Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
Exponential and Logarithm Functions

Exponential Functions

1. Given the function \( f(x) = 4^x \) evaluate each of the following.
   
   \( f(-2) \)  \( f\left(-\frac{1}{2}\right) \)  \( f(0) \)  \( f(1) \)  \( f\left(\frac{3}{2}\right) \)

   (a) \( f(-2) \)
   
   All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.
   
   \[ f(-2) = 4^{-2} = \frac{1}{4^2} = \frac{1}{16} \]

   (b) \( f\left(-\frac{1}{2}\right) \)
   
   All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.
   
   \[ f\left(-\frac{1}{2}\right) = 4^{-\frac{1}{2}} = \frac{1}{\sqrt{4}} = \frac{1}{2} \]

   (c) \( f(0) \)
   
   All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.
   
   \[ f(0) = 4^0 = 1 \]

   (d) \( f(1) \)
   
   All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.
   
   \[ f(1) = 4^1 = 4 \]

   (e) \( f\left(\frac{3}{2}\right) \)
   
   All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.
   
   \[ f\left(\frac{3}{2}\right) = 4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8 \]
2. Given the function \( f(x) = \left( \frac{1}{5} \right)^x \) evaluate each of the following.

(a) \( f(-3) \)  
(b) \( f(-1) \)  
(c) \( f(0) \)  
(d) \( f(2) \)  
(e) \( f(3) \)

(a) \( f(-3) \)
All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.

\[
f(-3) = \left( \frac{1}{5} \right)^{-3} = \left( \frac{5}{1} \right)^3 = \frac{5^3}{1^3} = 125
\]

(b) \( f(-1) \)
All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.

\[
f(-1) = \left( \frac{1}{5} \right)^{-1} = \left( \frac{5}{1} \right)^1 = 5
\]

(c) \( f(0) \)
All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.

\[
f(0) = \left( \frac{1}{5} \right)^0 = 1
\]

(d) \( f(2) \)
All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.

\[
f(2) = \left( \frac{1}{5} \right)^2 = \frac{1^2}{5^2} = \frac{1}{25}
\]

(e) \( f(3) \)
All we need to do here is plug in the \( x \) and do any quick arithmetic we need to do.

\[
f(3) = \left( \frac{1}{5} \right)^3 = \frac{1^3}{5^3} = \frac{1}{125}
\]

3. Sketch each of the following.

(a) \( f(x) = 6^x \)  
(b) \( g(x) = 6^x - 9 \)  
(c) \( g(x) = 6^{x+1} \)

(a) \( f(x) = 6^x \)
We can build up a quick table of values that we can plot for the graph of this function.
Here is a quick sketch of the graph of the function.

\[
\begin{array}{c|c}
 x & f(x) \\
\hline
-2 & f(-2) = 6^{-2} = \frac{1}{36} \\
-1 & f(-1) = 6^{-1} = \frac{1}{6} \\
0 & f(0) = 6^0 = 1 \\
1 & f(1) = 6^1 = 6 \\
2 & f(2) = 6^2 = 36
\end{array}
\]

(b) \( g(x) = 6^x - 9 \)

For this part all we need to do is recall the Transformations section from a couple of chapters ago. Using the “base” function of \( f(x) = 6^x \) the function for this part can be written as,

\[ g(x) = 6^x - 9 = f(x) - 9 \]

Therefore, the graph for this part is just the graph of \( f(x) \) shifted down by 9.

The graph of this function is shown below. The blue dashed line is the “base” function, \( f(x) \), and the red solid line is the graph for this part, \( g(x) \).
(c) \( g(x) = 6^{x+1} \)

For this part all we need to do is recall the Transformations section from a couple of chapters ago. Using the “base” function of \( f(x) = 6^x \) the function for this part can be written as,

\[
g(x) = 6^{x+1} = f(x+1)
\]

Therefore, the graph for this part is just the graph of \( f(x) \) shifted left by 1.

The graph of this function is shown below. The blue dashed line is the “base” function, \( f(x) \), and the red solid line is the graph for this part, \( g(x) \).

4. Sketch the graph of \( f(x) = e^{-x} \).

Solution
For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the “base” function of \( f(x) = e^x \) the function for this part can be written as,

\[
g(x) = e^{-x} = f(-x)
\]

Therefore, the graph for this part is just the graph of \( f(x) \) reflected about the y-axis.

The graph of this function is shown below. The blue dashed line is the “base” function, \( f(x) \), and the red solid line is the graph for this part, \( g(x) \).

5. Sketch the graph of \( f(x) = e^{x-3} + 6 \).

Solution
For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the “base” function of \( f(x) = e^x \) the function for this part can be written as,

\[
f(x) = e^{x-3} + 6 = f(x-3) + 6
\]

Therefore, the graph for this part is just the graph of \( f(x) