]

In this step we rewrote the difference quotient a little to make the numerator a little easier to deal with. All that we’re doing here is using the fact that,

\[
\frac{a}{b} = (a) \left( \frac{1}{b} \right) = \left( \frac{1}{b} \right) (a)
\]
9. The difference quotient of a function \( f(x) \) is defined to be,
\[
\frac{f(x + h) - f(x)}{h}
\]
compute the difference quotient for \( A(t) = \frac{2t}{3-t} \).

Hint: Don’t get excited about the fact that the function is now named \( A(t) \), the difference quotient still works in the same manner it just has \( A \)’s and \( t \)’s instead of \( f \)’s and \( x \)’s now. So, compute \( A(t + h) \), then compute the numerator and finally compute the difference quotient.

Step 1
\[
A(t + h) = \frac{2(t + h)}{3 - (t + h)} = \frac{2t + 2h}{3 - t - h}
\]

Step 2
\[
A(t + h) - A(t) = \frac{2t + 2h}{3 - t - h} - \frac{2t}{3 - t} = \frac{(2t + 2h)(3 - t) - 2t(3 - t - h)}{(3 - t - h)(3 - t)}
\]
\[
= \frac{6t - 2t^2 + 6h - 2ht - (6t - 2t^2 - 2th)}{(3 - t - h)(3 - t)} = \frac{6h}{(3 - t - h)(3 - t)}
\]

Note that, when dealing with difference quotients, it will almost always be advisable to combine rational expressions into a single term in preparation of the next step. Also, when doing this don’t forget to simplify the numerator as much as possible. With most difference quotients you’ll see a lot of cancelation as we did here.

Step 3
\[
\frac{A(t + h) - A(t)}{h} = \frac{1}{h} \left( A(t + h) - A(t) \right) = \frac{1}{h} \left( \frac{6h}{(3 - t - h)(3 - t)} \right) = \frac{6}{(3 - t - h)(3 - t)}
\]

In this step we rewrote the difference quotient a little to make the numerator a little easier to deal with. All that we’re doing here is using the fact that,
\[
\frac{a}{b} = (a) \left( \frac{1}{b} \right) = \left( \frac{1}{b} \right)(a)
\]

10. Determine all the roots of \( f(x) = x^5 - 4x^4 - 32x^3 \).
Calculus I

[Solution]
Set the function equal to zero and factor the left side.

\[ x^5 - 4x^3 - 32x = x^3(x^2 - 4x - 32) = x^3(x - 8)(x + 4) = 0 \]

After factoring we can see that the three roots of this function are,
\[ x = -4, \quad x = 0, \quad x = 8 \]

11. Determine all the roots of \( R(y) = 12y^2 + 11y - 5 \).

[Solution]
Set the function equal to zero and factor the left side.

\[ 12y^2 + 11y - 5 = (4y + 5)(3y - 1) = 0 \]

After factoring we see that the two roots of this function are,
\[ y = -\frac{5}{4}, \quad y = \frac{1}{3} \]

12. Determine all the roots of \( h(t) = 18 - 3t - 2t^2 \).

[Solution]
Set the function equal to zero and because the left side will not factor we’ll need to use the quadratic formula to find the roots of the function.

\[ 18 - 3t - 2t^2 = 0 \]

\[ t = \frac{3 \pm \sqrt{(-3)^2 - 4(-2)(18)}}{2(-2)} = \frac{3 \pm \sqrt{\sqrt{153}}}{-4} = \frac{3 \pm \sqrt{17}}{-4} = \frac{-3 \pm \sqrt{17}}{4} \]

So, the quadratic formula gives the following two roots of the function,
\[ -\frac{3}{4}(1 + \sqrt{17}) = 2.342329 \quad \text{and} \quad -\frac{3}{4}(1 - \sqrt{17}) = -3.842329 \]
13. Determine all the roots of \( g(x) = x^3 + 7x^2 - x \).

[Solution]
Set the equation equal to zero and factor the left side as much as possible.
\[
x^3 + 7x^2 - x = x(x^2 + 7x - 1) = 0
\]
So, we can see that one root is \( x = 0 \) and because the quadratic doesn’t factor we’ll need to use the quadratic formula on that to get the remaining two roots.
\[
x = \frac{-7 \pm \sqrt{(7)^2 - 4(1)(-1)}}{2(1)} = \frac{-7 \pm \sqrt{53}}{2}
\]
We then have the following three roots of the function,
\[
x = 0, \quad -\frac{7 + \sqrt{53}}{2} = 0.140055, \quad -\frac{7 - \sqrt{53}}{2} = -7.140055
\]

14. Determine all the roots of \( W(x) = x^4 + 6x^2 - 27 \).

[Solution]
Set the function equal to zero and factor the left side as much as possible.
\[
x^4 + 6x^2 - 27 = (x^2 - 3)(x^2 + 9) = 0
\]
Don’t so locked into quadratic equations that the minute you see an equation that is not quadratic you decide you can’t deal with it. While this function was not a quadratic it still factored in an obvious manner.

Now, the second term will never be zero (for any real value of \( x \) anyway and in this class those tend to be the only ones we are interested in) and so we can ignore that term. The first will be zero if,
\[
x^2 - 3 = 0 \quad \Rightarrow \quad x^2 = 3 \quad \Rightarrow \quad x = \pm \sqrt{3}
\]
So, we have two real roots of this function. Note that if we allowed complex roots (which again, we aren’t really interested in for this course) there would also be two complex roots from the second term as well.
15. Determine all the roots of \( f(t) = t^3 - 7t^{\frac{4}{3}} - 8t \).

[Solution]
Set the function equal to zero and factor the left side as much as possible.

\[
5t^\frac{5}{3} - 7t^\frac{4}{3} - 8t = t \left( t^\frac{2}{3} - 7t^\frac{1}{3} - 8 \right) = t \left( t^\frac{1}{3} - 8 \right) \left( t^\frac{1}{3} + 1 \right) = 0
\]

Don’t so locked into quadratic equations that the minute you see an equation that is not quadratic you decide you can’t deal with it. While this function was not a quadratic it still factored, it just wasn’t as obvious that it did in this case. You could have clearly seen that if factored if it had been,

\[ t \left( t^2 - 7t - 8 \right) \]

but notice that the only real difference is that the exponents are fractions now, but it still has the same basic form and so can be factored.

Okay, back to the problem. From the factored form we get,

\[
t = 0
\]

\[
t^\frac{1}{3} - 8 = 0 \quad \Rightarrow \quad t^\frac{1}{3} = 8 \quad \Rightarrow \quad t = 8^3 = 512
\]

\[
t^\frac{1}{3} + 1 = 0 \quad \Rightarrow \quad t^\frac{1}{3} = -1 \quad \Rightarrow \quad t = (-1)^3 = -1
\]

So, the function has three roots,

\[ t = -1, \quad t = 0, \quad t = 512 \]

16. Determine all the roots of \( h(z) = \frac{z}{z - 5} - \frac{4}{z - 8} \).

[Solution]
Set the function equal to zero and clear the denominator by multiplying by the least common denominator, \((z - 5)(z - 8)\), and then solve the resulting equation.
Calculus I

\[(z-5)(z-8)\left(\frac{z}{z-5} - \frac{4}{z-8}\right) = 0\]
\[z(z-8)-4(z-5) = 0\]
\[z^2-12z+20 = 0\]
\[(z-10)(z-2) = 0\]

So, it looks like the function has two roots, \(z = 2\) and \(z = 10\) however recall that because we started off with a function that contained rational expressions we need to go back to the original function and make sure that neither of these will create a division by zero problem in the original function. In this case neither do and so the two roots are,

\[z = 2 \quad z = 10\]

17. Determine all the roots of \(g(w) = \frac{2w}{w+1} + \frac{w-4}{2w-3}\).

[Solution]
Set the function equal to zero and clear the denominator by multiplying by the least common denominator, \((w+1)(2w-3)\), and then solve the resulting equation.

\[(w+1)(2w-3)\left(\frac{2w}{w+1} + \frac{w-4}{2w-3}\right) = 0\]
\[2w(2w-3) + (w-4)(w+1) = 0\]
\[5w^2 - 9w - 4 = 0\]

This quadratic doesn’t factor so we’ll need to use the quadratic formula to get the solution.

\[w = \frac{9 \pm \sqrt{(-9)^2 - 4(5)(-4)}}{2(5)} = \frac{9 \pm \sqrt{161}}{10}\]

So, it looks like this function has the following two roots,

\[\frac{9 + \sqrt{161}}{10} = 2.168858 \quad \frac{9 - \sqrt{161}}{10} = -0.368858\]

Recall that because we started off with a function that contained rational expressions we need to go back to the original function and make sure that neither of these will create a division by zero problem in the original function. Neither of these do and so they are the two roots of this function.
18. Find the domain and range of \( Y(t) = 3t^2 - 2t + 1 \).

[Solution]
This is a polynomial (a 2nd degree polynomial in fact) and so we know that we can plug any value of \( t \) into the function and so the domain is all real numbers or,

\[
\text{Domain} : \quad -\infty < t < \infty \quad \text{or} \quad (-\infty, \infty)
\]

The graph of this 2nd degree polynomial (or quadratic) is a parabola that opens upwards (because the coefficient of the \( t^2 \) is positive) and so we know that the vertex will be the lowest point on the graph. This also means that the function will take on all values greater than or equal to the \( y \)-coordinate of the vertex which will in turn give us the range.

So, we need the vertex of the parabola. The \( t \)-coordinate is,

\[
t = -\frac{-2}{2(3)} = \frac{1}{3}
\]

and the \( y \) coordinate is then, \( Y\left(\frac{1}{3}\right) = \frac{2}{3} \).

The range is then,

\[
\text{Range} : \quad \left[ \frac{2}{3}, \infty \right)
\]

19. Find the domain and range of \( g(z) = -z^2 - 4z + 7 \).

[Solution]
This is a polynomial (a 2nd degree polynomial in fact) and so we know that we can plug any value of \( z \) into the function and so the domain is all real numbers or,

\[
\text{Domain} : \quad -\infty < z < \infty \quad \text{or} \quad (-\infty, \infty)
\]

The graph of this 2nd degree polynomial (or quadratic) is a parabola that opens downwards (because the coefficient of the \( z^2 \) is negative) and so we know that the vertex will be the highest
point on the graph. This also means that the function will take on all values less than or equal to the $y$-coordinate of the vertex which will in turn give us the range.

So, we need the vertex of the parabola. The $z$-coordinate is,

$$z = -\frac{-4}{2(-1)} = -2$$

and the $y$ coordinate is then, $g(-2) = 11$.

The range is then,

$$\text{Range} : (-\infty, 11]$$

20. Find the domain and range of $f(z) = 2 + \sqrt{z^2 + 1}$.

[Solution]

We know that when we have square roots that we can’t take the square root of a negative number. However, because,

$$z^2 + 1 \geq 1$$

we will never be taking the square root of a negative number in this case and so the domain is all real numbers or,

$$\text{Domain} : -\infty < z < \infty \text{ or } (-\infty, \infty)$$

For the range we need to recall that square roots will only return values that are positive or zero and in fact the only way we can get zero out of a square root will be if we take the square root of zero. For our function, as we’ve already noted, the quantity that is under the root is always at least 1 and so this root will never be zero. Also recall that we have the following fact about square roots,

$$\text{If } x \geq 1 \text{ then } \sqrt{x} \geq 1$$

So, we now know that,

$$\sqrt{z^2 + 1} \geq 1$$

Finally, we are adding 2 onto the root and so we know that the function must always be greater than or equal to 3 and so the range is,

$$\text{Range} : [3, \infty)$$
21. Find the domain and range of \( h(y) = -3\sqrt{14 + 3y} \).

[Solution]
In this case we need to require that,
\[
14 + 3y \geq 0 \Rightarrow y \geq -\frac{14}{3}
\]
in order to make sure that we don't take the square root of negative numbers. The domain is then,
\[
\text{Domain : } -\frac{14}{3} \leq y < \infty \text{ or } \left[ -\frac{14}{3}, \infty \right)
\]
For the range for this function we can notice that the quantity under the root can be zero (if \( y = -\frac{14}{3} \)). Also note that because the quantity under the root is a line it will take on all positive values and so the square root will in turn take on all positive value and zero. The square root is then multiplied by -3. This won’t change the fact that the root can be zero, but the minus sign will change the sign of the non-zero values from positive to negative. The 3 will only affect the general size of the square root but it won’t change the fact that the square root will still take on all positive (or negative after we add in the minus sign) values.

The range is then,
\[
\text{Range : } (-\infty,0]
\]

22. Find the domain and range of \( M(x) = 5 - |x + 8| \).

[Solution]
We’re dealing with an absolute value here and the quantity inside is a line, which we can plug all values of \( x \) into, and so the domain is all real numbers or,
\[
\text{Domain : } -\infty < x < \infty \text{ or } (-\infty,\infty)
\]
For the range let’s again note that the quantity inside the absolute value is a linear function that will take on all real values. We also know that absolute value functions will never be negative and will only be zero if we take the absolute value of zero. So we now know that,
\[
|x + 8| \geq 0
\]
However, we are subtracting this from 5 and so we’ll be subtracting a positive or zero number from 5 and so the range is,

\[ \text{Range : } (-\infty, 5] \]

23. Find the domain of \( f(w) = \frac{w^3 - 3w + 1}{12w - 7} \).

[Solution]
In this case we need to avoid division by zero issues and so we’ll need to determine where the denominator is zero. To do this we will solve,

\[
12w - 7 = 0 \quad \Rightarrow \quad w = \frac{7}{12}
\]

We can plug all other values of \( w \) into the function without any problems and so the domain is,

\[ \text{Domain : All real numbers except } w = \frac{7}{12} \]

24. Find the domain of \( R(z) = \frac{5}{z^3 + 10z^2 + 9z} \).

[Solution]
In this case we need to avoid division by zero issues and so we’ll need to determine where the denominator is zero. To do this we will solve,

\[
z^3 + 10z^2 + 9z = z(z^2 + 10z + 9) = z(z + 1)(z + 9) = 0 \quad \Rightarrow \quad z = 0, \ z = -1, \ z = -9
\]

The three values above are the only values of \( z \) that we can’t plug into the function. All other values of \( z \) can be plugged into the function and will return real values. The domain is then,

\[ \text{Domain : All real numbers except } z = 0, \ z = -1, \ z = -9 \]

25. Find the domain of \( g(t) = \frac{6t - t^3}{7 - t - 4t^2} \).
In this case we need to avoid division by zero issues and so we’ll need to determine where the denominator is zero. To do this we will solve,

$$7 - t - 4t^2 = 0 \quad \Rightarrow \quad t = \frac{1 \pm \sqrt{(-1)^2 - 4(-4)(7)}}{2(-4)} = -\frac{1}{8}(1 \pm \sqrt{113})$$

The two values above are the only values of $t$ that we can’t plug into the function. All other values of $t$ can be plugged into the function and will return real values. The domain is then,

$$\text{Domain} : \text{All real numbers except } t = -\frac{1}{8}(1 \pm \sqrt{113})$$

26. Find the domain of $g(x) = \sqrt{25 - x^2}$.

[Solution]
In this case we need to avoid square roots of negative numbers and so we need to require,

$$25 - x^2 \geq 0$$

Note that once we have the original inequality written down we can do a little rewriting of things as follows to make things a little easier to see.

$$x^2 \leq 25 \quad \Rightarrow \quad -5 \leq x \leq 5$$

At this point it should be pretty easy to find the values of $x$ that will keep the quantity under the radical positive or zero and so we won’t need to do a numberline or sign table to determine the range.

The domain is then,

$$\text{Domain} : -5 \leq x \leq 5$$

27. Find the domain of $h(x) = \sqrt{x^4 - x^3 - 20x^2}$.
Step 1 Hint: We need to avoid negative numbers under the square root and because the quantity under the root is a polynomial we know that it can only change sign if it goes through zero and so we first need to determine where it is zero.

In this case we need to avoid square roots of negative numbers and so we need to require,

\[ x^4 - x^3 - 20x^2 = x^2(x^2 - x - 20) = x^2(x - 5)(x + 4) \geq 0 \]

Once we have the polynomial in factored form we can see that the left side will be zero at \( x = 0 \), \( x = -4 \) and \( x = 5 \). Because the quantity under the radical is a polynomial we know that it can only change sign if it goes through zero and so these are the only points the only places where the polynomial on the left can change sign.

Step 2 Hint: Because the polynomial can only change sign at these points we know that it will be the same sign in each region defined by these points and so all we need to know is the value of the polynomial as a single point in each region.

Here is a number line giving the value/sign of the polynomial at a test point in each of the region defined by these three points. To make it a little easier to read the number line let’s define the polynomial under the radical to be,

\[ R(x) = x^4 - x^3 - 20x^2 = x^2(x - 5)(x + 4) \]

Now, here is the number line,

\[ \begin{align*}
R(-5) &= 250 & R(-1) &= -18 & R(1) &= -20 & R(6) &= 360 \\
R(x) &> 0 & R(x) &< 0 & R(x) &< 0 & R(x) &> 0 \\
-6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7
\end{align*} \]

Step 3 Hint: Now all we need to do is write down the values of \( x \) where the polynomial under the root will be positive or zero and we’ll have the domain. Be careful with the points where the polynomial is zero.

The domain will then be all the points where the polynomial under the root is positive or zero and so the domain is,

\[ \text{Domain} : -\infty < x \leq -4, \ x = 0, \ 5 \leq x < \infty \]
In this case we need to be very careful and not miss $x = 0$. This is the point separating two regions which give negative values of the polynomial, but it will give zero and so it also part of the domain. This point is often very is very easy to miss.

28. Find the domain of $P(t) = \frac{5t + 1}{\sqrt{t^3 - t^2 - 8t}}$.

Step 1 Hint : We need to avoid negative numbers under the square root and because the quantity under the root is a polynomial we know that it can only change sign if it goes through zero and so we first need to determine where it is zero.

In this case we need to avoid square roots of negative numbers and because the square root is in the denominator we’ll also need to avoid division by zero issues. We can satisfy both needs by requiring,

$$t^3 - t^2 - 8t = t(t^2 - t - 8) > 0$$

Note that there is nothing wrong with the square root of zero, but we know that the square root of zero is zero and so if we require that the polynomial under the root is strictly positive we’ll know that we won’t have square roots of negative numbers and we’ll avoid division by zero.

Now, despite the fact that we need to avoid where the polynomial is zero we know that it will only change signs if it goes through zero and so we’ll next need to determine where the polynomial is zero.

Clearly one value is $t = 0$ and because the quadratic does not factor we can use the quadratic formula on it to get the following two additional points.

$$t = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(-8)}}{2} = \frac{1 \pm \sqrt{33}}{2}$$

$$t = \frac{1 + \sqrt{33}}{2} = 3.372281$$

$$t = \frac{1 - \sqrt{33}}{2} = -2.372281$$

So, these three points ($t = 0$, $t = -2.372281$ and $t = 3.372281$) are the only places that the polynomial under the root can change sign.

Step 2 Hint : Because the polynomial can only change sign at these points we know that it will be the same sign in each region defined by these points and so all we need to know is the value of the polynomial as a single point in each region.
Here is a number line giving the value/sign of the polynomial at a test point in each of the region defined by these three points. To make it a little easier to read the number line let’s define the polynomial under the radical to be,

\[ R(t) = t^3 - t^2 - 8t = t(t^2 - t - 8) > 0 \]

Now, here is the number line,

\[
\begin{array}{cccccc}
R(-3) = -12 & | & R(-1) = 6 & | & R(1) = -8 & | & R(4) = 16 \\
R(t) < 0 & | & R(t) > 0 & | & R(t) < 0 & | & R(t) > 0 \\
-5 & | & -4 & | & -3 & | & -2 & | & -1 & | & 0 & | & 1 & | & 2 & | & 3 & | & 4 & | & 5 & | & 6
\end{array}
\]

Step 3 Hint : Now all we need to do is write down the values of x where the polynomial under the root will be positive (recall we need to avoid division by zero) and we’ll have the domain.

The domain will then be all the points where the polynomial under the root is positive, but not zero as we also need to avoid division by zero, and so the domain is,

\[
\text{Domain} : \frac{1 - \sqrt{33}}{2} < t < 0, \quad \frac{1 + \sqrt{33}}{2} < t < \infty
\]

29. Find the domain of \( f(z) = \sqrt{z - 1} + \sqrt{z + 6} \).

Hint Step 1 : The domain of this function will be the set of all values of \( z \) that will work in both terms of this function.

The domain of this function will be the set of all \( z \)’s that we can plug into both terms in this function and get a real number back as a value. This means that we first need to determine the domain of each of the two terms.

For the first term we need to require,

\[ z - 1 \geq 0 \quad \Rightarrow \quad z \geq 1 \]

For the second term we need to require,

\[ z + 6 \geq 0 \quad \Rightarrow \quad z \geq -6 \]
Hint Step 2 : What values of $z$ are in both of these?
[Show Step 2]
Now, we just need the set of $z$’s that are in both conditions above. In this case notice that all the $z$ that satisfy $z \geq 1$ will also satisfy $z \geq -6$. The reverse is not true however. Any $z$ that is in the range $-6 \leq z < 1$ will satisfy $z \geq 6$ but will not satisfy $z \geq 1$.

So, in this case, the domain is in fact just the first condition above or,

Domain : $z \geq 1$

30. Find the domain of $h(y) = \sqrt{2y + 9} - \frac{1}{\sqrt{2 - y}}$.

Hint Step 1 : The domain of this function will be the set of all values of $y$ that will work in both terms of this function.
[Show Step 1]
The domain of this function will be the set of all $y$’s that we can plug into both terms in this function and get a real number back as a value. This means that we first need to determine the domain of each of the two terms.

For the first term we need to require,

$$2y + 9 \geq 0 \quad \Rightarrow \quad y \geq -\frac{9}{2}$$

For the second term we need to require,

$$2 - y > 0 \quad \Rightarrow \quad y < 2$$

Note that we need the second condition to be strictly positive to avoid division by zero as well.

Hint Step 2 : What values of $y$ are in both of these?
[Show Step 2]
Now, we just need the set of $y$’s that are in both conditions above. In this case we need all the $y$’s that will be greater than or equal to $-\frac{9}{2}$ AND less than 2. The domain is then,

Domain : $-\frac{9}{2} \leq y < 2$
31. Find the domain of \( A(x) = \frac{4}{x-9} - \sqrt{x^2 - 36} \).

Hint Step 1: The domain of this function will be the set of all values of \( x \) that will work in both terms of this function.

[Show Step 1]

The domain of this function will be the set of all \( x \)'s that we can plug into both terms in this function and get a real number back as a value. This means that we first need to determine the domain of each of the two terms.

For the first term we need to require,
\[
x - 9 \neq 0 \quad \Rightarrow \quad x \neq 9
\]

For the second term we need to require,
\[
x^2 - 36 \geq 0 \quad \Rightarrow \quad x \leq -6 \quad \text{or} \quad x \geq 6
\]

Hint Step 2: What values of \( x \) are in both of these?

[Show Step 2]

Now, we just need the set of \( x \)'s that are in both conditions above. In this case the second condition gives us most of the domain as it is the most restrictive. The first term is okay as long as we avoid \( x = 9 \) and because this point will in fact satisfy the second condition we'll need to make sure and exclude it. The domain is then,

\[
\text{Domain} : x \leq -6 \quad \text{or} \quad x \geq 6, \quad x \neq 9
\]

32. Find the domain of \( Q(y) = \sqrt{y^2 + 1} - \sqrt{1 - y} \).

[Solution]

The domain of this function will be the set of \( y \)'s that will work in both terms of this function. So, we need the domain of each of the terms.

For the first term let's note that,
\[
y^2 + 1 \geq 1
\]

and so will always be positive. The domain of the first term is then all real numbers.

For the second term we need to notice that we're dealing with the cube root in this case and we can plug all real numbers into a cube root and so the domain of this term is again all real numbers.
So, the domain of both terms is all real numbers and so the domain of the function as a whole must also be all real numbers or,

\[
\text{Domain } : -\infty < y < \infty
\]

33. Compute \((f \circ g)(x)\) and \((g \circ f)(x)\) for \(f(x) = 4x - 1\), \(g(x) = \sqrt{6 + 7x}\).

[Solution]
Not much to do here other than to compute each of these.

\[
\begin{align*}
(f \circ g)(x) &= f\left[g(x)\right] = f\left[\sqrt{6 + 7x}\right] = 4\sqrt{6 + 7x} - 1 \\
(g \circ f)(x) &= g\left[f(x)\right] = g\left[4x - 1\right] = \sqrt{6 + 7(4x - 1)} = \sqrt{28x - 1}
\end{align*}
\]

34. Compute \((f \circ g)(x)\) and \((g \circ f)(x)\) for \(f(x) = 5x + 2\), \(g(x) = x^2 - 14x\).

[Solution]
Not much to do here other than to compute each of these.

\[
\begin{align*}
(f \circ g)(x) &= f\left[g(x)\right] = f\left[x^2 - 14x\right] = 5\left(x^2 - 14x\right) + 2 = 5x^2 - 70x + 2 \\
(g \circ f)(x) &= g\left[f(x)\right] = g\left[5x + 2\right] = (5x + 2)^2 - 14(5x + 2) = 25x^2 - 50x - 24
\end{align*}
\]

35. Compute \((f \circ g)(x)\) and \((g \circ f)(x)\) for \(f(x) = x^2 - 2x + 1\), \(g(x) = 8 - 3x^2\).

[Solution]
Not much to do here other than to compute each of these.
(f \circ g)(x) = f[g(x)] = f[8 - 3x^2] = (8 - 3x^2)^2 - 2(8 - 3x^2) + 1 = 9x^4 - 42x^2 + 49

(g \circ f)(x) = g[f(x)] = g[x^2 - 2x + 1]
= 8 - 3(x^2 - 2x + 1)^2 = -3x^4 + 12x^3 - 18x^2 + 12x + 5

36. Compute \((f \circ g)(x)\) and \((g \circ f)(x)\) for \(f(x) = x^2 + 3, \ g(x) = \sqrt{5 + x^2}\).

[Solution]
Not much to do here other than to compute each of these.

\[(f \circ g)(x) = f[g(x)] = f[\sqrt{5 - x^2}] = \left(\sqrt{5 + x^2}\right)^2 + 3 = 8 + x^2\]

\[(g \circ f)(x) = g[f(x)] = g[x^2 + 3] = \sqrt{5 + (x^2 + 3)^2} = \sqrt{x^4 + 6x^2 + 14}\]

**Review: Inverse Functions**

1. Find the inverse for \(f(x) = 6x + 15\). Verify your inverse by computing one or both of the composition as discussed in this section.

   **Hint:** Remember the process described in this section. Replace the \(f(x)\), interchange the \(x\)'s and \(y\)'s, solve for \(y\) and the finally replace the \(y\) with \(f^{-1}(x)\).

   **Step 1**
   
   \[y = 6x + 15\]

   **Step 2**
   
   \[x = 6y + 15\]

   **Step 3**
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\[ x - 15 = 6y \]
\[ y = \frac{1}{6}(x - 15) \quad \rightarrow \quad f^{-1}(x) = \frac{1}{6}(x - 15) \]

Finally, compute either \((f \circ f^{-1})(x)\) or \((f^{-1} \circ f)(x)\) to verify our work.

**Step 4**

Either composition can be done so let’s do \((f \circ f^{-1})(x)\) in this case.

\[
(f \circ f^{-1})(x) = f \left[ f^{-1}(x) \right] \\
= 6 \left[ \frac{1}{6}(x - 15) \right] + 15 \\
= x - 15 + 15 \\
= x
\]

So, we got \(x\) out of the composition and so we know we’ve done our work correctly.

---

2. Find the inverse for \(h(x) = 3 - 29x\). Verify your inverse by computing one or both of the composition as discussed in this section.

**Hint:** Remember the process described in this section. Replace the \(h(x)\), interchange the \(x\)'s and \(y\)'s, solve for \(y\) and the finally replace the \(y\) with \(h^{-1}(x)\).

**Step 1**

\[ y = 3 - 29x \]

**Step 2**

\[ x = 3 - 29y \]

**Step 3**

\[ x - 3 = -29y \]
\[ y = -\frac{1}{29}(x - 3) \quad \rightarrow \quad h^{-1}(x) = \frac{1}{29}(3 - x) \]

Notice that we multiplied the minus sign into the parenthesis. We did this in order to avoid potentially losing the minus sign if it had stayed out in front. This does not need to be done in order to get the inverse.
Finally, compute either \((h \circ h^{-1})(x)\) or \((h^{-1} \circ h)(x)\) to verify our work.

Step 4
Either composition can be done so let’s do \((h \circ h^{-1})(x)\) in this case.

\[
(h \circ h^{-1})(x) = h\left[h^{-1}(x)\right]
\]

\[
= 3 - 29 \left[\frac{1}{29}(3 - x)\right]
\]

\[
= 3 - (3 - x)
\]

\[
= x
\]

So, we got \(x\) out of the composition and so we know we’ve done our work correctly.

3. Find the inverse for \(R(x) = x^3 + 6\). Verify your inverse by computing one or both of the composition as discussed in this section.

Hint: Remember the process described in this section. Replace the \(R(x)\), interchange the \(x\)’s and \(y\)’s, solve for \(y\) and the finally replace the \(y\) with \(R^{-1}(x)\).

Step 1
\[y = x^3 + 6\]

Step 2
\[x = y^3 + 6\]

Step 3
\[x - 6 = y^3\]
\[y = \sqrt[3]{x - 6}\]

\[R^{-1}(x) = \sqrt[3]{x - 6}\]

Finally, compute either \((R \circ R^{-1})(x)\) or \((R^{-1} \circ R)(x)\) to verify our work.

Step 4
Either composition can be done so let’s do \((R^{-1} \circ R)(x)\) in this case.
\[ \left( R^{-1} \circ R \right)(x) = R^{-1} \left[ R(x) \right] \]
\[ = \sqrt[3]{x^3 + 6} - 6 \]
\[ = \sqrt[3]{x^3} \]
\[ = x \]

So, we got \( x \) out of the composition and so we know we’ve done our work correctly.

4. Find the inverse for \( g(x) = 4(x - 3)^5 + 21 \). Verify your inverse by computing one or both of the composition as discussed in this section.

Hint: Remember the process described in this section. Replace the \( g(x) \), interchange the \( x \)'s and \( y \)'s, solve for \( y \) and the finally replace the \( y \) with \( g^{-1}(x) \).

Step 1
\[ y = 4(x - 3)^5 + 21 \]

Step 2
\[ x = 4(y - 3)^5 + 21 \]

Step 3
\[ x - 21 = 4(y - 3)^5 \]
\[ \frac{1}{4}(x - 21) = (y - 3)^5 \]
\[ \sqrt[5]{\frac{1}{4}(x - 21)} = y - 3 \]
\[ y = 3 + \sqrt[5]{\frac{1}{4}(x - 21)} \]

Finally, compute either \( (g \circ g^{-1})(x) \) or \( (g^{-1} \circ g)(x) \) to verify our work.

Step 4
Either composition can be done so let’s do \( (g \circ g^{-1})(x) \) in this case.
\[(g \circ g^{-1})(x) = g[g^{-1}(x)]\]

\[= 4 \left[ 3 + \sqrt[4]{\frac{1}{4}(x - 21)} \right]^5 + 21\]

\[= 4 \left( \frac{1}{4}(x - 21) \right)^5 + 21\]

\[= 4 \left( \frac{1}{4}(x - 21) \right) + 21\]

\[= (x - 21) + 21\]

\[= x\]

So, we got $x$ out of the composition and so we know we’ve done our work correctly.

5. Find the inverse for $W(x) = \sqrt[3]{9 - 11x}$. Verify your inverse by computing one or both of the composition as discussed in this section.

Hint: Remember the process described in this section. Replace the $W(x)$, interchange the x’s and y’s, solve for $y$ and the finally replace the $y$ with $W^{-1}(x)$.

Step 1

\[y = \sqrt[3]{9 - 11x}\]

Step 2

\[x = \sqrt[3]{9 - 11y}\]

Step 3

\[x = \sqrt[3]{9 - 11y}\]

\[x^3 = 9 - 11y\]

\[x^3 - 9 = -11y\]

\[y = -\frac{1}{11}(x^3 - 9)\] →

\[W^{-1}(x) = \frac{1}{11}(9 - x^3)\]

Notice that we multiplied the minus sign into the parenthesis. We did this in order to avoid potentially losing the minus sign if it had stayed out in front. This does not need to be done in order to get the inverse.

Finally, compute either $(W \circ W^{-1})(x)$ or $(W^{-1} \circ W)(x)$ to verify our work.
Step 4
Either composition can be done so let’s do \((W^{-1} \circ W)(x)\) in this case.

\[
(W^{-1} \circ W)(x) = W^{-1}[W(x)]
\]

\[
= \frac{1}{11} \left(9 - \left[\sqrt[9]{9-11x}\right]^5\right)
\]

\[
= \frac{1}{11} \left(9 - [9-11x]\right)
\]

\[
= \frac{1}{11} (11x)
\]

\[
= x
\]

So, we got \(x\) out of the composition and so we know we’ve done our work correctly.

6. Find the inverse for \(f(x) = \sqrt[5]{5x + 8}\). Verify your inverse by computing one or both of the composition as discussed in this section.

Hint: Remember the process described in this section. Replace the \(f(x)\), interchange the \(x\)'s and \(y\)'s, solve for \(y\) and the finally replace the \(y\) with \(f^{-1}(x)\).

Step 1

\[
y = \sqrt[5]{5x + 8}
\]

Step 2

\[
x = \sqrt[5]{5y + 8}
\]

Step 3

\[
x = \sqrt[5]{5y + 8}
\]

\[
x^7 = 5y + 8
\]

\[
x^7 - 8 = 5y
\]

\[
y = \frac{1}{5}(x^7 - 8) \quad \rightarrow \quad f^{-1}(x) = \frac{1}{5}(x^7 - 8)
\]

Finally, compute either \((f \circ f^{-1})(x)\) or \((f^{-1} \circ f)(x)\) to verify our work.

Step 4
Either composition can be done so let’s do \((f \circ f^{-1})(x)\) in this case.

\[
(f \circ f^{-1})(x) = f\left[f^{-1}(x)\right]
= \sqrt[7]{5 \left[\frac{1}{5}(x^7 - 8)\right] + 8}
= \sqrt[7]{x^7 - 8} + 8
= \sqrt[7]{x^7}
= x
\]

So, we got \(x\) out of the composition and so we know we’ve done our work correctly.

7. Find the inverse for \(h(x) = \frac{1 + 9x}{4 - x}\). Verify your inverse by computing one or both of the composition as discussed in this section.

Hint: Remember the process described in this section. Replace the \(h(x)\), interchange the \(x\)’s and \(y\)’s, solve for \(y\) and the finally replace the \(y\) with \(h^{-1}(x)\).

Step 1

\[
y = \frac{1 + 9x}{4 - x}
\]

Step 2

\[
x = \frac{1 + 9y}{4 - y}
\]

Step 3

\[
x = \frac{1 + 9y}{4 - y}
\]
\[
x(4 - y) = 1 + 9y
\]
\[
4x - xy = 1 + 9y
\]
\[
4x - 1 = 9y + xy
\]
\[
4x - 1 = (9 + x)y
\]
\[
y = \frac{4x - 1}{9 + x} \quad \rightarrow \quad h^{-1}(x) = \frac{4x - 1}{9 + x}
\]
Note that the Algebra in these kinds of problems can often be fairly messy, but don’t let that make you decide that you can’t do these problems. Messy Algebra will be a fairly common occurrence in a Calculus class so you’ll need to get used to it!

Finally, compute either \( (h \circ h^{-1})(x) \) or \( (h^{-1} \circ h)(x) \) to verify our work.

**Step 4**
Either composition can be done so let’s do \( (h^{-1} \circ h)(x) \) in this case. As with the previous step, the Algebra here is going to be messy and in fact will probably be messier.

\[
(h^{-1} \circ h)(x) = h^{-1}[h(x)]
\]

\[
= 4 \left[ \frac{1+9x}{4-x} \right] - 1 \quad 4 - x
\]

\[
= 9 + \frac{1+9x}{4-x} \quad 4 - x
\]

\[
= 4(1+9x) - (4-x)
\]

\[
= 9(4-x) + 1 + 9x
\]

\[
= \frac{4 + 36x - 4 + x}{36 - 9x + 1 + 9x}
\]

\[
= \frac{37x}{37}
\]

\[
= x
\]

In order to do the simplification we multiplied the numerator and denominator of the initial fraction by \( 4 - x \) in order to clear out some of the denominators. This in turn allowed a fair amount of simplification.

So, we got \( x \) out of the composition and so we know we’ve done our work correctly.

8. Find the inverse for \( f(x) = \frac{6-10x}{8x+7} \). Verify your inverse by computing one or both of the composition as discussed in this section.

**Hint:** Remember the process described in this section. Replace the \( f(x) \), interchange the \( x \)'s and \( y \)'s, solve for \( y \) and the finally replace the \( y \) with \( f^{-1}(x) \).

**Step 1**

\[
y = \frac{6-10x}{8x+7}
\]
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Step 2

\[ x = \frac{6 - 10y}{8y + 7} \]

Step 3

\[ x = \frac{6 - 10y}{8y + 7} \]
\[ x(8y + 7) = 6 - 10y \]
\[ 8xy + 7x = 6 - 10y \]
\[ 8xy + 10y = 6 - 7x \]
\[ (8x + 10)y = 6 - 7x \]
\[ y = \frac{6 - 7x}{8x + 10} \]
\[ \rightarrow \]

\[ f^{-1}(x) = \frac{6 - 7x}{8x + 10} \]

Note that the Algebra in these kinds of problems can often be fairly messy, but don’t let that make you decide that you can’t do these problems. Messy Algebra will be a fairly common occurrence in a Calculus class so you’ll need to get used to it!

Finally, compute either \((f \circ f^{-1})(x)\) or \((f^{-1} \circ f)(x)\) to verify our work.

Step 4

Either composition can be done so let’s do \((f \circ f^{-1})(x)\) in this case. As with the previous step, the Algebra here is going to be messy and in fact will probably be messier.

\[ (f \circ f^{-1})(x) = f\left[f^{-1}(x)\right] \]
\[ = \frac{6 - 10\left[\frac{6 - 7x}{8x + 10}\right]}{8\left[\frac{6 - 7x}{8x + 10}\right] + 7} \]
\[ = \frac{6(8x + 10) - 10(6 - 7x)}{8(6 - 7x) + 7(8x + 10)} \]
\[ = \frac{48x + 120 - 60 + 70x}{48 - 56x + 56x + 70} \]
\[ = \frac{118x}{118} \]
\[ = x \]

So, we got \(x\) out of the composition and so we know we’ve done our work correctly.

**Review : Trig Functions**

1. Determine the exact value of \( \cos \left( \frac{5\pi}{6} \right) \) without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1
First we can notice that \( \pi - \frac{\pi}{6} = \frac{5\pi}{6} \) and so the terminal line for \( \frac{5\pi}{6} \) will form an angle of \( \frac{\pi}{6} \) with the negative x-axis in the second quadrant and we’ll have the following unit circle for this problem.
Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{5\pi}{6}$ to the coordinates of the line representing $\frac{\pi}{6}$ and use those to answer the question.

Step 2
The coordinates of the line representing $\frac{5\pi}{6}$ will be the same as the coordinates of the line representing $\frac{\pi}{6}$ except that the $x$ coordinate will now be negative. So, our new coordinates will then be \(-\frac{\sqrt{3}}{2}, \frac{1}{2}\) and so the answer is,

$$\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$$

2. Determine the exact value of $\sin\left(-\frac{4\pi}{3}\right)$ without using a calculator.

Hint 1: Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1
First we can notice that $-\pi - \frac{\pi}{3} = -\frac{4\pi}{3}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{4\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the negative $x$-axis in the second quadrant and we’ll have the following unit circle for this problem.
Hint 2 : Given the obvious symmetry in the unit circle relate the coordinates of the line representing $-\frac{4\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use those to answer the question.

Step 2

The coordinates of the line representing $-\frac{4\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the $x$ coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\sin \left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
3. Determine the exact value of $\sin\left(\frac{7\pi}{4}\right)$ without using a calculator.

Hint 1: Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1
First we can notice that $2\pi - \frac{\pi}{4} = \frac{7\pi}{4}$ and so the terminal line for $\frac{7\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the positive $x$-axis in the fourth quadrant and we’ll have the following unit circle for this problem.

Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{7\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and use those to answer the question.
Step 2
The coordinates of the line representing $\frac{7\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the $y$ coordinate will now be negative. So, our new coordinates will then be $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\sin\left(\frac{7\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

4. Determine the exact value of $\cos\left(-\frac{2\pi}{3}\right)$ without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1
First we can notice that $-\pi + \frac{\pi}{3} = -\frac{2\pi}{3}$ so (recalling that negative angles rotate clockwise and positive angles rotation counter clockwise) the terminal line for $-\frac{2\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the negative $x$-axis in the third quadrant and we’ll have the following unit circle for this problem.
Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing $-\frac{2\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use those to answer the question.

Step 2

The line representing $-\frac{2\pi}{3}$ is a mirror image of the line representing $\frac{\pi}{3}$ and so the coordinates for $-\frac{2\pi}{3}$ will be the same as the coordinates for $\frac{\pi}{3}$ except that both coordinates will now be negative. So, our new coordinates will then be $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and so the answer is,

$$\cos\left(-\frac{2\pi}{3}\right) = -\frac{1}{2}$$
5. Determine the exact value of \( \tan \left( \frac{3\pi}{4} \right) \) without using a calculator.

Hint 1: Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1
First we can notice that \( \pi - \frac{\pi}{4} = \frac{3\pi}{4} \) and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for \( \frac{3\pi}{4} \) will form an angle of \( \frac{\pi}{4} \) with the negative x-axis in the second quadrant and we’ll have the following unit circle for this problem.
Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{3\pi}{4}$ to the coordinates of the line representing $\frac{\pi}{4}$ and then recall how tangent is defined in terms of sine and cosine to answer the question.

Step 2

The coordinates of the line representing $\frac{3\pi}{4}$ will be the same as the coordinates of the line representing $\frac{\pi}{4}$ except that the $x$ coordinate will now be negative. So, our new coordinates will then be $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ and so the answer is,

$$\tan \left(\frac{3\pi}{4}\right) = \frac{\sin \left(\frac{3\pi}{4}\right)}{\cos \left(\frac{3\pi}{4}\right)} = \frac{\sqrt{2}/2}{-\sqrt{2}/2} = -1$$

6. Determine the exact value of $\sec \left(-\frac{11\pi}{6}\right)$ without using a calculator.

Hint 1: Even though a unit circle only tells us information about sine and cosine it is still useful for secant so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that $\frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$ and so (remembering that negative angles are rotated clockwise) we can see that the terminal line for $-\frac{11\pi}{6}$ will form an angle of $\frac{\pi}{6}$ with the positive $x$-axis in the first quadrant. In other words $-\frac{11\pi}{6}$ and $\frac{\pi}{6}$ represent the same angle. So, we’ll have the following unit circle for this problem.
Hint 2 : Given the obvious symmetry here use the definition of secant in terms of cosine to write down the solution.

Step 2
Because the two angles $-\frac{11\pi}{6}$ and $\frac{\pi}{6}$ have the same coordinates the answer is,

$$\sec\left(-\frac{11\pi}{6}\right) = \frac{1}{\cos\left(-\frac{11\pi}{6}\right)} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

7. Determine the exact value of $\cos\left(\frac{8\pi}{3}\right)$ without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.
Step 1

First we can notice that \(2\pi + \frac{2\pi}{3} = \frac{8\pi}{3}\) and because \(2\pi\) is one complete revolution the angles \(\frac{8\pi}{3}\) and \(\frac{2\pi}{3}\) are the same angle. Also, note that \(\pi - \frac{\pi}{3} = \frac{2\pi}{3}\) and so the terminal line for \(\frac{8\pi}{3}\) will form an angle of \(\frac{\pi}{3}\) with the negative \(x\)-axis in the second quadrant and we’ll have the following unit circle for this problem.

![Unit Circle Diagram](image)

Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing \(\frac{8\pi}{3}\) to the coordinates of the line representing \(\frac{2\pi}{3}\) and use those to answer the question.

Step 2
The coordinates of the line representing $\frac{8\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the $x$ coordinate will now be negative. So, our new coordinates will then be $\left( -\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$ and so the answer is,

$$\cos\left( \frac{8\pi}{3} \right) = -\frac{1}{2}$$

8. Determine the exact value of $\tan\left( -\frac{\pi}{3} \right)$ without using a calculator.

Hint 1 : Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

To do this problem all we need to notice is that $-\frac{\pi}{3}$ will form an angle of $\frac{\pi}{3}$ with the positive $x$-axis in the fourth quadrant and we’ll have the following unit circle for this problem.
Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing $\frac{5\pi}{3}$ to the coordinates of the line representing $\frac{\pi}{3}$ and use the definition of tangent in terms of sine and cosine to answer the question.

Step 2
The coordinates of the line representing $\frac{5\pi}{3}$ will be the same as the coordinates of the line representing $\frac{\pi}{3}$ except that the $y$ coordinate will now be negative. So, our new coordinates will then be $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and so the answer is,
\[ \tan \left( -\frac{\pi}{3} \right) = \frac{\sin \left( -\frac{\pi}{3} \right)}{\cos \left( -\frac{\pi}{3} \right)} = \frac{-\sqrt{3}}{2} = -\sqrt{3} \]

9. Determine the exact value of \( \tan \left( \frac{15\pi}{4} \right) \) without using a calculator.

Hint 1: Even though a unit circle only tells us information about sine and cosine it is still useful for tangents so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1

First we can notice that \( 4\pi - \frac{\pi}{4} = \frac{15\pi}{4} \) and also note that \( 4\pi \) is two complete revolutions so the terminal line for \( \frac{15\pi}{4} \) and \( -\frac{\pi}{4} \) represent the same angle. Also note that \( -\frac{\pi}{4} \) will form an angle of \( \frac{\pi}{4} \) with the positive \( x \)-axis in the fourth quadrant and we’ll have the following unit circle for this problem.
Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing \( \frac{15\pi}{4} \) to the coordinates of the line representing \( \frac{\pi}{4} \) and the definition of tangent in terms of sine and cosine to answer the question.

Step 2

The coordinates of the line representing \( \frac{15\pi}{4} \) will be the same as the coordinates of the line representing \( \frac{\pi}{4} \) except that the \( y \) coordinate will now be negative. So, our new coordinates will then be \( \left( \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \) and so the answer is,
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\[
\tan \left( \frac{15\pi}{4} \right) = \frac{\sin \left( \frac{15\pi}{4} \right)}{\cos \left( \frac{15\pi}{4} \right)} = \frac{-\sqrt{2}}{2

10. Determine the exact value of \( \sin \left( -\frac{11\pi}{3} \right) \) without using a calculator.

Hint 1 : Sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1
First we can notice that \( \frac{\pi}{3} - 4\pi = -\frac{11\pi}{3} \) and note that \( 4\pi \) is two complete revolutions (also, remembering that negative angles are rotated clockwise) we can see that the terminal line for \( -\frac{11\pi}{3} \) and \( \frac{\pi}{3} \) are the same angle and so we’ll have the following unit circle for this problem.
Hint 2: Given the very obvious symmetry here write down the answer to the question.

Step 2
Because $-\frac{11\pi}{3}$ and $\frac{\pi}{3}$ are the same angle the answer is,

$$\sin\left(-\frac{11\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

11. Determine the exact value of $\sec\left(\frac{29\pi}{4}\right)$ without using a calculator.

Hint 1: Even though a unit circle only tells us information about sine and cosine it is still useful for secant so sketch a unit circle and relate the angle to one of the standard angles in the first quadrant.

Step 1
First we can notice that $\frac{5\pi}{4} + 6\pi = \frac{25\pi}{4}$ and recalling that $6\pi$ is three complete revolutions we can see that $\frac{25\pi}{4}$ and $\frac{5\pi}{4}$ represent the same angle. Next, note that $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$ and so the line representing $\frac{5\pi}{4}$ will form an angle of $\frac{\pi}{4}$ with the negative $x$-axis in the third quadrant and we’ll have the following unit circle for this problem.
Hint 2: Given the obvious symmetry in the unit circle relate the coordinates of the line representing \( \frac{25\pi}{4} \) to the coordinates of the line representing \( \frac{\pi}{4} \) and the recall how secant is defined in terms of cosine to answer the question.

Step 2

The line representing \( \frac{25\pi}{4} \) is a mirror image of the line representing \( \frac{\pi}{4} \) and so the coordinates for \( \frac{25\pi}{4} \) will be the same as the coordinates for \( \frac{\pi}{4} \) except that both coordinates will now be negative. So, our new coordinates will then be \( \left( -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right) \) and so the answer is,
\[
\sec\left(\frac{29\pi}{4}\right) = \frac{1}{\sec\left(\frac{29\pi}{4}\right)} = \frac{1}{\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}
\]

**Review : Solving Trig Equations**

1. Without using a calculator find all the solutions to \(4 \sin(3t) = 2\).

Hint 1 : Isolate the sine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the sine (with a coefficient of one) on one side of the equation gives,
\[
\sin(3t) = \frac{1}{2}
\]

Hint 2 : Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which sine will have this value.

Step 2
Because we’re dealing with sine in this problem and we know that the \(y\)-axis represents sine on a unit circle we’re looking for angles that will have a \(y\) coordinate of \(\frac{1}{2}\). This means we’ll have an angle in the first quadrant and an angle in the second quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.
Clearly the angle in the first quadrant is $\frac{\pi}{6}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{6}$ with the negative $x$-axis as shown above and so it will be: $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Hint 3: Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+ 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.

This then means that we must have,
Finally, to get all the solutions to the equation all we need to do is divide both sides by 3.

\[
\begin{align*}
t &= \frac{\pi + 2\pi n}{18} \quad \text{OR} \quad t &= \frac{5\pi + 2\pi n}{18} \\
n &= 0, \pm 1, \pm 2, \ldots
\end{align*}
\]

2. Without using a calculator find the solution(s) to \(4 \sin(3t) = 2\) that are in \([0, \frac{4\pi}{3}]\).

Hint 1 : First, find all the solutions to the equation without regard to the given interval.

Step 1
Because we found all the solutions to this equation in Problem 1 of this section we’ll just list the result here. For full details on how these solutions were obtained please see the solution to Problem 1.

All solutions to the equation are,

\[
\begin{align*}
t &= \frac{\pi + 2\pi n}{18} \quad \text{OR} \quad t &= \frac{5\pi + 2\pi n}{18} \\
n &= 0, \pm 1, \pm 2, \ldots
\end{align*}
\]

Hint 2 : Now all we need to do is plug in values of \(n\) to determine which solutions will actually fall in this interval.

Step 2
Note that because at least some of the solutions will have a denominator of 18 it will probably be convenient to also have the interval written in terms of fractions with denominators of 18. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[
\left[0, \frac{4\pi}{3}\right] = \left[0, \frac{24\pi}{18}\right]
\]

With the interval written in this form, if our potential solutions have a denominator of 18, all we need to do is compare numerators. As long as the numerators are positive and less than \(24\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \(n\) it will be much easier to have both fractions in the solutions have denominators of 18. So the solutions, written in this form, are.
Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \( n = 0 \) and see what we get.

\[
\begin{align*}
\text{n = 0: } & \quad t = \frac{\pi}{18} \quad \text{OR} \quad t = \frac{5\pi}{18} \\
\text{n = 1: } & \quad t = \frac{13\pi}{18} \quad \text{OR} \quad t = \frac{17\pi}{18} \\
\text{n = 2: } & \quad t = \frac{25\pi}{18} \quad \text{OR} \quad t = \frac{29\pi}{18}
\end{align*}
\]

Note that we didn’t really need to plug in \( n = 2 \) above to see that they would not work. With each increase in \( n \) we were really just adding another \( \frac{12\pi}{18} \) onto the previous results and by a quick inspection we could see that adding \( 12\pi \) to the numerator of either solution from the \( n = 1 \) step would result in a numerator that is larger than \( 24\pi \) and so would result in a solution that is outside of the interval. This is not something that must be noticed in order to work the problem, but noticing this would definitely help reduce the amount of actual work.

So, it looks like we have the four solutions to this equation in the given interval.

\[
\begin{align*}
t & = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}
\end{align*}
\]

3. Without using a calculator find all the solutions to \( 2\cos\left(\frac{x}{3}\right) + \sqrt{2} = 0 \).

Hint 1: Isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos\left(\frac{x}{3}\right) = -\frac{\sqrt{2}}{2}
\]
Hint 2 : Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle we’re looking for angles that will have a x coordinate of $-\frac{\sqrt{2}}{2}$. This means that we’ll have angles in the second and third quadrant.

Because of the negative value we can’t just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.

If we didn’t have the negative value then the angle would be $\frac{\pi}{4}$. Now, based on the symmetry in the unit circle, the terminal line for both of the angles will form an angle of $\frac{\pi}{4}$ with the negative
The angle in the second quadrant will then be \( \pi - \frac{\pi}{4} = \frac{3\pi}{4} \) and the angle in the third quadrant will be \( \pi + \frac{\pi}{4} = \frac{5\pi}{4} \).

Hint 3: Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding \( +2\pi n \) for \( n = 0, \pm 1, \pm 2, \ldots \) onto each of these.

This then means that we must have,
\[
\frac{x}{3} = \frac{3\pi}{4} + 2\pi n \quad \text{OR} \quad \frac{x}{3} = \frac{5\pi}{4} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 3.
\[
x = \frac{9\pi}{4} + 6\pi n \quad \text{OR} \quad x = \frac{15\pi}{4} + 6\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

4. Without using a calculator find the solution(s) to \( 2\cos \left( \frac{x}{3} \right) + \sqrt{2} = 0 \) that are in \([-7\pi, 7\pi]\).

Hint 1: First, find all the solutions to the equation without regard to the given interval.

Step 1
Because we found all the solutions to this equation in Problem 3 of this section we’ll just list the result here. For full details on how these solutions were obtained please see the solution to Problem 3.

All solutions to the equation are,
\[
x = \frac{9\pi}{4} + 6\pi n \quad \text{OR} \quad x = \frac{15\pi}{4} + 6\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 2: Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in this interval.

Step 2
Note that because at least some of the solutions will have a denominator of 4 it will probably be convenient to also have the interval written in terms of fractions with denominators of 4. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[ [-7\pi, 7\pi] = \left[ -\frac{28\pi}{4}, \frac{28\pi}{4} \right] \]

With the interval written in this form, if our potential solutions have a denominator of 4, all we need to do is compare numerators. As long as the numerators are between \(-28\pi\) and \(28\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \(n\) it will be much easier to have both fractions in the solutions have denominators of 4. So the solutions, written in this form, are.

\[ x = \frac{9\pi}{4} + \frac{24\pi n}{4} \quad \text{OR} \quad x = \frac{15\pi}{4} + \frac{24\pi n}{4} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions.

\[
\begin{align*}
n = -2: & \quad x = -\frac{39\pi}{4} < -\frac{28\pi}{4} \quad \text{OR} \quad x = -\frac{33\pi}{4} < -\frac{28\pi}{4} \\
n = -1: & \quad x = -\frac{15\pi}{4} \quad \text{OR} \quad x = -\frac{9\pi}{4} \\
n = 0: & \quad x = \frac{9\pi}{4} \quad \text{OR} \quad x = \frac{15\pi}{4} \\
n = 1: & \quad x = \frac{39\pi}{4} > \frac{28\pi}{4} \quad \text{OR} \quad x = \frac{33\pi}{4} > \frac{28\pi}{4}
\end{align*}
\]

Note that we didn’t really need to plug in \(n = 1\) or \(n = -2\) above to see that they would not work. With each increase in \(n\) we were really just adding (for positive \(n\)) or subtracting (for negative \(n\)) another \(\frac{24\pi}{4}\) from the previous results. By a quick inspection we could see that adding \(24\pi\) to the numerator of either solution from the \(n = 1\) step would result in a numerator that is larger than \(28\pi\) and so would result in a solution that is outside of the interval. Likewise, for the \(n = -2\) case, subtracting \(24\pi\) from each of the numerators will result in numerators that will be less than \(-28\pi\) and so will not be in the interval. This is not something that must be noticed in order to work the problem, but noticing this would definitely help reduce the amount of actual work.

So, it looks like we have the four solutions to this equation in the given interval.

\[
\begin{align*}
x = & \frac{15\pi}{4}, -\frac{9\pi}{4}, -\frac{9\pi}{4}, \frac{15\pi}{4}
\end{align*}
\]
5. Without using a calculator find the solution(s) to \(4 \cos(6z) = \sqrt{12}\) that are in \([0, \frac{\pi}{2}]\).

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos(6z) = \frac{\sqrt{12}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}
\]

Notice that we needed to do a little simplification of the root to get the value into a more recognizable form. This kind of simplification is usually a good thing to do.

Hint 2: Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle we’re looking for angles that will have a x coordinate of \(\frac{\sqrt{3}}{2}\). This means we’ll have an angle in the first quadrant and an angle in the fourth quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.
Clearly the angle in the first quadrant is $\frac{\pi}{6}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{6}$ with the positive $x$-axis as shown above and so it will be: $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{6}$ for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.
Hint 3: Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “+2\pi n for \( n = 0, \pm 1, \pm 2, \ldots \)” onto each of these.

This then means that we must have,

\[ 6z = \frac{\pi}{6} + 2\pi n \quad \text{OR} \quad 6z = \frac{11\pi}{6} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 6.

\[ z = \frac{\pi}{36} + \frac{\pi n}{3} \quad \text{OR} \quad z = \frac{11\pi}{36} + \frac{\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4: Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 36 it will probably be convenient to also have the interval written in terms of fractions with denominators of 36. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[ \left[ 0, \frac{\pi}{2} \right] = \left[ 0, \frac{18\pi}{36} \right] \]

With the interval written in this form, if our potential solutions have a denominator of 36, all we need to do is compare numerators. As long as the numerators are positive and less than \( 18\pi \) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 36. So the solutions, written in this form, are.

\[ z = \frac{\pi}{36} + \frac{12\pi n}{36} \quad \text{OR} \quad z = \frac{11\pi}{36} + \frac{12\pi n}{36} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \( n = 0 \) and see what we get.
There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of $n$ will not necessarily be in the given interval. It is completely possible, as this problem shows, that we will only get one or the other solution from a given value of $n$ to fall in the given interval.

Next notice that with each increase in $n$ we were really just adding another $\frac{12\pi}{36}$ onto the previous results and by a quick inspection we could see that adding $12\pi$ to the numerator of the first solution from the $n = 1$ step would result in a numerator that is larger than $\frac{18\pi}{36}$ and so would result in a solution that is outside of the interval. Therefore, there was no reason to plug in $n = 2$ into the first set of solutions. Of course, we also didn’t plug $n = 2$ into the second set because once we’ve gotten out of the interval adding anything else on will remain out of the interval.

So, it looks like we have the three solutions to this equation in the given interval.

$$z = \frac{\pi}{36}, \frac{11\pi}{36}, \frac{13\pi}{36}$$

6. Without using a calculator find the solution(s) to $2 \sin \left( \frac{3\gamma}{2} \right) + \sqrt{3} = 0$ that are in $\left[ -\frac{7\pi}{3}, 0 \right]$.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the sine (with a coefficient of one) on one side of the equation gives,

$$\sin \left( \frac{3\gamma}{2} \right) = -\frac{\sqrt{3}}{2}$$

Hint 2: Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2
Because we’re dealing with sine in this problem and we know that the $y$-axis represents sine on a unit circle we’re looking for angles that will have a $y$ coordinate of $-\frac{\sqrt{3}}{2}$. This means that we’ll have angles in the third and fourth quadrant.

Because of the negative value we can’t just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.

If we didn’t have the negative value then the angle would be $\frac{\pi}{3}$. Now, based on the symmetry in the unit circle, the terminal line for the first angle will form an angle of $\frac{\pi}{3}$ with the negative $x$-axis and the terminal line for the second angle will form an angle of $\frac{\pi}{3}$ with the positive $x$-axis.
The angle in the third quadrant will then be \( \pi + \frac{\pi}{3} = \frac{4\pi}{3} \) and the angle in the fourth quadrant will be \( 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \).

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used \( -\frac{\pi}{3} \) for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.

Hint 3 : Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “\( + 2\pi n \) for \( n = 0, \pm 1, \pm 2, \ldots \)” onto each of these.

This then means that we must have,

\[
\frac{3y}{2} = \frac{4\pi}{3} + 2\pi n \quad \text{OR} \quad \frac{3y}{2} = \frac{5\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by \( \frac{2}{3} \).

\[
y = \frac{8\pi}{9} + \frac{4\pi n}{3} \quad \text{OR} \quad y = \frac{10\pi}{9} + \frac{4\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4 : Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 9 it will probably be convenient to also have the interval written in terms of fractions with denominators of 9. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[
\left[ -\frac{7\pi}{3}, 0 \right] = \left[ -\frac{21\pi}{9}, 0 \right]
\]

With the interval written in this form, if our potential solutions have a denominator of 9, all we need to do is compare numerators. As long as the numerators are negative and greater than \(-21\pi\) we’ll know that the solution is in the interval.
Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 9. So the solutions, written in this form, are.

\[ y = \frac{8\pi}{9} + \frac{12\pi n}{9} \quad \text{OR} \quad y = \frac{10\pi}{9} + \frac{12\pi n}{9} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions. First notice that, in this case, if we plug in positive values of \( n \) or zero we will get positive solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \( n = -1 \) and see what we get.

\[ n = -1: \quad y = -\frac{4\pi}{9} \quad \text{OR} \quad y = -\frac{2\pi}{9} \]

\[ n = -2: \quad y = -\frac{16\pi}{9} \quad \text{OR} \quad y = -\frac{14\pi}{9} \]

Notice that with each increase (in the negative sense anyway) in \( n \) we were really just subtracting another \( \frac{12\pi}{9} \) from the previous results and by a quick inspection we could see that subtracting \( 12\pi \) from either of the numerators from the \( n = -2 \) solutions the numerators will be less than \(-21\pi\) and so will be out of the interval. There is no reason to write down the \( n = -3 \) solutions since we know that they will not be in the given interval.

So, it looks like we have the four solutions to this equation in the given interval.

\[ y = -\frac{16\pi}{9}, -\frac{14\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9} \]

7. Without using a calculator find the solution(s) to \( 8 \tan(2x) - 5 = 3 \) that are in \( \left[ -\frac{\pi}{2}, \frac{3\pi}{2} \right] \).

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

Step 1
Isolating the tangent (with a coefficient of one) on one side of the equation gives,

\[ \tan(2x) = 1 \]

Hint 2: Determine all the angles in the range \( [0, 2\pi] \) for which tangent will have this value.
Step 2
If tangent has a value of 1 then we know that sine and cosine must be the same. This means that, in the first quadrant, the solution is $\frac{\pi}{4}$. We also know that sine and cosine will be the same in the third quadrant and we can use the basic symmetry on our unit circle to determine this value. Here is a unit circle for this situation.

By basic symmetry we can see that the line terminal line for the second angle must form an angle of $\frac{\pi}{4}$ with the negative $x$-axis as shown above and so it will be: $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Hint 3: Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.
This then means that we must have,

\[ 2x = \frac{\pi}{4} + 2\pi n \quad \text{OR} \quad 2x = \frac{5\pi}{4} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 2.

\[ x = \frac{\pi}{8} + \pi n \quad \text{OR} \quad x = \frac{5\pi}{8} + \pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4: Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 8 it will probably be convenient to also have the interval written in terms of fractions with denominators of 8. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[ \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8} \]

With the interval written in this form, if our potential solutions have a denominator of 8, all we need to do is compare numerators. As long as the numerators are between \(-4\pi\) and \(12\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 8. So the solutions, written in this form, are.

\[ x = \frac{\pi}{8} + \frac{8\pi n}{8} \quad \text{OR} \quad x = \frac{5\pi}{8} + \frac{8\pi n}{8} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions.

\begin{align*}
  n = -1: & \quad x = \frac{-7\pi}{8} < \frac{-4\pi}{8} \quad \text{OR} \quad x = -\frac{3\pi}{8} \\
  n = 0: & \quad x = \frac{\pi}{8} \quad \text{OR} \quad x = \frac{5\pi}{8} \\
  n = 1: & \quad x = \frac{9\pi}{8} \quad \text{OR} \quad x = \frac{13\pi}{8} > \frac{12\pi}{8}
\end{align*}

There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of \( n \) will not necessarily be in the given interval. It
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is completely possible, as this problem shows, that we will only get one or the other solution from a given value of \( n \) to fall in the given interval.

Next notice that with each increase in \( n \) we were really just adding/subtracting (depending upon the sign of \( n \)) another \( \frac{8\pi}{8} \) from the previous results and by a quick inspection we could see that adding \( 8\pi \) to the numerator of the \( n = 1 \) solutions would result in numerators that are larger than \( 12\pi \) and so would result in solutions that are outside of the interval. Likewise, subtracting \( 8\pi \) from the \( n = -1 \) solutions would result in numerators that are smaller than \( -4\pi \) and so would result in solutions that are outside the interval. Therefore, there is no reason to even go past the values of \( n \) listed here.

So, it looks like we have the four solutions to this equation in the given interval.

\[
\begin{align*}
x &= -\frac{3\pi}{8}, \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}
\end{align*}
\]

8. Without using a calculator find the solution(s) to \( 16 = -9 \sin(7x) - 4 \) that are in \( \left[-2\pi, \frac{9\pi}{4}\right] \).

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1

Isolating the sine (with a coefficient of one) on one side of the equation gives,

\[
\sin(7x) = -\frac{20}{9} < -1
\]

Okay, at this point we can stop all work. We know that \(-1 \leq \sin \theta \leq 1\) for any argument and so in this case there is no solution. This will happen on occasion and we shouldn’t get to excited about it when it happens.

9. Without using a calculator find the solution(s) to \( \sqrt{3} \tan\left(\frac{t}{4}\right) + 5 = 4 \) that are in \( [0, 4\pi] \).

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

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Step 1
Isolating the tangent (with a coefficient of one) on one side of the equation gives,

\[ \tan \left( \frac{t}{4} \right) = -\frac{1}{\sqrt{3}} \]

Hint 2: Determine all the angles in the range \([0, 2\pi]\) for which tangent will have this value.

Step 2
To get the first angle here let’s recall the definition of tangent in terms of sine and cosine.

\[ \tan \left( \frac{t}{4} \right) = \frac{\sin \left( \frac{t}{4} \right)}{\cos \left( \frac{t}{4} \right)} = -\frac{1}{\sqrt{3}} \]

Now, because of the section we’re in, we know that the angle must be one of the “standard” angles and from a quick look at a unit circle (shown below) we know that for \( \frac{\pi}{6} \) we will have,

\[ \frac{\sin \left( \frac{\pi}{6} \right)}{\cos \left( \frac{\pi}{6} \right)} = \frac{1}{2} = \frac{1}{\sqrt{3}} \]

So, if we had a positive value on the tangent we’d have the first angle. We do have a negative value however, but this work will allow us to get the two angles we’re after. Because the value is negative this simply means that the sine and cosine must have the same values that they have for \( \frac{\pi}{6} \) except that one must be positive and the other must be negative. This means that the angles that we’re after must be in the second and fourth quadrants. Here is a unit circle for this situation.
By basic symmetry we can see that the terminal line for the angle in the second quadrant must form an angle of $\frac{\pi}{6}$ with the negative $x$-axis and the terminal line in the fourth quadrant must form an angle of $\frac{\pi}{6}$ with the positive $x$-axis as shown above. The angle in the second quadrant will then be: $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ while the angle in the fourth quadrant will be $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{6}$ for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.
Hint 3 : Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.

This then means that we must have,

$$t = \frac{5\pi}{6} + 2\pi n \quad \text{OR} \quad t = \frac{11\pi}{6} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 4.

$$t = \frac{10\pi}{3} + 8\pi n \quad \text{OR} \quad t = \frac{22\pi}{3} + 8\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

Hint 4 : Now all we need to do is plug in values of $n$ to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 3 it will probably be convenient to also have the interval written in terms of fractions with denominators of 3. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[ 0, 4\pi \right] = \left[ 0, \frac{12\pi}{3} \right]$$

With the interval written in this form, if our potential solutions have a denominator of 3, all we need to do is compare numerators. As long as the numerators are positive and less than $\frac{12\pi}{3}$ we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of $n$ it will be much easier to have both fractions in the solutions have denominators of 3. So the solutions, written in this form, are.

$$t = \frac{10\pi}{3} + \frac{24\pi n}{3} \quad \text{OR} \quad t = \frac{22\pi}{3} + \frac{24\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots$$

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of $n$ we will get negative solutions and these will not be in the interval and so there is no reason to even try these. Next, notice that for any positive $n$ we will be adding $\frac{24\pi}{3}$ onto a positive quantity and so are guaranteed to be greater than $\frac{12\pi}{3}$ and so will out of the given interval. This
leaves \( n = 0 \) and for this one we can notice that the only solution that will fall in the given interval is then,

\[
\frac{10\pi}{3}
\]

Before leaving this problem let’s note that on occasion we will only get a single solution out of all the possible solutions that will fall in the given interval. So, don’t get excited about it if this should happen.

10. Without using a calculator find the solution(s) to \( \sqrt{3} \csc (9z) - 7 = -5 \) that are in \( \left[ -\frac{\pi}{3}, \frac{4\pi}{9} \right] \).

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosecant (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosecant (with a coefficient of one) on one side of the equation gives,

\[
\csc (9z) = \frac{2}{\sqrt{3}}
\]

Hint 2 : We need to determine all the angles in the range \( \left[ 0, 2\pi \right] \) for which cosecant will have this value. The best way to do this is to rewrite this equation into one in terms of a different trig function that we can more easily deal with.

Step 2
The best way to do this is to recall the definition of cosecant in terms of sine and rewrite the equation in terms sine instead as that will be easier to deal with. Doing this gives,

\[
\csc (9z) = \frac{1}{\sin (9z)} = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad \sin (9z) = \frac{\sqrt{3}}{2}
\]

The solution(s) to the equation with sine in it are the same as the solution(s) to the equation with cosecant in it and so let’s work with that instead.

At this point we are now dealing with sine and we know that the \( y \)-axis represents sine on a unit circle. So we’re looking for angles that will have a \( y \) coordinate of \( \frac{\sqrt{3}}{2} \). This means we’ll have
an angle in the first quadrant and an angle in the second quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation

Clearly the angle in the first quadrant is $\frac{\pi}{3}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{3}$ with the negative $x$-axis as shown above and so it will be: $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Hint 3: Using the two angles above write down all the angles for which sine/cosecant will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.
This then means that we must have,

\[ 9z = \frac{\pi}{3} + 2\pi n \quad \text{OR} \quad 9z = \frac{2\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 9.

\[ z = \frac{\pi}{27} + \frac{2\pi n}{9} \quad \text{OR} \quad z = \frac{2\pi}{27} + \frac{2\pi n}{9} \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4 : Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 27 it will probably be convenient to also have the interval written in terms of fractions with denominators of 27. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[ \left[ -\frac{\pi}{3}, \frac{4\pi}{9} \right] = \left[ -\frac{9\pi}{27}, \frac{12\pi}{27} \right] \]

With the interval written in this form, if our potential solutions have a denominator of 27, all we need to do is compare numerators. As long as the numerators are between \(-9\pi\) and \(12\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 27. So the solutions, written in this form, are.

\[ z = \frac{\pi}{27} + \frac{6\pi n}{27} \quad \text{OR} \quad z = \frac{2\pi}{27} + \frac{6\pi n}{27} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions.

\[ n = -1 : \quad z = -\frac{5\pi}{27} \quad \text{OR} \quad z = -\frac{4\pi}{27} \]

\[ n = 0 : \quad z = \frac{\pi}{27} \quad \text{OR} \quad z = \frac{2\pi}{27} \]

\[ n = 1 : \quad z = \frac{7\pi}{27} \quad \text{OR} \quad z = \frac{8\pi}{27} \]

Notice that with each increase in \( n \) we were really just adding/subtracting (depending upon the sign of \( n \)) another \( \frac{6\pi}{27} \) from the previous results and by a quick inspection we could see that adding \( 6\pi \) to the numerator of the \( n = 1 \) solutions would result in numerators that are larger than \( 12\pi \) and so would result in solutions that are outside of the interval. Likewise, subtracting \( 6\pi \) from the \( n = -1 \) solutions would result in numerators that are smaller than \(-9\pi\) and so would
result in solutions that are outside the interval. Therefore, there is no reason to even go past the values of $n$ listed here.

So, it looks like we have the six solutions to this equation in the given interval.

\[
x = \frac{-5\pi}{27}, \frac{-4\pi}{27}, \frac{\pi}{27}, \frac{2\pi}{27}, \frac{7\pi}{27}, \frac{8\pi}{27}
\]

11. Without using a calculator find the solution(s) to \(1 - 14\cos\left(\frac{2x}{5}\right) = -6\) that are in \([5\pi, \frac{40\pi}{3}]\).

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos\left(\frac{2x}{5}\right) = \frac{1}{2}
\]

Hint 2: Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the $x$-axis represents cosine on a unit circle we’re looking for angles that will have a $x$ coordinate of $\frac{1}{2}$. This means we’ll have an angle in the first quadrant and an angle in the fourth quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.
Clearly the angle in the first quadrant is $\frac{\pi}{3}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{3}$ with the positive $x$-axis as shown above and so it will be: $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{3}$ for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.
Hint 3: Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “+2πn” for n = 0, ±1, ±2,…” onto each of these.

This then means that we must have,

\[ \frac{2x}{5} = \frac{\pi}{3} + 2\pi n \quad \text{OR} \quad \frac{2x}{5} = \frac{5\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by \( \frac{5}{2} \).

\[ x = \frac{5\pi}{6} + 5\pi n \quad \text{OR} \quad x = \frac{25\pi}{6} + 5\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4: Now all we need to do is plug in values of n to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 6 it will probably be convenient to also have the interval written in terms of fractions with denominators of 6. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[ \left[ \frac{5\pi}{3} , \frac{40\pi}{3} \right] = \left[ \frac{30\pi}{6} , \frac{80\pi}{6} \right] \]

With the interval written in this form, if our potential solutions have a denominator of 6, all we need to do is compare numerators. As long as the numerators are between \( 30\pi \) and \( 80\pi \) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of n it will be much easier to have both fractions in the solutions have denominators of 6. So the solutions, written in this form, are.

\[ x = \frac{5\pi}{6} + \frac{30\pi n}{6} \quad \text{OR} \quad x = \frac{25\pi}{6} + \frac{30\pi n}{6} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of n we will get negative solutions and these will not be in the interval and so there is no reason to even try these. We can also see from a quick inspection that \( n = 0 \) will result in solutions that are not in the interval and so let’s start at \( n = 1 \) and see what we get.
There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of $n$ will not necessarily be in the given interval. It is completely possible, as this problem shows, that we will only get one or the other solution from a given value of $n$ to fall in the given interval.

Next notice that with each increase in $n$ we were really just adding another $\frac{30\pi}{6}$ onto the previous results and by a quick inspection we could see that adding $30\pi$ to the numerator of the first solution from the $n = 2$ step would result in a numerator that is larger than $80\pi$ and so would result in a solution that is outside of the interval. Therefore, there was no reason to plug in $n = 3$ into the first set of solutions. Of course, we also didn’t plug $n = 3$ into the second set because once we’ve gotten out of the interval adding anything else on will remain out of the interval.

Finally, unlike most of the problems in this section $n = 0$ did not produce any solutions that were in the given interval. This will happen on occasion so don’t get excited about it when it happens.

So, it looks like we have the three solutions to this equation in the given interval.

$$z = \frac{35\pi}{6}, \frac{55\pi}{6}, \frac{65\pi}{6}$$

12. Without using a calculator find the solution(s) to $15 = 17 + 4\cos\left(\frac{y}{7}\right)$ that are in $[10\pi, 15\pi]$.

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos\left(\frac{y}{7}\right) = -\frac{1}{2}$$
Hint 2: Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the \(x\)-axis represents cosine on a unit circle we’re looking for angles that will have a \(x\) coordinate of \(-\frac{1}{2}\). This means that we’ll have angles in the second and third quadrant.

Because of the negative value we can’t just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.
Calculus I

If we didn’t have the negative value then the angle would be $\frac{\pi}{3}$. Now, based on the symmetry in the unit circle, the terminal line for both of the angles will form an angle of $\frac{\pi}{3}$ with the negative $x$-axis. The angle in the second quadrant will then be $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and the angle in the third quadrant will be $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$.

Hint 3 : Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+ 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.

This then means that we must have,

$$\frac{y}{7} = \frac{2\pi}{3} + 2\pi n \quad \text{OR} \quad \frac{y}{7} = \frac{4\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 7.

$$y = \frac{14\pi}{3} + 14\pi n \quad \text{OR} \quad y = \frac{28\pi}{3} + 14\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

Hint 4 : Now all we need to do is plug in values of $n$ to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 3 it will probably be convenient to also have the interval written in terms of fractions with denominators of 3. Doing this will make it much easier to identify solutions that fall inside the interval so,

$$\left[ \frac{30\pi}{3}, \frac{45\pi}{3} \right]$$

With the interval written in this form, if our potential solutions have a denominator of 3, all we need to do is compare numerators. As long as the numerators are between $30\pi$ and $45\pi$ we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of $n$ it will be much easier to have both fractions in the solutions have denominators of 3. So the solutions, written in this form, are.
Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. We can also see from a quick inspection that \( n = 0 \) will result in solutions that are not in the interval and so let’s start at \( n = 1 \) and see what we get.

\[
\begin{align*}
\text{for } n = 1: & \quad x = \frac{56\pi}{3} > \frac{45\pi}{3} \quad \text{OR} \quad x = \frac{80\pi}{3} > \frac{45\pi}{3}
\end{align*}
\]

So, by plugging in \( n = 1 \) we get solutions that are already outside of the interval and increasing \( n \) will simply mean adding another \( \frac{42\pi}{3} \) onto these and so will remain outside of the given interval. We also noticed earlier than all other value of \( n \) will result in solutions outside of the given interval.

What all this means is that while there are solutions to the equation none fall inside the given interval and so the official answer would then be **no solutions in the given interval**.

---

**Review : Solving Trig Equations with Calculators, Part I**

1. Find all the solutions to \( 7 \cos(4x) + 11 = 10 \). Use at least 4 decimal places in your work.

Hint 1 : Isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos(4x) = -\frac{1}{7}
\]

Hint 2 : Using a calculator and your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
First, using our calculator we can see that,
Now we’re dealing with cosine in this problem and we know that the \( x \)-axis represents cosine on a unit circle and so we’re looking for angles that will have a \( x \) coordinate of \( -\frac{1}{7} \). This means that we’ll have angles in the second (this is the one our calculator gave us) and third quadrant. Here is a unit circle for this situation.

From the symmetry of the unit circle we can see that we can either use \(-1.7141\) or \(2\pi - 1.7141 = 4.5691\) for the second angle. Each will give the same set of solutions. However, because it is easy to lose track of minus signs we will use the positive angle for our second solution.

Hint 3 : Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “\( +2\pi n \) for \( n = 0, \pm 1, \pm 2, \ldots \)” onto each of these.

This then means that we must have,

\[
4x = 1.7141 + 2\pi n \quad \text{OR} \quad 4x = 4.5691 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 4.
Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

2. Find the solution(s) to \(6 + 5\cos \left(\frac{x}{3}\right) = 10\) that are in \([0, 3\pi]\). Use at least 4 decimal places in your work.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos \left(\frac{x}{3}\right) = \frac{4}{5}
\]

Hint 2: Using a calculator and your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
First, using our calculator we can see that,

\[
\frac{x}{3} = \cos^{-1} \left(\frac{4}{5}\right) = 0.6435
\]

Now we’re dealing with cosine in this problem and we know that the \(x\)-axis represents cosine on a unit circle and so we’re looking for angles that will have a \(x\) coordinate of \(\frac{4}{5}\). This means that we’ll have angles in the first (this is the one our calculator gave us) and fourth quadrant. Here is a unit circle for this situation.
From the symmetry of the unit circle we can see that we can either use \(-0.6435\) or 
\(2\pi - 0.6435 = 5.6397\) for the second angle. Each will give the same set of solutions. However, 
because it is easy to lose track of minus signs we will use the positive angle for our second 
solution.

Hint 3: Using the two angles above write down all the angles for which cosine will have this 
value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we 
can get all possible angles by simply adding \(+2\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\) onto each of these.

This then means that we must have,

\[ \frac{x}{3} = 0.6435 + 2\pi n \quad \text{OR} \quad \frac{x}{3} = 5.6397 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 3 and 
we’ll convert everything to decimals to help with the final step.

\[ x = 1.9305 + 6\pi n \quad \text{OR} \quad x = 16.9191 + 6\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ = 1.9305 + 18.8496n \quad \text{OR} \quad = 16.9191 + 18.8496n \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4: Now all we need to do is plug in values of \(n\) to determine which solutions will actually 
fall in the given interval.

Step 4
Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \( n = 0 \) and see what we get.

\[
\begin{align*}
  n = 0 : & \quad x = 1.9305 \quad \text{OR} \quad x = 16.9191 \\
  n = 1 : & \quad x = 20.7801 \quad \text{OR} \quad x = 35.7687
\end{align*}
\]

Notice that with each increase in \( n \) we were really just adding another 18.8496 onto the previous results and by doing this to the results from the \( n = 1 \) step we will get solutions that are outside of the interval and so there is no reason to even plug in \( n = 2 \).

So, it looks like we have the four solutions to this equation in the given interval.

\[
\{ x = 1.9305, 16.9191, 20.7801, 35.7687 \}
\]

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

3. Find all the solutions to \( 3 = 6 - 11 \sin \left( \frac{t}{8} \right) \). Use at least 4 decimal places in your work.

Hint 1 : Isolate the sine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the sine (with a coefficient of one) on one side of the equation gives,

\[
\sin \left( \frac{t}{8} \right) = \frac{3}{11}
\]

Hint 2 : Using a calculator and your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which sine will have this value.

Step 2
First, using our calculator we can see that,

\[
\frac{t}{8} = \sin^{-1} \left( \frac{3}{11} \right) = 0.2762
\]

Now we’re dealing with sine in this problem and we know that the \( y \)-axis represents sine on a unit circle and so we’re looking for angles that will have a \( y \) coordinate of \( \frac{3}{11} \). This means that we’ll
have angles in the first (this is the one our calculator gave us) and second quadrant. Here is a unit circle for this situation.

From the symmetry of the unit circle we can see that \( \pi - 0.2762 = 2.8654 \) is the second angle.

Hint 3 : Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding \( +2\pi n \) for \( n = 0, \pm 1, \pm 2, \ldots \) onto each of these.

This then means that we must have,

\[
\frac{t}{8} = 0.2762 + 2\pi n \quad \text{OR} \quad \frac{t}{8} = 2.8654 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 8.

\[
t = 2.2096 + 16\pi n \quad \text{OR} \quad t = 22.9232 + 16\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.
4. Find the solution(s) to \(4\sin(6\theta) + \frac{13}{10} = -\frac{3}{10}\) that are in \([0, 2]\). Use at least 4 decimal places in your work.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the sine (with a coefficient of one) on one side of the equation gives,

\[
\sin(6\theta) = -\frac{2}{5}
\]

Hint 2: Using a calculator and your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which sine will have this value.

Step 2
First, using our calculator we can see that,

\[
6\theta = \sin^{-1}\left(-\frac{2}{5}\right) = -0.4115
\]

Now we're dealing with sine in this problem and we know that the \(y\)-axis represents sine on a unit circle and so we're looking for angles that will have a \(y\) coordinate of \(-\frac{2}{5}\). This means that we'll have angles in the fourth (this is the one our calculator gave us) and third quadrant. Here is a unit circle for this situation.
From the symmetry of the unit circle we can see that the second angle will make an angle of 0.4115 with the negative $x$-axis and so the second angle will be $\pi + 0.4115 = 3.5531$. Also, as noted on the unit circle above a positive angle that represents the first angle (i.e. the angle in the fourth quadrant) is $2\pi - 0.4115 = 5.8717$. We can use either the positive or the negative angle here and we’ll get the same solutions. However, because it is often easy to lose track of minus signs we will be using the positive angle in the fourth quadrant for our work here.

Hint 3 : Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.

This then means that we must have,

$$6z = 3.5531 + 2\pi n \quad \text{OR} \quad 6z = 5.8717 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is divide both sides by 6 and we’ll convert everything to decimals to help with the final step.

$$z = 0.5922 + \frac{\pi n}{3} \quad \text{OR} \quad z = 0.9786 + \frac{\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots$$

$$= 0.5922 + 1.0472n \quad \text{OR} \quad = 0.9786 + 1.0472n \quad n = 0, \pm 1, \pm 2, \ldots$$

Hint 4 : Now all we need to do is plug in values of $n$ to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of $n$ we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at $n = 0$ and see what we get.

$$n = 0: \quad z = 0.5922 \quad \text{OR} \quad z = 0.9786$$

$$n = 1: \quad z = 1.6394 \quad \text{OR} \quad = 2.0258 \quad > \ 2$$

Notice that with each increase in $n$ we were really just adding another 1.0472 onto the previous results and by doing this to the results from the $n = 1$ step we will get solutions that are outside of the interval and so there is no reason to even plug in $n = 2$. Also, as we’ve seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don’t worry about that when it happens.

So, it looks like we have the three solutions to this equation in the given interval.
Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

5. Find the solution(s) to \( 449\cos \left( \frac{4z}{9} \right) + 21\sin \left( \frac{4z}{9} \right) = 0 \) that are in \([-10, 10]\). Use at least 4 decimal places in your work.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to reduce the equation down to a single trig function (with a coefficient of one) on one side of the equation.

Step 1
Because we’ve got both a sine and a cosine here it makes some sense to reduce this down to tangent. So, reducing to a tangent (with a coefficient of one) on one side of the equation gives,

\[ \tan \left( \frac{4z}{9} \right) = \frac{3}{7} \]

Hint 2: Using a calculator and your knowledge of solving trig equations involving tangents to determine all the angles in the range \([0, 2\pi]\) for which tangent will have this value.

Step 2
First, using our calculator we can see that,

\[ \frac{4z}{9} = \tan^{-1} \left( \frac{3}{7} \right) = -0.4049 \]

As we discussed in Example 5 of this section the second angle for equations involving tangent will always be the \( \pi \) plus the first angle. Therefore, \( \pi + (-0.4049) = 2.7367 \) will be the second angle.

Also, because it is very easy to lose track of minus signs we’ll use the fact that we know that any angle plus \( 2\pi \) will give another angle whose terminal line is identical to the original angle to eliminate the minus sign on the first angle. So, another angle that will work for the first angle is \( 2\pi + (-0.4049) = 5.8783 \). Note that there is nothing wrong with using the negative angle and if you chose to work with that you will get the same solutions. We are using the positive angle only to make sure we don’t accidentally lose the minus sign on the angle we received from our calculator.
Hint 3: Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.

This then means that we must have,

$$\frac{4z}{9} = 2.7367 + 2\pi n \quad \text{OR} \quad \frac{4z}{9} = 5.8783 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

Finally, to get all the solutions to the equation all we need to do is multiply both sides by $\frac{9}{4}$ and we’ll convert everything to decimals to help with the final step.

$$z = 6.1576 + \frac{9\pi n}{2} \quad \text{OR} \quad z = 13.2262 + \frac{9\pi n}{2} \quad n = 0, \pm 1, \pm 2, \ldots$$

$$= 6.1576 + 14.1372 n \quad \text{OR} \quad = 13.2262 + 14.1372 n \quad n = 0, \pm 1, \pm 2, \ldots$$

Hint 4: Now all we need to do is plug in values of $n$ to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions.

$$n = -1: \quad z = -7.9796 \quad \text{OR} \quad z = -0.9110$$

$$n = 0: \quad z = 6.1576 \quad \text{OR} \quad \frac{z}{10} = 13.2262 > 10$$

Notice that with each increase in $n$ we were really just adding/subtracting (depending on the sign of $n$) another 14.1372 onto the previous results. A quick inspection of the results above will quickly show us that we don’t need to go any farther and we won’t bother with any other values of $n$. Also, as we’ve seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don’t worry about that when it happens.

So, it looks like we have the three solutions to this equation in the given interval.

$$z = -7.9796, -0.9110, 6.1576$$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the $4^{th}$ decimal place or so however.
6. Find the solution(s) to \(3 \tan \left( \frac{w}{4} \right) - 1 = 11 - 2 \tan \left( \frac{w}{4} \right)\) that are in \([-50, 0]\). Use at least 4 decimal places in your work.

**Hint 1:** Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

**Step 1**
Isolating the tangent (with a coefficient of one) on one side of the equation gives,
\[
\tan \left( \frac{w}{4} \right) = \frac{12}{5}
\]

**Hint 2:** Using a calculator and your knowledge of solving trig equations involving tangents to determine all the angles in the range \([0, 2\pi]\) for which tangent will have this value.

**Step 2**
First, using our calculator we can see that,
\[
\frac{w}{4} = \tan^{-1} \left( \frac{12}{5} \right) = 1.1760
\]

As we discussed in Example 5 of this section the second angle for equations involving tangent will always be the \(\pi\) plus the first angle. Therefore, \(\pi + 1.1760 = 4.3176\) will be the second angle.

**Hint 3:** Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

**Step 3**
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding \(+2\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\) onto each of these.

This then means that we must have,
\[
\frac{w}{4} = 1.1760 + 2\pi n \quad \text{OR} \quad \frac{w}{4} = 4.3176 + 2\pi n \quad \text{where } n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 4 and we’ll convert everything to decimals to help with the final step.

\[
\begin{align*}
w &= 4.7040 + 8\pi n & \text{OR} & \quad w &= 17.2704 + 8\pi n \quad & n = 0, \pm 1, \pm 2, \ldots \\
\text{or} & \quad w &= 4.7040 + 25.1327n & \text{OR} & \quad w &= 17.2704 + 25.1327n \quad & n = 0, \pm 1, \pm 2, \ldots
\end{align*}
\]
Calculus I

Hint 4 : Now all we need to do is plug in values of $n$ to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions. First, notice that if we plug in positive $n$ or $n = 0$ we will have positive solutions and these solutions will be out of the interval. Therefore, we’ll start with $n = -1$.

\[
\begin{align*}
  n = -1 : & \quad w = -20.4287 \quad \text{OR} \quad w = -7.8623 \\
  n = -2 : & \quad w = -45.5614 \quad \text{OR} \quad w = -32.9950
\end{align*}
\]

Notice that with each increase in $n$ we were really just subtracting another 25.1327 from the previous results. A quick inspection of the results above will quickly show us that we don’t need to go any farther and we won’t bother with any other values of $n$.

So, it looks like we have the four solutions to this equation in the given interval.

\[
w = -45.5614, -32.9950, -20.4287, -7.8623
\]

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

7. Find the solution(s) to $17 - 3\sec\left(\frac{z}{2}\right) = 2$ that are in $[20, 45]$. Use at least 4 decimal places in your work.

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the secant (with a coefficient of one) on one side of the equation.

Step 1
Isolating the secant (with a coefficient of one) on one side of the equation gives,

\[
\sec\left(\frac{z}{2}\right) = 5
\]

Hint 2 : Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which secant will have this value. The best way to do this is to rewrite the equation into one in terms of a different trig function that we can more easily deal with.

Step 2
In order to get the solutions it will be much easier to recall the definition of secant in terms of cosine and rewrite the equation into one involving cosine. Doing this gives,
The solution(s) to the equation involving the cosine are the same as the solution(s) to the equation involving the secant and so working with that will be easier. Using our calculator we can see that,

\[
\frac{z}{2} = \cos^{-1}\left(\frac{1}{5}\right) = 1.3694
\]

Now we’re dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle and so we’re looking for angles that will have a x coordinate of \(\frac{1}{5}\). This means that we’ll have angles in the first (this is the one our calculator gave us) and fourth quadrant. Here is a unit circle for this situation.

![Unit circle diagram](image)

From the symmetry of the unit circle we can see that we can either use \(-1.3694\) or \(2\pi - 1.3694 = 4.9138\) for the second angle. Each will give the same set of solutions. However, because it is easy to lose track of minus signs we will use the positive angle for our second solution.

Hint 3 : Using the two angles above write down all the angles for which cosine/secant will have this value and use these to write down all the solutions to the equation.
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+ 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.

This then means that we must have,

\[
\frac{z}{2} = 1.3694 + 2\pi n \quad \text{OR} \quad \frac{z}{2} = 4.9138 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 2 and we’ll convert everything to decimals to help with the final step.

\[
z = 2.7388 + 4\pi n \quad \text{OR} \quad z = 9.8276 + 4\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
= 2.7388 + 12.5664n \quad \text{OR} \quad = 9.8276 + 12.5664n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4: Now all we need to do is plug in values of $n$ to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of $n$ we will get negative solutions and these will not be in the interval and so there is no reason to even try these. Also note that if we use $n = 0$ we will still not be in the interval and so let’s start things off at $n = 1$.

\[
n = 1: \quad \overline{z = 15.3052} < 20 \quad \text{OR} \quad z = 22.3940
\]

\[
n = 2: \quad z = 27.8716 \quad \text{OR} \quad z = 34.9604
\]

\[
n = 3: \quad z = 40.4380 \quad \text{OR} \quad \overline{z = 47.5268} > 45
\]

Notice that with each increase in $n$ we were really just adding another 12.5664 onto the previous results and by a quick inspection of the results above we can see that we don’t need to go any farther. Also, as we’ve seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don’t worry about that when it happens.

So, it looks like we have the four solutions to this equation in the given interval.

\[
\overline{z = 22.3940, 27.8716, 34.9604, 40.4380}
\]

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.
8. Find the solution(s) to \(12 \sin(7y) + 11 = 3 + 4 \sin(7y)\) that are in \([-2, -\frac{1}{2}]\). Use at least 4 decimal places in your work.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the sine (with a coefficient of one) on one side of the equation gives,
\[
\sin(7y) = -1
\]

Hint 2: Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which sine will have this value.

Step 2
If you need to use a calculator to get the solution for this that is fine, but this is also one of the standard angles as we can see from the unit circle below.

Because we’re dealing with sine in this problem and we know that the \(y\)-axis represents sine on a unit circle we’re looking for angle(s) that will have a \(y\) coordinate of \(-1\). The only angle that will have this \(y\) coordinate will be \(\frac{3\pi}{2} = 4.7124\).

Note that unlike all the other problems that we’ve worked to this point this will be the only angle. There is simply not another angle in the range \([0, 2\pi]\) for which sine will have this value. Don’t
get so locked into the *usual* case where we get two possible angles in the \([0, 2\pi]\) that when these single solution cases roll around you decide you must have done something wrong. They happen on occasion and we need to be able to deal with them when they occur.

Hint 3 : Using the angle above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have the angle above we can get all possible angles by simply adding “\(+2\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\)” onto the angle.

This then means that we must have,

\[
y = 4.7124 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 7 and we’ll convert everything to decimals to help with the final step.

\[
y = 0.6732 + \frac{2\pi n}{7} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4 : Now all we need to do is plug in values of \(n\) to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions. First notice that, in this case, if we plug in positive values of \(n\) or \(n = 0\) we will get positive solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \(n = -1\) and see what we get.

\[
\begin{align*}
n = -1 & : \quad y = -0.2244 > -0.5 \\
n = -2 & : \quad y = -1.122 \\
n = -3 & : \quad y = -2.0196 < -2
\end{align*}
\]

So, it looks like we have only a single solution to this equation in the given interval.

\[
y = -1.122
\]

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.
9. Find the solution(s) to \( 5 - 14 \tan (8x) = 30 \) that are in \([-1, 1]\). Use at least 4 decimal places in your work.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

Step 1
Isolating the tangent (with a coefficient of one) on one side of the equation gives,
\[
5 - 14 \tan (8x) = 30
\]
\[
8 \tan (8x) = 30 - 5 = 25
\]
\[
\tan (8x) = \frac{25}{14}
\]

Hint 2: Using a calculator and your knowledge of solving trig equations involving tangents to determine all the angles in the range \([0, 2\pi]\) for which tangent will have this value.

Step 2
First, using our calculator we can see that,
\[
8x = \tan^{-1}\left(\frac{25}{14}\right) = -1.0603
\]
As we discussed in Example 5 of this section the second angle for equations involving tangent will always be the \(\pi\) plus the first angle. Therefore, \(\pi + (-1.0603) = 2.0813\) will be the second angle.

Hint 3: Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding \(+2\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\) onto each of these.

This then means that we must have,
\[
8x = -1.0603 + 2\pi n \quad \text{OR} \quad 8x = 2.0813 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]
Finally, to get all the solutions to the equation all we need to do is divide both sides by 8 and we’ll convert everything to decimals to help with the final step.

\[
x = -0.1325 + \frac{\pi n}{4} \quad \text{OR} \quad x = 0.2602 + \frac{\pi n}{4} \quad n = 0, \pm 1, \pm 2, \ldots
\]
\[
= -0.1325 + 0.7854n \quad \text{OR} \quad = 0.2602 + 0.7854n \quad n = 0, \pm 1, \pm 2, \ldots
\]
Hint 4: Now all we need to do is plug in values of $n$ to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions.

$n = -1$: $x = -0.9179$ OR $x = -0.5252$

$n = 0$: $x = -0.1325$ OR $x = 0.2602$

$n = 1$: $x = 0.6529$ OR $x = 1.0456 > 1$

Notice that with each increase in $n$ we were really just adding/subtracting another 0.7854 from the previous results. A quick inspection of the results above will quickly show us that we don’t need to go any farther and we won’t bother with any other values of $n$. Also, as we’ve seen in this problem it is completely possible for only one of the solutions from a given interval to be in the given interval so don’t worry about that when it happens.

So, it looks like we have the five solutions to this equation in the given interval.

$x = -0.9179, -0.5252, -0.1325, 0.2602, 0.6529$

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

10. Find the solution(s) to $0 = 18 + 2\csc\left(\frac{t}{3}\right)$ that are in $[0, 5]$. Use at least 4 decimal places in your work.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosecant (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosecant (with a coefficient of one) on one side of the equation gives,

\[\csc\left(\frac{t}{3}\right) = -9\]

Hint 2: Using a calculator and your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosecant will have this value. The best way to do this is to rewrite the equation into one in terms of a different trig function that we can more easily deal with.

Step 2
Calculus I

In order to get the solutions it will be much easier to recall the definition of cosecant in terms of sine and rewrite the equation into one involving sine. Doing this gives,

\[
\csc \left( \frac{t}{3} \right) = \frac{1}{\sin \left( \frac{t}{3} \right)} = -9 \quad \Rightarrow \quad \sin \left( \frac{t}{3} \right) = -\frac{1}{9}
\]

The solution(s) to the equation involving the sine are the same as the solution(s) to the equation involving the cosecant and so working with that will be easier. Using our calculator we can see that,

\[
\frac{t}{3} = \sin^{-1} \left( -\frac{1}{9} \right) = -0.1113
\]

Now we’re dealing with sine in this problem and we know that the \(y\)-axis represents sine on a unit circle and so we’re looking for angles that will have a \(y\) coordinate of \(-\frac{1}{9}\). This means that we’ll have angles in the fourth (this is the one our calculator gave us) and third quadrant. Here is a unit circle for this situation.

From the symmetry of the unit circle we can see that the second angle will make an angle of 0.1113 with the negative \(x\)-axis and so the second angle will be \(\pi + 0.1113 = 3.2529\). Also, as noted on the unit circle above a positive angle that represents the first angle (\(i.e.\) the angle in the fourth quadrant) is \(2\pi - 0.1113 = 6.1719\). We can use either the positive or the negative angle here and we’ll get the same solutions. However, because it is often easy to lose track of minus signs we will be using the positive angle in the fourth quadrant for our work here.

Hint 3 : Using the two angles above write down all the angles for which sine/cosecant will have this value and use these to write down all the solutions to the equation.
Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “\(+2\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\)” onto each of these.

This then means that we must have,

\[
\frac{t}{3} = 3.2529 + 2\pi n \quad \text{OR} \quad \frac{t}{3} = 6.1719 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 3 and we’ll convert everything to decimals to help with the final step.

\[
t = 9.7587 + 6\pi n \quad \text{OR} \quad t = 18.5157 + 6\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
= 9.7587 + 18.8496n \quad \text{OR} \quad = 18.5157 + 18.8496n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4 : Now all we need to do is plug in values of \(n\) to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \(n\) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. Also note that even if we start off with \(n = 0\) we will get solutions that are already out of the given interval.

So, despite the fact that there are solutions to this equation none of them fall in the given interval and so there are no solutions to this equation. Do not get excited about the answer here. This kind of situation will happen on occasion and so we need to be aware of it and able to deal with it.

11 Find the solution(s) to \(\frac{1}{2}\cos\left(\frac{x}{8}\right) + \frac{1}{4} = \frac{2}{3}\) that are in \([0,100]\). Use at least 4 decimal places in your work.

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos\left(\frac{x}{8}\right) = \frac{5}{6}
\]
Hint 2 : Using a calculator and your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
First, using our calculator we can see that,

\[
\frac{x}{8} = \cos^{-1} \left( \frac{5}{6} \right) = 0.5859
\]

Now we’re dealing with cosine in this problem and we know that the \(x\)-axis represents cosine on a unit circle and so we’re looking for angles that will have a \(x\) coordinate of \(\frac{5}{6}\). This means that we’ll have angles in the first (this is the one our calculator gave us) and fourth quadrant. Here is a unit circle for this situation.

From the symmetry of the unit circle we can see that we can either use \(-0.5859\) or \(2\pi - 0.5859 = 5.6973\) for the second angle. Each will give the same set of solutions. However, because it is easy to lose track of minus signs we will use the positive angle for our second solution.

Hint 3 : Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding \(+2\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\) onto each of these.
This then means that we must have,
\[
\frac{x}{8} = 0.5859 + 2\pi n \quad \text{OR} \quad \frac{x}{8} = 5.6973 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 8 and we’ll convert everything to decimals to help with the final step.

\[
x = 4.6872 + 16\pi n \quad \text{OR} \quad x = 45.5784 + 16\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]
\[
x = 4.6872 + 50.2655n \quad \text{OR} \quad x = 45.5784 + 50.2655n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4 : Now all we need to do is plug in values of \(n\) to determine which solutions will actually fall in the given interval.

Step 4
Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \(n\) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \(n = 0\) and see what we get.

\[
n = 0: \quad x = 4.6872 \quad \text{OR} \quad x = 45.5784
\]
\[
n = 1: \quad x = 54.9527 \quad \text{OR} \quad x = 95.8439
\]

Notice that with each increase in \(n\) we were really just adding another 50.2655 onto the previous results and by doing this to the results from the \(n = 1\) step we will get solutions that are outside of the interval and so there is no reason to even plug in \(n = 2\).

So, it looks like we have the four solutions to this equation in the given interval.

\[
x = 4.6872, 45.5784, 54.9527, 95.8439
\]

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

12. Find the solution(s) to \(\frac{4}{3} = 1 + 3\sec(2t)\) that are in \([-4, 6]\). Use at least 4 decimal places in your work.

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the secant (with a coefficient of one) on one side of the equation.

Step 1
Isolating the secant (with a coefficient of one) on one side of the equation gives,
Calculus I

\[ \sec(2t) = \frac{1}{9} \]

At this point we can stop. We know that 

\[ \sec \theta \leq -1 \quad \text{or} \quad \sec \theta \leq 1 \]

This means that it is impossible for secant to ever be \( \frac{1}{9} \) and so there will be no solution to this equation.

Note that if you didn’t recall the restrictions on secant the next step would have been to convert this to cosine so let’s do that.

\[ \sec(2t) = \frac{1}{\cos(2t)} = \frac{1}{9} \quad \Rightarrow \quad \cos(2t) = 9 \]

At this point we can note that \(-1 \leq \cos \theta \leq 1\) and so again there is no way for cosine to be 9 and again we get that there will be no solution to this equation.

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**Review : Solving Trig Equations with Calculators, Part II**

1. Find all the solutions to \( 3 - 14 \sin(12t + 7) = 13 \). Use at least 4 decimal places in your work.

Hint : With the exception of the argument, which is a little more complex, this is identical to the equations that we solved in the previous section.

Solution

The argument of the sine is a little more complex in this equation than those we saw in the previous section, but the solution process is identical. Therefore, we will be assuming that you recall the process from the previous section and do not need all the hints or quite as many details as we put into the solutions there. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the section.

First, isolating the sine on one side of the equation gives,

\[ \sin(12t + 7) = -\frac{5}{7} \]

Using a calculator we get,
\[ 12t + 7 = \sin^{-1}\left(-\frac{5}{7}\right) = -0.7956 \]

From our knowledge of the unit circle we can see that a positive angle that corresponds to this angle is \(2\pi - 0.7956 = 5.4876\). Either these angles can be used here but we’ll use the positive angle to avoid the possibility of losing the minus sign. Also, from a quick look at a unit circle we can see that a second angle in the range \([0, 2\pi]\) will be \(\pi + 0.7965 = 3.9372\).

Now, all possible angles for which sine will have this value are,
\[
12t + 7 = 3.9372 + 2\pi n \quad \text{OR} \quad 12t + 7 = 5.4876 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

At this point all we need to do is solve each of these for \(t\) and we’ll have all the solutions to the equation. Doing this gives,
\[
t = -0.2552 + \frac{\pi n}{6} \quad \text{OR} \quad t = -0.1260 + \frac{\pi n}{6} \quad n = 0, \pm 1, \pm 2, \ldots
\]

If an interval had been given we would next proceed with plugging in values of \(n\) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4\(^{th}\) decimal place or so however.

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2. Find all the solutions to \(3\sec(4 - 9z) - 24 = 0\). Use at least 4 decimal places in your work.

Hint: With the exception of the argument, which is a little more complex, this is identical to the equations that we solved in the previous section.

Solution
The argument of the secant is a little more complex in this equation than those we saw in the previous section, but the solution process is identical. Therefore, we will be assuming that you recall the process from the previous section and do not need all the hints or quite as many details as we put into the solutions there. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the section.

First, isolating the secant on one side of the equation gives and converting the equation into one involving cosine (to make the work a little easier) gives,
\[
\sec(4 - 9z) = 8 \quad \Rightarrow \quad \cos(4 - 9z) = \frac{1}{8}
\]
Using a calculator we get,

\[ 4 - 9z = \cos^{-1}\left(\frac{1}{8}\right) = 1.4455 \]

From a quick look at a unit circle we can see that a second angle in the range \( [0, 2\pi] \) will be \( 2\pi - 1.4455 = 4.8377 \). Now, all possible angles for which secant will have this value are,

\[ 4 - 9z = 1.4455 + 2\pi n \quad \text{OR} \quad 4 - 9z = 4.8377 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

At this point all we need to do is solve each of these for \( z \) and we’ll have all the solutions to the equation. Doing this gives,

\[ z = 0.2838 - \frac{2\pi n}{9} \quad \text{OR} \quad z = -0.09308 - \frac{2\pi n}{9} \quad n = 0, \pm 1, \pm 2, \ldots \]

If an interval had been given we would next proceed with plugging in values of \( n \) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

3. Find all the solutions to \( 4\sin(x + 2) - 15\sin(x + 2)\tan(4x) = 0 \). Use at least 4 decimal places in your work.

Hint 1 : Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1

Notice that each term has a sine in it and so we can factor this out of each term to get,

\[ \sin(x + 2) \left( 4 - 15\tan(4x) \right) = 0 \]

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

\[ \sin(x + 2) = 0 \quad \text{OR} \quad 4 - 15\tan(4x) = 0 \]

Hint 2 : Solve each of these two equations to attain all the solutions to the original equation.

Step 2
Each of these equations are similar to equations solved in the previous section or in the earlier problems of this section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

We’ll start with,

\[ \sin(x + 2) = 0 \]

From a unit circle we can see that we must have,

\[ x + 2 = 0 + 2\pi n \quad \text{OR} \quad x + 2 = \pi + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Notice that we can further reduce this down to,

\[ x + 2 = \pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, the solutions from this equation are,

\[ x = \pi n - 2 \quad n = 0, \pm 1, \pm 2, \ldots \]

The second equation will take a little more (but not much more) work. First, isolating the tangent gives,

\[ \tan(4x) = \frac{4}{15} \]

Using our calculator we get,

\[ 4x = \tan^{-1}\left(\frac{4}{15}\right) = 0.2606 \]

From our knowledge on solving equations involving tangents we know that the second angle in the range \([0, 2\pi]\) will be \(\pi + 0.2606 = 3.4022\).

Finally, the solutions to this equation are,

\[ x = 0.2606 + \frac{\pi n}{2} \quad \text{OR} \quad x = 3.4022 + \frac{\pi n}{2} \quad n = 0, \pm 1, \pm 2, \ldots \]

Putting all of this together gives the following set of solutions.

\[ x = \pi n - 2, \quad x = 0.06515 + \frac{\pi n}{2}, \quad \text{OR} \quad x = 0.8506 + \frac{\pi n}{2} \quad n = 0, \pm 1, \pm 2, \ldots \]

If an interval had been given we would next proceed with plugging in values of \(n\) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.
Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

4. Find all the solutions to \(3\cos\left(\frac{3y}{7}\right)\sin\left(\frac{y}{2}\right) + 14\cos\left(\frac{3y}{7}\right) = 0\). Use at least 4 decimal places in your work.

Hint 1: Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1
Notice that each term has a cosine in it and so we can factor this out of each term to get,

\[\cos\left(\frac{3y}{7}\right)\left(3\sin\left(\frac{y}{2}\right) + 14\right) = 0\]

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

\[\cos\left(\frac{3y}{7}\right) = 0 \quad \text{OR} \quad 3\sin\left(\frac{y}{2}\right) + 14 = 0\]

Hint 2: Solve each of these two equations to attain all the solutions to the original equation.

Step 2
Each of these equations are similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

We’ll start with,

\[\cos\left(\frac{3y}{7}\right) = 0\]

From a unit circle we can see that we must have,

\[\frac{3y}{7} = \frac{\pi}{2} + 2\pi n \quad \text{OR} \quad \frac{3y}{7} = \frac{3\pi}{2} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots\]

Notice that we can further reduce this down to,
Finally, the solutions from this equation are,
\[ y = \frac{7\pi}{6} + \frac{7\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \]

The second equation will take a little more (but not much more) work. First, isolating the sine gives,
\[ \sin\left(\frac{y}{2}\right) = -\frac{14}{3} < -1 \]

At this point recall that we know \(-1 \leq \sin \theta \leq 1\) and so this equation will have no solutions.

Therefore, the only solutions to this equation are,
\[ y = \frac{7\pi}{6} + \frac{7\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \]

Do get too excited about the fact that we only got solutions from one of the two equations we got after factoring. This will happen on occasion and so we need to be ready for these cases when they happen.

If an interval had been given we would next proceed with plugging in values of \(n\) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

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5. Find all the solutions to \(7\cos^2(3x) - \cos(3x) = 0\). Use at least 4 decimal places in your work.

Hint 1: Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1
Notice that we can factor a cosine out of each term to get,
\[ \cos(3x)(7\cos(3x) - 1) = 0 \]
Calculus I

Now, we have a product of two factors that equals zero and so by basic algebraic properties we
know that we must have,

\[ \cos(3x) = 0 \quad \text{OR} \quad 7\cos(3x) - 1 = 0 \]

Hint 2: Solve each of these two equations to attain all the solutions to the original equation.

Step 2
Each of these equations are similar to equations solved in the previous section. Therefore, we will
be assuming that you can recall the solution process for each and we will not be putting in as
many details. If you are unsure of the process you should go back to the previous section and
work some of the problems there before proceeding with the solution to this problem.

We’ll start with,

\[ \cos(3x) = 0 \]

From a unit circle we can see that we must have,

\[ 3x = \frac{\pi}{2} + 2\pi n \quad \text{OR} \quad 3x = \frac{3\pi}{2} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Notice that we can further reduce this down to,

\[ 3x = \frac{\pi}{2} + \pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, the solutions from this equation are,

\[ x = \frac{\pi}{6} + \frac{\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \]

The second equation will take a little more (but not much more) work. First, isolating the cosine
gives,

\[ \cos(3x) = \frac{1}{7} \]

Using our calculator we get,

\[ 3x = \cos^{-1}\left(\frac{1}{7}\right) = 1.4274 \]

From a quick look at a unit circle we know that the second angle in the range \([0, 2\pi]\) will be
\[ 2\pi - 1.4274 = 4.8558 \].

Finally, the solutions to this equation are,
Calculus I

\[ 3x = 1.4274 + 2\pi n \quad \text{OR} \quad 3x = 4.8558 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ x = 0.4758 + \frac{\pi n}{3} \quad \text{OR} \quad x = 1.6186 + \frac{\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \]

Putting all of this together gives the following set of solutions.

\[
\begin{align*}
x &= \frac{\pi}{6} + \frac{\pi n}{3}, \quad x = 0.4758 + \frac{\pi n}{3}, \quad \text{OR} \quad x &= 1.6186 + \frac{\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots
\end{align*}
\]

If an interval had been given we would next proceed with plugging in values of \( n \) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

6. Find all the solutions to \( \tan^2\left(\frac{w}{4}\right) = \tan\left(\frac{w}{4}\right) + 12 \). Use at least 4 decimal places in your work.

Hint 1 : Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques. If you’re not sure how to factor this think about how you would factor \( x^2 - x - 12 = 0 \).

Step 1
This equation may look very different from anything that we’ve ever been asked to factor, however it is something that we can factor. First think about factoring the following,

\[ x^2 = x + 12 \quad \Rightarrow \quad x^2 - x - 12 = (x - 4)(x + 3) = 0 \]

If we can factor this algebraic equation then we can factor the given equation in exactly the same manner.

\[
\begin{align*}
\tan^2\left(\frac{w}{4}\right) &= \tan\left(\frac{w}{4}\right) + 12 \\
\tan^2\left(\frac{w}{4}\right) - \tan\left(\frac{w}{4}\right) - 12 &= 0 \\
\left(\tan\left(\frac{w}{4}\right) - 4\right)\left(\tan\left(\frac{w}{4}\right) + 3\right) &= 0
\end{align*}
\]
Now, we have a product of two factors that equals zero and so by basic algebraic properties we 
know that we must have,
\[
\tan \left( \frac{w}{4} \right) - 4 = 0 \quad \text{OR} \quad \tan \left( \frac{w}{4} \right) + 3 = 0
\]

Hint 2 : Solve each of these two equations to attain all the solutions to the original equation.

Step 2
Each of these equations are similar to equations solved in the previous section. Therefore, we will 
be assuming that you can recall the solution process for each and we will not be putting in as 
many details. If you are unsure of the process you should go back to the previous section and 
work some of the problems there before proceeding with the solution to this problem.

We’ll start with the first equation and isolate the tangent to get,
\[
\tan \left( \frac{w}{4} \right) = 4
\]

Using our calculator we get,
\[
\frac{w}{4} = \tan^{-1}(4) = 1.3258
\]

From our knowledge on solving equations involving tangents we know that the second angle in 
the range \([0, 2\pi]\) will be \(\pi + 1.3258 = 4.4674\).

All the solutions to the first equation are then,
\[
\frac{w}{4} = 1.3258 + 2\pi n \quad \text{OR} \quad \frac{w}{4} = 4.4674 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]
\[
w = 5.3032 + 8\pi n \quad \text{OR} \quad w = 17.8696 + 8\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Now, let’s solve the second equation.
\[
\tan \left( \frac{w}{4} \right) = -3 \quad \rightarrow \quad \frac{w}{4} = \tan^{-1}(-3) = -1.2490
\]

From our knowledge of the unit circle we can see that a positive angle that corresponds to this 
angle is \(2\pi - 1.2490 = 5.0342\). Either these angles can be used here but we’ll use the positive 
angle to avoid the possibility of losing the minus sign. Also, the second angle in the range 
\([0, 2\pi]\) is \(\pi + (-1.2490) = 1.8926\).

All the solutions to the second equation are then,
Putting all of this together gives the following set of solutions.

\[
\begin{array}{l}
w = 5.3032 + 8\pi n, \quad w = 7.5704 + 8\pi n, \\
w = 17.8696 + 8\pi n, \quad w = 20.1368 + 8\pi n
\end{array}
\]

\(n = 0, \pm 1, \pm 2, \ldots\)

If an interval had been given we would next proceed with plugging in values of \(n\) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

7. Find all the solutions to \(4\csc^2(1-t) + 6 = 25\csc(1-t)\). Use at least 4 decimal places in your work.

Hint 1: Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques. If you’re not sure how to factor this think about how you would factor \(4x^2 - 25x + 6 = 0\).

Step 1
This equation may look very different from anything that we’ve ever been asked to factor, however it is something that we can factor. First think about factoring the following,

\[4x^2 + 6 = 25x \quad \rightarrow \quad 4x^2 - 25x + 6 = (4x - 1)(x - 6) = 0\]

If we can factor this algebraic equation then we can factor the given equation in exactly the same manner.

\[4\csc^2(1-t) + 6 = 25\csc(1-t)\]
\[4\csc^2(1-t) - 25\csc(1-t) + 6 = 0\]
\[(4\csc(1-t) - 1)(\csc(1-t) - 6) = 0\]

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

\[4\csc(1-t) - 1 = 0 \quad \text{OR} \quad \csc(1-t) - 6 = 0\]
Hint 2: Solve each of these two equations to attain all the solutions to the original equation.

Step 2
Each of these equations are similar to equations solved in the previous section and earlier in this
section. Therefore, we will be assuming that you can recall the solution process for each and we
will not be putting in as many details. If you are unsure of the process you should go back to the
previous section and work some of the problems there before proceeding with the solution to this
problem.

We’ll start with the first equation, isolate the cosecant and convert to an equation in terms of sine
for easier solving. Doing this gives,
\[
\csc(1-t) = \frac{1}{4} \quad \rightarrow \quad \sin(1-t) = 4 > 1
\]

We now know that there are now solutions to the first equation because we know
\(-1 \leq \sin \theta \leq 1\).

Now, let’s solve the second equation.
\[
\csc(1-t) = 6 \quad \rightarrow \quad \sin(1-t) = \frac{1}{6}
\]

Using our calculator we get,
\[
1-t = \sin^{-1} \left( \frac{1}{6} \right) = 0.1674
\]

A quick glance at a unit circle shows us that the second angle in the range \([0, 2\pi]\) is
\[
\pi - 0.1674 = 2.9742
\]

All the solutions to the second equation are then,
\[
1-t = 0.1674 + 2\pi n \quad \text{OR} \quad 1-t = 2.9742 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]
\[
t = 0.8326 - 2\pi n \quad \text{OR} \quad t = -1.9742 - 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Because we had not solutions to the first equation all the solutions to the original equation are
then,
\[
t = 0.8326 - 2\pi n \quad \text{OR} \quad t = -1.9742 - 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Do get too excited about the fact that we only got solutions from one of the two equations we got
after factoring. This will happen on occasion and so we need to be ready for these cases when
they happen.

If an interval had been given we would next proceed with plugging in values of \(n\) to determine
which solutions fall in that interval. Since we were not given an interval this is as far as we can
go.
Calculus I

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

8. Find all the solutions to $4y \sec(7y) = -21$. Use at least 4 decimal places in your work.

Hint 1: Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1
Notice that if we move all the terms to one side we can then factor a $y$ out of the equation. Doing this gives,

$$
(4 \sec(7y) + 21)y = 0
$$

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

$$
y = 0 \quad \text{OR} \quad 4 \sec(7y) + 21 = 0
$$

Be careful with this type of equation to not make the mistake of just canceling the $y$ from both sides in the initial step. Had you done that you would have missed the $y = 0$ solution.

When solving equations it is important to remember that you can’t cancel anything from both sides unless you know for a fact that what you are canceling will never be zero.

Hint 2: Solve each of these two equations to attain all the solutions to the original equation.

Step 2
There really isn’t anything that we need to do with the first equation and so we can move right on to the second equation. Note that this equation is similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

First, isolating the secant and converting to cosines (to make the solving a little easier) gives,

$$
\sec(7y) = -\frac{21}{4} \quad \rightarrow \quad \cos(7y) = -\frac{4}{21}
$$
Using our calculator we get,

\[ 7y = \cos^{-1}\left( -\frac{4}{21} \right) = 1.7624 \]

From a quick look at a unit circle we know that the second angle in the range \([0, 2\pi]\) will be

\[ 2\pi - 1.7624 = 4.5208 \]

Finally, the solutions to this equation are,

\[ 7y = 1.7624 + 2\pi n \quad \text{OR} \quad 7y = 4.5208 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ y = 0.2518 + \frac{2\pi n}{7} \quad \text{OR} \quad y = 0.6458 + \frac{2\pi n}{7} \quad n = 0, \pm 1, \pm 2, \ldots \]

Putting all of this together gives the following set of solutions.

\[
\begin{align*}
y &= 0, \quad y = 0.2518 + \frac{2\pi n}{7}, \quad \text{OR} \quad y = 0.6458 + \frac{2\pi n}{7} \quad n = 0, \pm 1, \pm 2, \ldots
\end{align*}
\]

If an interval had been given we would next proceed with plugging in values of \(n\) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.

9. Find all the solutions to \(10x^2 \sin(3x + 2) = 7x \sin(3x + 2)\). Use at least 4 decimal places in your work.

Hint 1 : Factor the equation and using basic algebraic properties get some equations that can be dealt with using known techniques.

Step 1
Notice that if we move all the terms to one side we can then factor an \(x\) and a sine out of the equation. Doing this gives,

\[ 10x^2 \sin(3x + 2) - 7x \sin(3x + 2) = 0 \]

\[ x(10x - 7) \sin(3x + 2) = 0 \]

Now, we have a product of three factors that equals zero and so by basic algebraic properties we know that we must have,
Be careful with this type of equation to not make the mistake of just canceling the \( x \) or the sine from both sides in the initial step. Had you done that you would have missed the \( x = 0 \) solution and the solutions we will get from solving \( \sin(3x + 2) = 0 \).

When solving equations it is important to remember that you can’t cancel anything from both sides unless you know for a fact that what you are canceling will never be zero.

Hint 2 : Solve each of these three equations to attain all the solutions to the original equation.

Step 2
There really isn’t anything that we need to do with the first equation and so we can move right on to the second equation (which also doesn’t really present any problems). Solving the second equation gives,

\[
x = \frac{7}{10}
\]

Now let’s take a look at the third equation. This equation is similar to equations solved earlier in this section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

From a unit circle we can see that we must have,

\[
3x + 2 = 0 + 2\pi n \quad \text{OR} \quad 3x + 2 = \pi + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Notice that we can further reduce this down to,

\[
3x + 2 = \pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, the solutions from this equation are,

\[
x = \frac{\pi n - 2}{3} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Putting all of this together gives the following set of solutions.

\[
x = 0, \quad x = \frac{7}{10}, \quad \text{OR} \quad x = \frac{\pi n - 2}{3} \quad n = 0, \pm 1, \pm 2, \ldots
\]

If an interval had been given we would next proceed with plugging in values of \( n \) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.
10. Find all the solutions to \( (2t - 3) \tan \left( \frac{6t}{11} \right) = 15 - 10t \). Use at least 4 decimal places in your work.

Hint 1: Factor the equation and using basic algebraic properties get two equations that can be dealt with using known techniques.

Step 1
This one may be a little trickier to factor than the others in this section, but it can be factored. First get everything on one side of the equation and then notice that we can factor out a \( 2t - 3 \) from the equation as follows,

\[
(2t - 3) \tan \left( \frac{6t}{11} \right) + 10t - 15 = 0 \\
(2t - 3) \tan \left( \frac{6t}{11} \right) + 5(2t - 3) = 0 \\
(2t - 3) \left( \tan \left( \frac{6t}{11} \right) + 5 \right) = 0
\]

Now, we have a product of two factors that equals zero and so by basic algebraic properties we know that we must have,

\[
2t - 3 = 0 \quad \text{OR} \quad \tan \left( \frac{6t}{11} \right) + 5 = 0
\]

Be careful with this type of equation to not make the mistake of just canceling the \( 2t - 3 \) from both sides. Had you done that you would have missed the solution from the first equation.

When solving equations it is important to remember that you can’t cancel anything from both sides unless you know for a fact that what you are canceling will never be zero.

Hint 2: Solve each of these two equations to attain all the solutions to the original equation.

Step 2
Solving the first equation gives,

\[
t = \frac{3}{2}
\]
Now we can move onto the second equation and note that this equation is similar to equations solved in the previous section. Therefore, we will be assuming that you can recall the solution process for each and we will not be putting in as many details. If you are unsure of the process you should go back to the previous section and work some of the problems there before proceeding with the solution to this problem.

First, isolating the tangent gives,

$$\tan\left(\frac{6t}{11}\right) = -5$$

Using our calculator we get,

$$\frac{6t}{11} = \tan^{-1}(-5) = -1.3734$$

From our knowledge of the unit circle we can see that a positive angle that corresponds to this angle is \(2\pi - 1.3734 = 4.9098\). Either these angles can be used here but we’ll use the positive angle to avoid the possibility of losing the minus sign. Also, the second angle in the range \([0, 2\pi]\) is \(\pi + (-1.3734) = 1.7682\).

Finally, the solutions to this equation are,

$$\frac{6t}{11} = 1.7682 + 2\pi n \quad \text{OR} \quad \frac{6t}{11} = 4.9098 + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots$$

$$t = 3.2417 + \frac{11\pi n}{3} \quad \text{OR} \quad t = 9.0013 + \frac{11\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots$$

Putting all of this together gives the following set of solutions.

$$t = \frac{3}{2}, \quad t = 3.2417 + \frac{11\pi n}{3}, \quad \text{OR} \quad t = 9.0013 + \frac{11\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots$$

If an interval had been given we would next proceed with plugging in values of \(n\) to determine which solutions fall in that interval. Since we were not given an interval this is as far as we can go.

Note that depending upon the amount of decimals you used here your answers may vary slightly from these due to round off error. Any differences should be slight and only appear around the 4th decimal place or so however.
Review: Exponential Functions

1. Sketch the graph of \( f(x) = 3^{1+2x} \).

Solution

There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we’ll be doing these the “old fashioned” way of plotting points. If you’d like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

So, with that out of the way here is a table of values for this function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>1/27</td>
</tr>
<tr>
<td>-1</td>
<td>1/3</td>
</tr>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>27</td>
</tr>
<tr>
<td>2</td>
<td>243</td>
</tr>
</tbody>
</table>

A natural question at this point is “how did we know to use these values of \( x \)?” That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is negative the function will quickly get close to zero. This means that often (but not always) we’ll want to keep the exponent in the range of about \([-4, 4]\) and by exponent we mean the value of \( 1+2x \) after we plug in the \( x \).

Note that we often won’t need the whole range given above to see what the curve looks like. As we plug in values of \( x \) we can look at our answers and if they aren’t changing much then we’ll know that the exponent has gone far enough in the negative direction so that the exponential is essentially zero. Likewise, once the value really starts changing fast we’ll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of \( x \) and nothing more.

Here is the sketch of the graph of this function.
2. Sketch the graph of \( h(x) = 2^{\frac{3-x}{4}} - 7 \).

Solution

There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we’ll be doing these the “old fashioned” way of plotting points. If you’d like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

So, with that out of the way here is a table of values for this function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( h(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10</td>
<td>38.2548</td>
</tr>
<tr>
<td>-6</td>
<td>15.6274</td>
</tr>
<tr>
<td>-2</td>
<td>4.3137</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-1.3431</td>
</tr>
<tr>
<td>6</td>
<td>-4.1716</td>
</tr>
<tr>
<td>10</td>
<td>-5.5858</td>
</tr>
</tbody>
</table>

A natural question at this point is “how did we know to use these values of \( x \)”’? That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is negative the function will quickly get close to zero. This means that often (but not always) we’ll want to keep the exponent in the range of about \([-4, 4]\) and by exponent we mean the value of \(3 - \frac{x}{4}\) after we plug in the \( x \).

Note that we often won’t need the whole range given above to see what the curve looks like. As we plug in values of \( x \) we can look at our answers and if they aren’t changing much then we’ll know that the exponent has gone far enough in the negative direction so that the exponential is
essentially zero. Likewise, once the value really starts changing fast we’ll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of $x$ and nothing more.

Here is the sketch of the graph of this function.

3. Sketch the graph of $h(t) = 8 + 3e^{2t-4}$.

Solution
There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we’ll be doing these the “old fashioned” way of plotting points. If you’d like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

So, with that out of the way here is a table of values for this function.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$h(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>8.0074</td>
</tr>
<tr>
<td>0</td>
<td>8.0549</td>
</tr>
<tr>
<td>1</td>
<td>8.4060</td>
</tr>
<tr>
<td>2</td>
<td>11.30</td>
</tr>
<tr>
<td>3</td>
<td>30.17</td>
</tr>
</tbody>
</table>

A natural question at this point is “how did we know to use these values of $t$”? That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is negative the function will quickly get close to zero. This means that often (but not always) we’ll want to keep the exponent in the range of about $[-4, 4]$ and by exponent we mean the value of $2t - 4$ after we plug in the $t$. 

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Note that we often won’t need the whole range given above to see what the curve looks like. As we plug in values of \( t \) we can look at our answers and if they aren’t changing much then we’ll know that the exponent has gone far enough in the negative direction so that the exponential is essentially zero. Likewise, once the value really starts changing fast we’ll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of \( t \) and nothing more.

Here is the sketch of the graph of this function.

\[
\text{Graph of } f(t) = e^{-2t}
\]

4. Sketch the graph of \( g(z) = 10 - \frac{1}{4}e^{-2-3z} \).

Solution
There are several methods that can be used for getting the graph of this function. One way would be to use some of the various algebraic transformations. The point of the problems in this section however are more to force you to do some evaluation of these kinds of functions to make sure you can do them. So, while you could use transformations, we’ll be doing these the “old fashioned” way of plotting points. If you’d like some practice of the transformations you can check out the practice problems for the Common Graphs section of this chapter.

So, with that out of the way here is a table of values for this function.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( g(z) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-3.6495</td>
</tr>
<tr>
<td>-1</td>
<td>9.3204</td>
</tr>
<tr>
<td>0</td>
<td>9.9662</td>
</tr>
<tr>
<td>1</td>
<td>9.9983</td>
</tr>
<tr>
<td>2</td>
<td>9.9999</td>
</tr>
</tbody>
</table>

A natural question at this point is “how did we know to use these values of \( z \)”? That is a good question and not always an easy one to answer. For exponential functions the key is to recall that when the exponent is positive the function will grow very quickly and when the exponent is
negative the function will quickly get close to zero. This means that often (but not always) we’ll want to keep the exponent in the range of about $[-4, 4]$ and by exponent we mean the value of $-2 - 3z$ after we plug in the $z$.

Note that we often won’t need the whole range given above to see what the curve looks like. As we plug in values of $z$ we can look at our answers and if they aren’t changing much then we’ll know that the exponent has gone far enough in the negative direction so that the exponential is essentially zero. Likewise, once the value really starts changing fast we’ll know that the exponent has gone far enough in the positive direction as well. The given above is just a way to give us some starting values of $z$ and nothing more.

Here is the sketch of the graph of this function.

![Graph of the function](image)

**Review : Logarithm Functions**

1. Without using a calculator determine the exact value of $\log_3 81$.

Hint : Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Solution
Converting the logarithm to exponential form gives,

$$\log_3 81 = \Rightarrow 3^y = 81$$

From this we can quickly see that $3^4 = 81$ and so we must have,
2. Without using a calculator determine the exact value of \( \log_5 125 \).

Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Solution

Converting the logarithm to exponential form gives,

\[ \log_5 125 = ? \quad \Rightarrow \quad 5^7 = 125 \]

From this we can quickly see that \( 5^3 = 125 \) and so we must have,

\[ \log_5 125 = 3 \]

3. Without using a calculator determine the exact value of \( \log_2 \frac{1}{8} \).

Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Solution

Converting the logarithm to exponential form gives,

\[ \log_2 \frac{1}{8} = ? \quad \Rightarrow \quad 2^{-3} = \frac{1}{8} \]

Now, we know that if we raise an integer to a negative exponent we’ll get a fraction and so we must have a negative exponent and then we know that \( 2^3 = 8 \). Therefore we can see that \( 2^{-3} = \frac{1}{8} \) and so we must have,

\[ \log_2 \frac{1}{8} = -3 \]

4. Without using a calculator determine the exact value of \( \log_{\frac{1}{4}} 16 \).
5. Without using a calculator determine the exact value of \( \ln e^4 \).

Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a natural logarithm.

Solution
Recalling that the base for a natural logarithm is \( e \) and converting the logarithm to exponential form gives,

\[
\ln e^4 = \log_e e^4 = ? \quad \Rightarrow \quad e^7 = e^4
\]

From this we can quickly see that \( e^4 = e^4 \) and so we must have,

\[
\ln e^4 = 4
\]

Note that an easier method of determining the value of this logarithm would have been to recall the properties of logarithm. In particular the property that states,

\[
\log_b b^x = x
\]

Using this we can also very quickly see what the value of the logarithm is.

6. Without using a calculator determine the exact value of \( \log \frac{1}{100} \).
Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a common logarithm.

Solution
Recalling that the base for a natural logarithm is 10 and converting the logarithm to exponential form gives,

\[
\log \frac{1}{100} = \log_{10} \frac{1}{100} = ? \quad \Rightarrow \quad 10^? = \frac{1}{100}
\]

Now, we know that if we raise an integer to a negative exponent we’ll get a fraction and so we must have a negative exponent and then we know that \(10^2 = 100\). Therefore we can see that \(10^{-2} = \frac{1}{100}\) and so we must have,

\[
\log \frac{1}{100} = -2
\]

7. Write \(\log (3x^4y^{-7})\) in terms of simpler logarithms.

Solution
So, we’re being asked here to use as many of the properties as we can to reduce this down into simpler logarithms. So, here is the work for this problem.

\[
\log (3x^4y^{-7}) = \log (3) + \log (x^4) + \log (y^{-7})
\]

\[
= \log (3) + 4 \log (x) - 7 \log (y)
\]

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn’t bring any of the exponents out of the logarithms until we had dealt with the product.

8. Write \(\ln (x\sqrt{y^2 + z^2})\) in terms of simpler logarithms.

Solution
So, we’re being asked here to use as many of the properties as we can to reduce this down into simpler logarithms. So, here is the work for this problem.
\[
\ln \left( x \sqrt{y^2 + z^2} \right) = \ln(x) + \ln \left( \left( y^2 + z^2 \right)^{\frac{1}{2}} \right) \\
= \ln(x) + \frac{1}{2} \ln(y^2 + z^2)
\]

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn’t bring any of the exponents out of the logarithms until we had dealt with the product. Also, in the second logarithm while each term is squared the whole argument is not squared, \(i.e.\) it’s not \((x + y)^2\) and so we can’t bring those 2’s out of the logarithm.

---

9. Write \(\log_4 \left( \frac{x - 4}{y^2 \sqrt[3]{z}} \right)\) in terms of simpler logarithms.

Solution
So, we’re being asked here to use as many of the properties as we can to reduce this down into simpler logarithms. So, here is the work for this problem.

\[
\log_4 \left( \frac{x - 4}{y^2 \sqrt[3]{z}} \right) = \log_4(x - 4) - \log_4 \left( y^2 \right) \left( \left( z^5 \right) \right) \\
= \log_4(x - 4) - \left( \log_4 \left( y^2 \right) + \log_4 \left( z^5 \right) \right) \\
= \log_4(x - 4) - 2 \log_4(y) - \frac{1}{5} \log_4(z)
\]

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn’t bring any of the exponents out of the logarithms until we had dealt with the quotient and product. Recall as well that we can’t split up an sum/difference in a logarithm. Finally, make sure that you are careful in dealing with the minus sign we get from breaking up the quotient when dealing with the product in the denominator.

---

10. Combine \(2 \log_4 x + 5 \log_4 y - \frac{1}{2} \log_4 z\) into a single logarithm with a coefficient of one.

Hint: The properties that we use to break up logarithms can be used in reverse as well.
Solution
To convert this into a single logarithm we’ll be using the properties that we used to break up 
logarithms in reverse. The first step in this process is to use the property,
\[ \log_b (x^r) = r \log_b x \]
to make sure that all the logarithms have coefficients of one. This needs to be done first because 
all the properties that allow us to combine sums/differences of logarithms require coefficients of 
one on individual logarithms. So, using this property gives,
\[ \log_4 (x^2) + \log_4 (y^5) - \log_4 \left( z^{\frac{1}{2}} \right) \]

Now, there are several ways to proceed from this point. We can use either of the two properties.
\[ \log_b (xy) = \log_b x + \log_b y \quad \quad \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \]
and in fact we’ll need to use both in the end. Which we use first does not matter as we’ll end up 
with the same result in the end. Here is the rest of the work for this problem.
\[ 2 \log_4 x + 5 \log_4 y - \frac{1}{2} \log_4 z = \log_4 \left( x^2 y^5 \right) - \log_4 \left( \sqrt{z} \right) \]
\[ = \log_4 \left( \frac{x^2 y^5}{\sqrt{z}} \right) \]

Note that the only reason we converted the fractional exponent to a root was to make the final 
answer a little nicer.

11. Combine \( 3 \ln (t + 5) - 4 \ln t - 2 \ln (s - 1) \) into a single logarithm with a coefficient of one.

Hint :The properties that we use to break up logarithms can be used in reverse as well.

Solution
To convert this into a single logarithm we’ll be using the properties that we used to break up 
logarithms in reverse. The first step in this process is to use the property,
\[ \log_b (x^r) = r \log_b x \]
to make sure that all the logarithms have coefficients of one. This needs to be done first because 
all the properties that allow us to combine sums/differences of logarithms require coefficients of 
one on individual logarithms. So, using this property gives,
\[ \ln (t + 5)^3 - \ln (t^4) - \ln (s - 1)^2 \]

Now, there are several ways to proceed from this point. We can use either of the two properties.
\[
\log_b (xy) = \log_b x + \log_b y \\
\log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y
\]

and in fact we’ll need to use both in the end.

We should also be careful with the fact that there are two minus signs in here as that sometimes adds confusion to the problem. They are easy to deal with however if we just factor a minus sign out of the last two terms and then proceed from there as follows.

\[
3 \ln (t + 5) - 4 t - 2 \ln (s - 1) = \ln (t + 5)^3 - \left( \ln (t^4) + \ln (s - 1)^2 \right)
\]

\[
= \ln (t + 5)^3 - \ln (t^4 (s - 1)^2) = \frac{\ln \left( \frac{(t + 5)^3}{t^4 (s - 1)^2} \right)}{}
\]

12. Combine \( \frac{1}{3} \log a - 6 \log b + 2 \) into a single logarithm with a coefficient of one.

Hint : The properties that we use to break up logarithms can be used in reverse as well. For the constant see if you figure out a way to write that as a logarithm.

Solution

To convert this into a single logarithm we’ll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

\[
\log_b \left( x^r \right) = r \log_b x
\]

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

\[
\log \left( a^{\frac{1}{3}} \right) - \log \left( b^6 \right) + 2
\]

Now, for the 2 let’s notice that we can write this in terms of a logarithm as,

\[
2 = \log 10^2 = \log 100
\]

Note that this is really just using the property,

\[
\log_b b^x = x
\]

So, we now have,

\[
\log \left( a^{\frac{1}{3}} \right) - \log \left( b^6 \right) + \log 100
\]
Now, there are several ways to proceed from this point. We can use either of the two properties.

\[
\log_b (xy) = \log_b x + \log_b y \quad \quad \quad \quad \quad \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y
\]

and in fact we’ll need to use both in the end. Which we use first does not matter as we’ll end up
with the same result in the end. Here is the rest of the work for this problem.

\[
\log \left( a^{\frac{1}{3}} \right) - \log \left( b^b \right) + \log 10^2 = \log \left( 100^{\frac{1}{3}} \right) - \log \left( b^b \right)
\]

\[
= \log \left( \frac{100^{\frac{1}{3}}}{b^b} \right)
\]

Note that the only reason we converted the fractional exponent to a root was to make the final
answer a little nicer.

---

13. Use the change of base formula and a calculator to find the value of \( \log_{12} 35 \).

Solution

We can use either the natural logarithm or the common logarithm to do this so we’ll do both.

\[
\log_{12} 35 = \frac{\ln 35}{\ln 12} = \frac{3.55534806}{2.48490665} = 1.43077731
\]

\[
\log_{12} 35 = \frac{\log_{10} 35}{\log_{10} 12} = \frac{1.54406804}{1.07918125} = 1.43077731
\]

So, as we noted at the start it doesn’t matter which logarithm we use we’ll get the same answer in
the end.

---

14. Use the change of base formula and a calculator to find the value of \( \log_{\frac{2}{3}} 53 \).

Solution

We can use either the natural logarithm or the common logarithm to do this so we’ll do both.

\[
\log_{\frac{2}{3}} 53 = \frac{\ln 53}{\ln \frac{2}{3}} = \frac{3.97029191}{-0.40546511} = -9.79194469
\]

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Calculus I

\[
\log_3 53 = \log_2 3 \frac{\log_{10} 53}{\log_{10} 2} = \frac{1.72427587}{-0.17609126} = -9.79194469
\]

So, as we noted at the start it doesn’t matter which logarithm we use we’ll get the same answer in the end.

---

**Review : Exponential and Logarithm Equations**

1. Find all the solutions to \( 7^{3x} - 4 = 7 \). If there are no solutions clearly explain why.

**Step 1**
There isn’t all that much to do here for this equation. First we need to isolate the exponential on one side by itself with a coefficient of one.

\[
7^{3x} = -5 \quad \Rightarrow \quad e^{x} = \frac{5}{4}
\]

**Step 2**
Now all we need to do is take the natural logarithm of both sides and then solve for \( x \).

\[
\ln(e^{3x}) = \ln\left(\frac{5}{4}\right)
\]

\[
7 + 3x = \ln\left(\frac{5}{4}\right)
\]

\[
x = \frac{1}{3} \left(\ln\left(\frac{5}{4}\right) - 7\right) = -2.25895
\]

Depending upon your preferences either the exact or decimal solution can be used.

---

2. Find all the solutions to \( 1 = 10 - 3e^{z^2 - 2z} \). If there are no solutions clearly explain why.

**Step 1**
There isn’t all that much to do here for this equation. First we need to isolate the exponential on one side by itself with a coefficient of one.

\[
-9 = -3e^{z^2 - 2z} \quad \Rightarrow \quad e^{z^2 - 2z} = 3
\]

**Step 2**
Now all we need to do is take the natural logarithm of both sides and then solve for \( z \).
Now, before proceeding with the solution here let’s pause and make sure that we don’t get too excited about the “strangeness” of the quadratic above. If we’d had the quadratic,

\[ z^2 - 2z - 5 = 0 \]

for instance, we’d know that all we would need to do is use the quadratic formula to get the solutions.

That’s all we need to as well for the quadratic that we have from our work. Of course we don’t have a 5 we have a \( \ln(3) \), but \( \ln(3) \) is just a number and so we can use the quadratic formula to find the solutions here as well. Here is the work for that.

\[
\begin{align*}
z &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-\ln(3))}}{2(1)} \\
&= \frac{2 \pm \sqrt{4 + 4\ln(3)}}{2} \\
&= \frac{2 \pm \sqrt{4(1 + \ln(3))}}{2} = \frac{2 \pm 2\sqrt{1 + \ln(3)}}{2} = 1 \pm \sqrt{1 + \ln(3)} = -0.4487, \ 2.4487
\end{align*}
\]

Notice that we did a little simplification on the root. This doesn’t need to be done, but can make the exact solution a little easier to deal with. Also, depending upon your preferences either the exact or decimal solution can be used.

Before leaving this solution we should again make a point that not all quadratics will be the “simple” type of quadratics that you may be used to solving from an Algebra class. They can, and often will be, messier that those. That doesn’t mean that you can’t solve them. They are, for all intents and purposes, identical to the types of problems you are used to working. The only real difference is that they numbers are a little messier.

So, don’t get too excited about this kind of problem. They will happen on occasion and you are capable of solving them!

3. Find all the solutions to \( 6t - te^{6t-1} = 0 \). If there are no solutions clearly explain why.

Hint : Be careful to not cancel terms that shouldn’t be canceled. Remember that you can’t cancel something unless you know for a fact that it won’t ever be zero. Also, note that if you can cancel something then it can be factored out of the equation.

Step 1
First notice that we can factor a \( t \) out of both terms to get,
Calculus I

\[ t(2 - e^{6t-1}) = 0 \]

Be careful to not cancel the \( t \) from both terms. When solving equations you can only cancel something if you know for a fact that it won’t be zero. If the term can be zero and you cancel it you will miss solutions and that will be the case here.

Step 2
We now have a product of terms that is equal to zero so we know,

\[ t = 0 \quad \text{OR} \quad 2 - e^{6t-1} = 0 \]

So, we have one solution already, \( t = 0 \), and again note that if we had canceled the \( t \) at the beginning we would have missed this solution. Now all we need to do is solve the equation involving the exponential.

Step 3
We can now solve the exponential equation in the same manner as the first couple of problems in this section.

\[ e^{6t-1} = 2 \]
\[ \ln(e^{6t-1}) = \ln(2) \]
\[ 6t - 1 = \ln(2) \]
\[ t = \frac{1}{6}(1 + \ln(2)) = 0.2822 \]

Depending upon your preferences either the exact or decimal solution can be used.

4. Find all the solutions to \( 4x + 1 = (12x + 3)e^{x^2-2} \). If there are no solutions clearly explain why.

Hint: Be careful to not cancel terms that shouldn’t be canceled. Remember that you can’t cancel something unless you know for a fact that it won’t ever be zero. Also, note that if you can cancel something then it can be factored out of the equation.

Step 1
It may not be apparent at first glance, but with some work we can do a little factoring on this equation. To do that first move everything to one side and then the factoring might become a little more apparent.
Calculus I

\[ 4x + 1 - (12x + 3)e^{x^2-2} = 0 \]
\[ (4x + 1) - 3(4x + 1)e^{x^2-2} = 0 \]
\[ (4x + 1)(1 - 3e^{x^2-2}) = 0 \]

Note that in the second step we put parenthesis around the first couple of terms solely to make the factoring in the next step a little more apparent. It does not need to be done in practice.

Be careful to not cancel the \(4x + 1\) from both terms. When solving equations you can only cancel something if you know for a fact that it won’t be zero. If the term can be zero and you cancel it you will miss solutions, and that will be the case here.

Step 2
We now have a product of terms that is equal to zero so we know,
\[ 4x + 1 = 0 \quad \text{OR} \quad 1 - 3e^{x^2-2} = 0 \]

From the first equation we can quickly arrive at one solution, \(x = -\frac{1}{4}\), and again note that if we had canceled the \(4x + 1\) at the beginning we would have missed this solution. Now all we need to do is solve the equation involving the exponential.

Step 3
We can now solve the exponential equation in the same manner as the first couple of problems in this section.

\[ e^{x^2-2} = \frac{1}{3} \]
\[ \ln\left(e^{x^2-2}\right) = \ln\left(\frac{1}{3}\right) \]
\[ x^2 - 2 = \ln\left(\frac{1}{3}\right) \]
\[ x^2 = 2 + \ln\left(\frac{1}{3}\right) \]
\[ x = \pm \sqrt{2 + \ln\left(\frac{1}{3}\right)} = \pm 0.9494 \]

Depending upon your preferences either the exact or decimal solution can be used.

5. Find all the solutions to \(2e^{3y^8} - 11e^{5-10y} = 0\). If there are no solutions clearly explain why.
Hint: The best way to proceed here is to reduce the equation down to a single exponential.

Step 1
With both exponentials in the equation this may be a little difficult to solve, so let’s do some work to reduce this down to an equation with a single exponential.

\[ 2e^{3y+8} = 11e^{5-10y} \]
\[ e^{3y+8} = \frac{11}{2} \]
\[ e^{5-10y} = \frac{11}{2} \]
\[ e^{13y+3} = \frac{11}{2} \]

Note that we could have divided by either exponential but by dividing by the one that we did we avoid a negative exponent on the \( y \), which is sometimes easy to lose track of.

Step 2
Now all we need to do is take the logarithm of both sides and solve for \( y \).

\[ \ln(e^{13y+3}) = \ln\left(\frac{11}{2}\right) \]
\[ 13y + 3 = \ln\left(\frac{11}{2}\right) \]
\[ y = \frac{1}{13} \left( \ln\left(\frac{11}{2}\right) - 3 \right) = -0.09963 \]

Depending upon your preferences either the exact or decimal solution can be used.

6. Find all the solutions to \( 14e^{6-x} + e^{12x-7} = 0 \). If there are no solutions clearly explain why.

Hint: The best way to proceed here is to reduce the equation down to a single exponential.

Step 1
With both exponentials in the equation this may be a little difficult to solve, so let’s do some work to reduce this down to an equation with a single exponential.

\[ 14e^{6-x} = -e^{12x-7} \]
\[ e^{12x-7} = -14 \]
\[ e^{6-x} = -14 \]
\[ e^{13x-13} = -14 \]
At this point we can stop. We know that exponential functions are always positive and there is no way for this to be negative and therefore there is no solution to this equation.

Note that if we hadn’t caught the exponent being negative our next step would have been to take the logarithm of both side and we also know that we can only take the logarithm of positive numbers and so again we’d see that there is no solution to this equation.

7. Find all the solutions to $1 - 8\ln \left( \frac{2x - 1}{7} \right) = 14$. If there are no solutions clearly explain why.

Step 1
There isn’t all that much to do here for this equation. First we need to isolate the logarithm on one side by itself with a coefficient of one.

$$1 - 8\ln \left( \frac{2x - 1}{7} \right) = 14 \quad \Rightarrow \quad \ln \left( \frac{2x - 1}{7} \right) = -\frac{13}{8}$$

Step 2
Now all we need to do is exponentiate both sides using $e$ (because we’re working with the natural logarithm) and then solve for $x$.

$$e^{\ln \left( \frac{2x - 1}{7} \right)} = e^{-\frac{13}{8}}$$

$$\frac{2x - 1}{7} = e^{-\frac{13}{8}}$$

$$x = \frac{1}{2} \left( 1 + 7e^{-\frac{13}{8}} \right) = 1.1892$$

Step 3
We’re dealing with logarithms and so we need to make sure that we won’t have any problems with any of our potential solutions. In other words, we need to make sure that if we plug in the potential solution into the original equation we won’t end up taking the logarithm of a negative number or zero.

Plugging in we can see that we won’t be taking the logarithm of a negative number and so the solution is,

$$x = \frac{1}{2} \left( 1 + 7e^{-\frac{13}{8}} \right) = 1.1892$$

Depending upon your preferences either the exact or decimal solution can be used.
8. Find all the solutions to \( \ln(y - 1) = 1 + \ln(3y + 2) \). If there are no solutions clearly explain why.

Hint: Don’t forget about the basic logarithm properties and how they can be used to combine multiple logarithms into a single logarithm.

Step 1
We need to reduce this down to an equation with a single logarithm and to do that we first should rewrite it a little. Upon doing that we can use the basic logarithm properties to combine the two logarithms into a single logarithm as follows,

\[
\ln(y - 1) - \ln(3y + 2) = 1
\]

\[
\ln\left(\frac{y - 1}{3y + 2}\right) = 1
\]

Step 2
Now all we need to do is exponentiate both sides using \( e \) (because we’re working with the natural logarithm) and then solve for \( y \).

\[
e^{\ln\left(\frac{y - 1}{3y + 2}\right)} = e^1
\]

\[
\frac{y - 1}{3y + 2} = e
\]

\[
y - 1 = e(3y + 2) = 3ey + 2e
\]

\[
(1 - 3e)y = 1 + 2e
\]

\[
y = \frac{1 + 2e}{1 - 3e} = -0.8996
\]

Step 3
We’re dealing with logarithms and so we need to make sure that we won’t have any problems with any of our potential solutions. In other words, we need to make sure that if we plug in the potential solutions into the original equation we won’t end up taking the logarithm of a negative number or zero.

Upon inspection we can quickly see that if we plug in our potential solution into the first logarithm we’ll be taking the logarithm of a negative number. The same will be true for the second logarithm and so \( y = -0.8996 \) can’t be a solution.

Because this was our only potential solution we know now that there will be **no solutions** to this equation.
9. Find all the solutions to \( \log(w) + \log(w - 21) = 2 \). If there are no solutions clearly explain why.

Hint: Don’t forget about the basic logarithm properties and how they can be used to combine multiple logarithms into a single logarithm.

Step 1
We need to reduce this down to an equation with a single logarithm and to do that we first should rewrite it a little. Upon doing that we can use the basic logarithm properties to combine the two logarithms into a single logarithm as follows,

\[
\log(w(w - 21)) = 2 \\
\log(w^2 - 21w) = 2
\]

Step 2
Now all we need to do is exponentiate both sides using 10 (because we’re working with the common logarithm) and then solve for \( y \).

\[
\log(w^2 - 21w) = 2 \\
10^{\log(w^2 - 21w)} = 10^2 \\
w^2 - 21w = 100 \\
w^2 - 21w - 100 = 0 \\
(w - 25)(w + 4) = 0 \quad \Rightarrow \quad w = -4, \ w = 25
\]

Step 3
We’re dealing with logarithms and so we need to make sure that we won’t have any problems with any of our potential solutions. In other words, we need to make sure that if we plug either of the two potential solutions into the original equation we won’t end up taking the logarithm of a negative number or zero.

Upon inspection we can quickly see that if we plug in \( w = -4 \) we will be taking a logarithm of a negative number (in both of the logarithms in this case) and so \( w = -4 \) can’t be a solution. On the other hand, if we plug in \( w = 25 \) we won’t be taking logarithms of negative numbers and so \( w = 25 \) is a solution.

In summary then, the only solution to the equation is \( w = 25 \).

10. Find all the solutions to \( 2\log(z) - \log(7z - 1) = 0 \). If there are no solutions clearly explain why.
Hint: This problem can be worked in the same manner as the previous two or because each term is a logarithm an easier solution would be to use the fact that,

$$\text{If } \log_b x = \log_b y \text{ then } x = y$$

Step 1
While we could use the same method we used in the previous couple of examples to solve this equation there is an easier method. Because each of the terms is a logarithm and it’s all equal to zero we can use the fact that,

$$\text{If } \log_b x = \log_b y \text{ then } x = y$$

So, a quick rewrite of the equation gives,

$$\begin{align*} 2 \log(z) &= \log(7z - 1) \\ \log(z^2) &= \log(7z - 1) \end{align*}$$

Note that in order to use the fact above we need both logarithms to have coefficients of one and so we also had to make quick use of one of the logarithm properties to make sure we had a coefficient of one.

Step 2
Now all we need to do use the fact and solve for $z$.

$$\begin{align*} z^2 &= 7z - 1 \\ z^2 - 7z + 1 &= 0 \end{align*}$$

In this case we’ll need to use the quadratic formula to finish this out.

$$z = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(1)}}{2(1)} = \frac{7 \pm \sqrt{45}}{2} = 0.1459, \ 6.8541$$

Step 3
We’re dealing with logarithms and so we need to make sure that we won’t have any problems with any of our potential solutions. In other words, we need to make sure that if we plug either of the two potential solutions into the original equation we won’t end up taking the logarithm of a negative number or zero.

In this case it is pretty easy to plug them in and see that neither of the two potential solutions will result in taking logarithms of negative numbers and so both are solutions to the equation.

In summary then, the solutions to the equation ar,

$$z = \frac{7 \pm \sqrt{45}}{2} = 0.1459, \ 6.8541$$
Depending upon your preferences either the exact or decimal solution can be used.

Before leaving this solution we should again make a point that not all quadratics will be the “simple” type of quadratics that you may be used to solving from an Algebra class. They can, and often will be, messier that those. That doesn’t mean that you can’t solve them. They are, for all intents and purposes, identical to the types of problems you are used to working. The only real difference is that they numbers are a little messier.

So, don’t get too excited about this kind of problem. They will happen on occasion and you are capable of solving them!

11. Find all the solutions to \( 216 - 17 + 11 \). If there are no solutions clearly explain why.

Hint : These look a little different from the first few problems in this section, but they work in essentially the same manner. The main difference is that we’re not dealing with \( e^{\text{power}} \) or \( 10^{\text{power}} \) and so there is no obvious logarithm to use and so can use any logarithm.

Step 1
First we need to isolate the term with the exponent in it on one side by itself.

\[
17^{t-2} = 5
\]

Step 2
At this point we need to take the logarithm of both sides so we can use logarithm properties to get the \( t \) out of the exponent. It doesn’t matter which logarithm we use, but if we want a decimal value for the answer it will need to be one that we can work with. For this solution we’ll use the natural logarithm.

Upon taking the logarithm we then need to use logarithm properties to get the \( t \)'s out of the exponent at which point we can solve for \( t \). Here is the rest of the work for this problem,

\[
\ln(17^{t-2}) = \ln(5) \\
(t-2)\ln(17) = \ln(5) \\
t-2 = \frac{\ln(5)}{\ln(17)} \\
x = 2 + \frac{\ln(5)}{\ln(17)} = 2.5681
\]

Depending upon your preferences either the exact or decimal solution can be used. Also note that if you had used, say the common logarithm, you would get exactly the same answer.
12. Find all the solutions to \( 2^{3-8w} - 7 = 11 \). If there are no solutions clearly explain why.

Hint: These look a little different from the first few problems in this section, but they work in essentially the same manner. The main difference is that we’re not dealing with \( e^{\text{power}} \) or \( 10^{\text{power}} \) and so there is no obvious logarithm to use and so can use any logarithm.

Step 1
First we need to isolate the term with the exponent in it on one side by itself.
\[
2^{3-8w} = 18
\]

Step 2
At this point we need to take the logarithm of both sides so we can use logarithm properties to get the \( w \) out of the exponent. It doesn’t matter which logarithm we use, but if we want a decimal value for the answer it will need to be one that we can work with. For this solution we’ll use the natural logarithm.

Upon taking the logarithm we then need to use logarithm properties to get the \( w \)’s out of the exponent at which point we can solve for \( w \). Here is the rest of the work for this problem,
\[
\ln\left(2^{3-8w}\right) = \ln(18)
\]
\[
(3-8w)\ln(2) = \ln(18)
\]
\[
3-8w = \frac{\ln(18)}{\ln(2)}
\]
\[
w = \frac{1}{8}\left(3 - \frac{\ln(18)}{\ln(2)}\right) = -0.1462
\]

Depending upon your preferences either the exact or decimal solution can be used. Also note that if you had used, say the common logarithm, you would get exactly the same answer.

---

**Compound Interest.** If we put \( P \) dollars into an account that earns interest at a rate of \( r \) (written as a decimal as opposed to the standard percent) for \( t \) years then,

a. if interest is compounded \( m \) times per year we will have,
\[
A = P\left(1 + \frac{r}{m}\right)^{tm}
\]
dollars after \( t \) years.

b. if interest is compounded continuously we will have,
\[
A = Pe^{rt}
\]
dollars after \( t \) years.
13. We have $10,000 to invest for 44 months. How much money will we have if we put the money into an account that has an annual interest rate of 5.5% and interest is compounded, 

Hint: There really isn’t a whole lot to these other than to identify the quantities and then plug into the appropriate equation and compute the amount. Also note that you’ll need to make sure that you don’t do too much in the way of rounding with the numbers here. A little rounding can lead to very large errors in these kinds of computations.

(a) quarterly  [Solution]
From the problem statement we can see that,

\[ P = 10000 \quad r = \frac{5.5}{100} = 0.055 \quad t = \frac{44}{12} = \frac{11}{3} \]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also remember that \( t \) must be in years and so we’ll need to convert to years.

For this part we are compounding interest rate quarterly and that means it will compound 4 times per year and so we also then know that,

\[ m = 4 \]

At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we’ll have.

\[ A = 10000 \left( 1 + \frac{0.055}{4} \right)^{\frac{11}{3}} = 10000 \left( 1.01375 \right)^{\frac{44}{3}} = 10000 \left( 1.221760422 \right) = 12217.60 \]

So, we’ll have $12,217.60 in the account after 44 months.

(b) monthly  [Solution]
From the problem statement we can see that,

\[ P = 10000 \quad r = \frac{5.5}{100} = 0.055 \quad t = \frac{44}{12} = \frac{11}{3} \]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also remember that \( t \) must be in years and so we’ll need to convert to years.

For this part we are compounding interest rate monthly and that means it will compound 12 times per year and so we also then know that,

\[ m = 12 \]
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At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we’ll have.

\[
A = 10000 \left( 1 + \frac{0.055}{12} \right)^{11(12)} = 10000 \left( 1.00453333 \right)^{44} = 10000 \left( 1.222876562 \right) = 12228.77
\]

So, we’ll have $12,228.77 in the account after 44 months.

(c) continuously  [Solution]

From the problem statement we can see that,

\[
(11.0) = 5.5 \times 1110000 \times 0.055100 \times 123
\]

Remember that the value of \( r \) must be given as a decimal, i.e. the percentage divided by 100. Also remember that \( t \) must be in years and so we’ll need to convert to years.

For this part we are compounding continuously and so we won’t have an \( m \) and will be using the other equation and all we have all we need to do the computation so,

\[
A = 10000e^{0.055 \times \frac{11}{3}} = 10000e^{0.2016666667} = 10000 \left( 1.223440127 \right) = 12234.40
\]

So, we’ll have $12,234.40 in the account after 44 months.

---

**Compound Interest.** If we put \( P \) dollars into an account that earns interest at a rate of \( r \) (written as a decimal as opposed to the standard percent) for \( t \) years then,

c. if interest is compounded \( m \) times per year we will have,

\[
A = P \left( 1 + \frac{r}{m} \right)^{tm}
\]
dollars after \( t \) years.

d. if interest is compounded continuously we will have,

\[
A = Pe^{rt}
\]
dollars after \( t \) years.

14. We are starting with $5000 and we’re going to put it into an account that earns an annual interest rate of 12%. How long should we leave the money in the account in order to double our money if interest is compounded,

Hint : Identify the given quantities, plug into the appropriate equation and use the techniques from earlier problem to solve for \( t \).
(a) quarterly [Solution]
From the problem statement we can see that,
\[ A = 10000 \quad P = 5000 \quad r = \frac{12}{100} = 0.12 \]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also, for this part we are compounding interest rate quarterly and that means it will compound 4 times per year and so we also then know that,
\[ m = 4 \]

Plugging into the equation gives us,
\[ 10000 = 5000 \left( 1 + \frac{0.12}{4} \right)^{4t} = 5000(1.03)^{4t} \]

Using the techniques from this section we can solve for \( t \).
\[
\begin{align*}
2 &= 1.03^{4t} \\
\ln(2) &= \ln(1.03^{4t}) \\
\ln(2) &= 4t \ln(1.03) \\
t &= \frac{\ln(2)}{4 \ln(1.03)} = 5.8624
\end{align*}
\]

So, we’ll double our money in approximately 5.8624 years.

(b) monthly [Solution]
From the problem statement we can see that,
\[ A = 10000 \quad P = 5000 \quad r = \frac{12}{100} = 0.12 \]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also, for this part we are compounding interest rate monthly and that means it will compound 12 times per year and so we also then know that,
\[ m = 12 \]

Plugging into the equation gives us,
\[ 10000 = 5000 \left( 1 + \frac{0.12}{12} \right)^{12t} = 5000(1.01)^{12t} \]

Using the techniques from this section we can solve for \( t \).
\[ 2 = 1.01^{12t} \]
\[ \ln(2) = \ln\left(1.01^{12t}\right) \]
\[ \ln(2) = 12t \ln(1.01) \]
\[ t = \frac{\ln(2)}{12 \ln(1.01)} = 5.8051 \]

So, we’ll double our money in approximately 5.8051 years.

(c) continuously [Solution]
From the problem statement we can see that,
\[ A = 10000 \quad P = 5000 \quad r = \frac{12}{100} = 0.12 \]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. For this part we are compounding continuously and so we won’t have an \( m \) and will be using the other equation.

Plugging into the continuously compounding interest equation gives,
\[ 10000 = 5000e^{0.12t} \]

Now, solving this gives,
\[ 2 = e^{0.12t} \]
\[ \ln(2) = \ln\left(e^{0.12t}\right) \]
\[ \ln(2) = 0.12t \]
\[ t = \frac{\ln(2)}{0.12} = 5.7762 \]

So, we’ll double our money in approximately 5.7762 years.

---

**Exponential Growth/Decay.** Many quantities in the world can be modeled (at least for a short time) by the exponential growth/decay equation.
\[ Q = Q_0 e^{kt} \]
If \( k \) is positive then we will get exponential growth and if \( k \) is negative we will get exponential decay.

15. A population of bacteria initially has 250 present and in 5 days there will be 1600 bacteria present.
(a) Determine the exponential growth equation for this population.

Hint: We have an equation with two unknowns and two values of the population at two times so use these values to find the two unknowns.

[Solution]
We can start off here by acknowledging that we know,

\[ Q(0) = 250 \quad \text{and} \quad Q(5) = 1600 \]

If we use the first condition in the equation we get,

\[ 250 = Q(0) = Q_0 e^{k(0)} = Q_0 \quad \rightarrow \quad Q_0 = 250 \]

We now know the first unknown in the equation. Plugging this as well as the second condition into the equation gives us,

\[ 1600 = Q(5) = 250e^{5k} \]

We can use techniques from earlier problems in this section to determine the value of \( k \).

\[ 1600 = 250e^{5k} \]
\[ \frac{1600}{250} = e^{5k} \]
\[ \ln\left(\frac{32}{5}\right) = 5k \]
\[ k = \frac{1}{5} \ln\left(\frac{32}{5}\right) = 0.3712596 \]

Depending upon your preferences we can use either the exact value or the decimal value. Note however that because \( k \) is in the exponent of an exponential function we’ll need to use quite a few decimal places to avoid potentially large differences in the value that we’d get if we rounded off too much.

Putting all of this together the exponential growth equation for this population is,

\[ Q = 250e^{\frac{1}{5} \ln\left(\frac{32}{5}\right) t} \]

(b) How long will it take for the population to grow from its initial population of 250 to a population of 2000? [Solution]
What we’re really being asked to do here is to solve the equation,

\[ 2000 = Q(t) = 250e^{\frac{1}{5} \ln\left(\frac{32}{5}\right) t} \]
and we know from earlier problems in this section how to do that. Here is the solution work for this part.

\[
\frac{2000}{250} = e^{\frac{1}{5} \ln \left( \frac{32}{5} \right) t}
\]

\[
\ln(8) = \frac{1}{5} \ln \left( \frac{32}{5} \right) t
\]

\[
t = \frac{5 \ln(8)}{\ln \left( \frac{32}{5} \right)} = 5.6010
\]

It will take 5.601 days for the population to reach 2000.

---

**Exponential Growth/Decay.** Many quantities in the world can be modeled (at least for a short time) by the exponential growth/decay equation.

\[Q = Q_0 e^{kt}\]

If \(k\) is positive then we will get exponential growth and if \(k\) is negative we will get exponential decay.

16. We initially have 100 grams of a radioactive element and in 1250 years there will be 80 grams left.

(a) Determine the exponential decay equation for this element.

Hint: We have an equation with two unknowns and two values of the amount of the element left at two times so use these values to find the two unknowns.

[Solution]

We can start off here by acknowledging that we know,

\[Q(0) = 100 \quad \text{and} \quad Q(1250) = 80\]

If we use the first condition in the equation we get,

\[100 = Q(0) = Q_0 e^{k(0)} = Q_0 \quad \rightarrow \quad Q_0 = 100\]

We now know the first unknown in the equation. Plugging this as well as the second condition into the equation gives us,

\[80 = Q(1250) = 100e^{1250k}\]

We can use techniques from earlier problems in this section to determine the value of \(k\).
Depending upon your preferences we can use either the exact value or the decimal value. Note however that because \( k \) is in the exponent of an exponential function we’ll need to use quite a few decimal places to avoid potentially large differences in the value that we’d get if we rounded off too much.

Putting all of this together the exponential decay equation for this population is,

\[
Q = 100e^{\frac{1}{1250} \ln \left( \frac{4}{5} \right) t}
\]

(b) How long will it take for half of the element to decay? [Solution]

What we’re really being asked to do here is to solve the equation,

\[
50 = Q(t) = 100e^{\frac{1}{1250} \ln \left( \frac{4}{5} \right) t}
\]

and we know from earlier problems in this section how to do that. Here is the solution work for this part.

\[
\frac{50}{100} = e^{\frac{1}{1250} \ln \left( \frac{4}{5} \right) t}
\]

\[
\ln \left( \frac{1}{2} \right) = \frac{1}{1250} \ln \left( \frac{4}{5} \right) t
\]

\[
\frac{1250 \ln \left( \frac{1}{2} \right)}{\ln \left( \frac{4}{5} \right)} = 3882.8546
\]

It will take 3882.8546 years for half of the element to decay. On a side note this time is called the half-life of the element.

(c) How long will it take until there is only 1 gram of the element left? [Solution]

In this part we’re being asked to solve the equation,

\[
1 = Q(t) = 100e^{\frac{1}{1250} \ln \left( \frac{4}{5} \right) t}
\]

and we know from earlier problems in this section how to do that. Here is the solution work for this part.
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\[
\frac{1}{100} = e^{1250 \ln \left( \frac{4}{5} \right) t}
\]

\[
\ln \left( \frac{1}{100} \right) = \frac{1}{1250} \ln \left( \frac{4}{5} \right) t
\]

\[
t = \frac{1250 \ln \left( \frac{1}{100} \right)}{\ln \left( \frac{4}{5} \right)} = 25797.1279
\]

There will only be 1 gram of the element left after 25,797.1279 years.

**Review: Common Graphs**

1. Without using a graphing calculator sketch the graph of \( y = \frac{4}{3} x - 2 \).

Solution

This is just a line with slope \( \frac{4}{3} \) and \( y \)-intercept \((0, -2)\) so here is the graph.

![Graph of \( y = \frac{4}{3} x - 2 \)](image)

2. Without using a graphing calculator sketch the graph of \( f(x) = |x - 3| \).

Hint: Recall that the graph of \( g(x + c) \) is simply the graph of \( g(x) \) shifted right by \( c \) units if \( c < 0 \) or shifted left by \( c \) units if \( c > 0 \).
Recall the basic Algebraic transformations. If we know the graph of \( g(x) \) then the graph of \( g(x + c) \) is simply the graph of \( g(x) \) shifted right by \( c \) units if \( c < 0 \) or shifted left by \( c \) units if \( c > 0 \).

So, in our case if \( g(x) = |x| \) we can see that,

\[
f(x) = |x - 3| = g(x - 3)
\]

and so the graph we’re being asked to sketch is the graph of the absolute value function shifted right by 3 units.

Here is the graph of \( f(x) = |x - 3| \) and note that to help see the transformation we have also sketched in the graph of \( g(x) = |x| \).

3. Without using a graphing calculator sketch the graph of \( g(x) = \sin(x) + 6 \).

Hint: Recall that the graph of \( f(x) + c \) is simply the graph of \( f(x) \) shifted down by \( c \) units if \( c < 0 \) or shifted up by \( c \) units if \( c > 0 \).

Solution
Recall the basic Algebraic transformations. If we know the graph of \( f(x) \) then the graph of \( f(x) + c \) is simply the graph of \( f(x) \) shifted down by \( c \) units if \( c < 0 \) or shifted up by \( c \) units if \( c > 0 \).
So, in our case if \( f(x) = \sin(x) \) we can see that,

\[
g(x) = \sin(x) + 6 = f(x) + 6
\]

and so the graph we’re being asked to sketch is the graph of the sine function shifted up by 6 units.

Here is the graph of \( g(x) = \sin(x) + 6 \) and note that to help see the transformation we have also sketched in the graph of \( f(x) = \sin(x) \).

4. Without using a graphing calculator sketch the graph of \( f(x) = \ln(x) - 5 \).

Hint : Recall that the graph of \( g(x) + c \) is simply the graph of \( g(x) \) shifted down by \( c \) units if \( c < 0 \) or shifted up by \( c \) units if \( c > 0 \).

Solution
Recall the basic Algebraic transformations. If we know the graph of \( g(x) \) then the graph of \( g(x) + c \) is simply the graph of \( g(x) \) shifted down by \( c \) units if \( c < 0 \) or shifted up by \( c \) units if \( c > 0 \).

So, in our case if \( g(x) = \ln(x) \) we can see that,

\[
f(x) = \ln(x) - 5 = g(x) - 5
\]

and so the graph we’re being asked to sketch is the graph of the natural logarithm function shifted down by 5 units.
Here is the graph of \( f(x) = \ln(x) - 5 \) and note that to help see the transformation we have also sketched in the graph of \( g(x) = \ln(x) \).

5. Without using a graphing calculator sketch the graph of \( h(x) = \cos \left( x + \frac{\pi}{2} \right) \).

Hint: Recall that the graph of \( g(x + c) \) is simply the graph of \( g(x) \) shifted right by \( c \) units if \( c < 0 \) or shifted left by \( c \) units if \( c > 0 \).

Solution
Recall the basic Algebraic transformations. If we know the graph of \( g(x) \) then the graph of \( g(x + c) \) is simply the graph of \( g(x) \) shifted right by \( c \) units if \( c < 0 \) or shifted left by \( c \) units if \( c > 0 \).

So, in our case if \( g(x) = \cos(x) \) we can see that,

\[
h(x) = \cos \left( x + \frac{\pi}{2} \right) = g \left( x + \frac{\pi}{2} \right)
\]

and so the graph we’re being asked to sketch is the graph of the cosine function shifted left by \( \frac{\pi}{2} \) units.

Here is the graph of \( h(x) = \cos \left( x + \frac{\pi}{2} \right) \) and note that to help see the transformation we have also sketched in the graph of \( g(x) = \cos(x) \).
6. Without using a graphing calculator sketch the graph of \( h(x) = (x - 3)^2 + 4 \).

Hint : The Algebraic transformations that we used to help us graph the first few graphs in this section can be used together to shift the graph of a function both up/down and right/left at the same time.

Solution
The Algebraic transformations we were using in the first few problems of this section can be combined to shift a graph up/down and right/left at the same time. If we know the graph of \( g(x) \) then the graph of \( g(x + c) + k \) is simply the graph of \( g(x) \) shifted right by \( c \) units if \( c < 0 \) or shifted left by \( c \) units if \( c > 0 \) and shifted up by \( k \) units if \( k > 0 \) or shifted down by \( k \) units if \( k < 0 \).

So, in our case if \( g(x) = x^2 \) we can see that,

\[
h(x) = (x - 3)^2 + 4 = g(x - 3) + 4
\]

and so the graph we’re being asked to sketch is the graph of \( g(x) = x^2 \) shifted right by 3 units and up by 4 units.

Here is the graph of \( h(x) = (x - 3)^2 + 4 \) and note that to help see the transformation we have also sketched in the graph of \( g(x) = x^2 \).
7. Without using a graphing calculator sketch the graph of \( W(x) = e^{x^2/2} - 3 \).

Hint: The Algebraic transformations that we used to help us graph the first few graphs in this section can be used together to shift the graph of a function both up/down and right/left at the same time.

Solution
The Algebraic transformations we were using in the first few problems of this section can be combined to shift a graph up/down and right/left at the same time. If we know the graph of \( g(x) \) then the graph of \( g(x + c) + k \) is simply the graph of \( g(x) \) shifted right by \( c \) units if \( c < 0 \) or shifted left by \( c \) units if \( c > 0 \) and shifted up by \( k \) units if \( k > 0 \) or shifted down by \( k \) units if \( k < 0 \).

So, in our case if \( g(x) = e^x \) we can see that,

\[
W(x) = e^{x^2/2} - 3 = g(x + 2) - 3
\]

and so the graph we’re being asked to sketch is the graph of \( g(x) = e^x \) shifted left by 2 units and down by 3 units.

Here is the graph of \( W(x) = e^{x^2/2} - 3 \) and note that to help see the transformation we have also sketched in the graph of \( g(x) = e^x \).
In this case the resulting sketch of $W(x)$ that we get by shifting the graph of $g(x)$ is not really the best, as it pretty much cuts off at $x = 0$ so in this case we should probably extend the graph of $W(x)$ a little. Here is a better sketch of the graph.

8. Without using a graphing calculator sketch the graph of $f(y) = (y - 1)^2 + 2$.

Hint : The Algebraic transformations can also be used to help us sketch graphs of functions in the form $x = f(y)$, but we do need to remember that we’re now working with functions in which the variables have been interchanged.

Solution
Even though our function is in the form \( x = f(y) \) we can still use the Algebraic transformations to help us sketch this graph. We do need to be careful however and remember that we’re working with interchanged variables and so the transformations will also switch.

In this case if we know the graph of \( h(y) \) then the graph of \( h(y + c) + k \) is simply the graph of \( h(x) \) shifted up by \( c \) units if \( c < 0 \) or shifted down by \( c \) units if \( c > 0 \) and shifted right by \( k \) units if \( k > 0 \) or shifted left by \( k \) units if \( k < 0 \).

So, in our case if \( h(y) = y^2 \) we can see that,

\[
f(y) = (y - 1)^2 + 2 = h(y - 1) + 2
\]

and so the graph we’re being asked to sketch is the graph of \( h(y) = y^2 \) shifted up by 1 units and right by 2 units.

Here is the graph of \( f(y) = (y - 1)^2 + 2 \) and note that to help see the transformation we have also sketched in the graph of \( h(y) = y^2 \).

9. Without using a graphing calculator sketch the graph of \( R(x) = -\sqrt{x} \).

Hint : Recall that the graph of \(-f(x)\) is the graph of \( f(x) \) reflected about the \( x \)-axis.

Solution
Recall the basic Algebraic transformations. If we know the graph of \( f(x) \) then the graph of \( -f(x) \) is simply the graph of \( f(x) \) reflected about the \( x \)-axis.

So, in our case if \( f(x) = \sqrt{x} \) we can see that,

\[
R(x) = -\sqrt{x} = -f(x)
\]

and so the graph we’re being asked to sketch is the graph of the square root function reflected about the \( x \)-axis.

Here is the graph of \( R(x) = -\sqrt{x} \) (the solid curve) and note that to help see the transformation we have also sketched in the graph of \( f(x) = \sqrt{x} \) (the dashed curve).

10. Without using a graphing calculator sketch the graph of \( g(x) = -\sqrt{x} \).

Hint: Recall that the graph of \( f(-x) \) is the graph of \( f(x) \) reflected about the \( y \)-axis.

Solution
First, do not get excited about the minus sign under the root. We all know that we won’t get real numbers if we take the square root of a negative number, but that minus sign doesn’t necessarily mean that we’ll be taking the square root of negative numbers. If we plug in positive value of \( x \) then clearly we will be taking the square root of negative numbers, but if we plug in negative values of \( x \) we will now be taking the square root of positive numbers and so there really is nothing wrong with the function as written. We’ll just be using a different set of \( x \)-s than what we may be used to working with when dealing with square roots.
Now, recall the basic Algebraic transformations. If we know the graph of \( f(x) \) then the graph of \( f(-x) \) is simply the graph of \( f(x) \) reflected about the \( y \)-axis.

So, in our case if \( f(x) = \sqrt{x} \) we can see that,

\[
g(x) = \sqrt{-x} = f(-x)
\]

and so the graph we’re being asked to sketch is the graph of the square root function reflected about the \( y \)-axis.

Here is the graph of \( g(x) = \sqrt{-x} \) and note that to help see the transformation we have also sketched in the graph of \( f(x) = \sqrt{x} \).

11. Without using a graphing calculator sketch the graph of \( h(x) = 2x^2 - 3x + 4 \).

Hint: Recall that the graph of \( f(x) = ax^2 + bx + c \) is the graph of a parabola with vertex \((-\frac{b}{2a}, f\left(-\frac{b}{2a}\right))\) that opens upwards if \( a > 0 \) and downwards if \( a < 0 \) and \( y \)-intercept at \((0,c)\).

Solution
We know that the graph of \( f(x) = ax^2 + bx + c \) will be a parabola that opens upwards if \( a > 0 \) and opens downwards if \( a < 0 \). We also know that its vertex is at,

\[
\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)
\]
The $y$-intercept of the parabola is the point $(0, f(0)) = (0, c)$ and the $x$-intercepts (if any) are found by solving $f(x) = 0$

So, or our case we know we have a parabola that opens upwards and that its vertex is at,

$$\left(-\frac{3}{2(2)}, f\left(-\frac{3}{2(2)}\right)\right) = \left(\frac{3}{4}, f\left(\frac{3}{4}\right)\right) = \left(\frac{3}{4}, \frac{23}{8}\right) = (0.75, 2.875)$$

We can also see that the $y$-intercept is $(0, 4)$. Because the vertex is above the $x$-axis and the parabola opens upwards we can see that there will be no $x$-intercepts.

It is usually best to have at least one point on either side of the vertex and we know that parabolas are symmetric about the vertical line running through the vertex. Therefore, because we know that the $y$-intercept is 0.75 units to the left of the vertex that we must also have a point that is 0.75 to the right of the vertex with the same $y$-value and this point is : $(1.5, 4)$.

Here is a sketch of this parabola.

12. Without using a graphing calculator sketch the graph of $f(y) = -4y^2 + 8y + 3$.

Hint : Recall that the graph of $f(y) = ay^2 + by + c$ is the graph of a parabola with vertex

$$\left(f\left(-\frac{b}{2a}\right), -\frac{b}{2a}\right)$$

that opens towards the right if $a > 0$ and towards the left if $a < 0$ and $x$-intercept at $(c, 0)$.
Solution
We know that the graph of \( f(y) = ay^2 + by + c \) will be a parabola that opens towards the right if \( a > 0 \) and opens towards the left if \( a < 0 \). We also know that its vertex is at,

\[
\left( f \left( -\frac{b}{2a} \right), -\frac{b}{2a} \right)
\]

The \( x \)-intercept of the parabola is the point \( \left( f(0), 0 \right) = (c, 0) \) and the \( x \)-intercepts (if any) are found by solving \( f(y) = 0 \)

So, or our case we know we have a parabola that opens towards the left and that its vertex is at,

\[
\left( f \left( -\frac{8}{2(-4)} \right), -\frac{8}{2(-4)} \right) = (f(1), 1) = (7, 1)
\]

We can also see that the \( y \)-intercept is \( (0, 0) \).

To find the \( y \)-intercepts all we need to do is solve : \(-4y^2 + 8y + 3 = 0\).

\[
y = \frac{-8 \pm \sqrt{8^2 - 4(-4)(3)}}{2(-4)} = \frac{-8 \pm \sqrt{112}}{-8} = \frac{-8 \pm 4\sqrt{7}}{-8} = \frac{2 \pm \sqrt{7}}{2} = -0.3229, 2.3229
\]

So, the two \( y \)-intercepts are : \( (0, -0.3229) \) and \( (0, 2.3229) \).

Here is a sketch of this parabola.
13. Without using a graphing calculator sketch the graph of \((x+1)^2 + (y-5)^2 = 9\).

Solution
This is just a circle in standard form and so we can see that it has a center of \((-1,5)\) and a radius of 3. Here is a quick sketch of the circle.

![Graph of a circle](image)

14. Without using a graphing calculator sketch the graph of \(x^2 - 4x + y^2 - 6y - 87 = 0\).

Hint: Complete the square a couple of times to put this into standard form. This will allow you to identify the type of graph this will be.

Solution
The first thing that we should do is complete the square on the \(x\)'s and the \(y\)'s to see what we've got here. This could be a circle, ellipse, or hyperbola and completing the square a couple of times will put it into standard form and we'll be able to identify the graph at that point.

Here is the completing the square work.
15. Without using a graphing calculator sketch the graph of \( 25(x + 2)^2 + \frac{y^2}{16} = 1 \).

Solution
This is just an ellipse that is almost in standard form. With a little rewrite we can put it into standard form as follows,

\[
\frac{(x + 2)^2}{25} + \frac{y^2}{4} = 1
\]

We can now see that the ellipse has a center of \((-2, 0)\) while the left/right most points will be \(\frac{1}{5} = 0.2\) units away from the center and the top/bottom most points will be 2 units away from the center. Here is a quick sketch of the ellipse.
16. Without using a graphing calculator sketch the graph of \( x^2 + \frac{(y-6)^2}{9} = 1 \).

Solution

This is just an ellipse that is in standard form (if it helps rewrite the first term as \( \frac{x^2}{1} \)) and so we can see that it has a center of \((0, 6)\) while the left/right most points will be 1 unit away from the center and the top/bottom most points will be 3 units away from the center.

Here is a quick sketch of the ellipse.
17. Without using a graphing calculator sketch the graph of \( \frac{x^2}{36} - \frac{y^2}{49} = 1 \).

Solution
This is a hyperbola in standard form with the minus sign in front of the \( y \) term and so will open right and left. The center of the hyperbola is at \((0,0)\), the two vertices are at \((-6,0)\) and \((6,0)\), and the slope of the two asymptotes are \(\pm \frac{7}{6}\).

Here is a quick sketch of the hyperbola.
18. Without using a graphing calculator sketch the graph of \((y + 2)^2 - \frac{(x + 4)^2}{16} = 1\).

Solution
This is a hyperbola in standard form with the minus sign in front of the \(x\) term and so will open up and down. The center of the hyperbola is at \((-4, -2)\), the two vertices are at \((-1, -1)\) and \((-1, -3)\), and the slope of the two asymptotes are \(\pm \frac{1}{4}\).

Here is a quick sketch of the hyperbola.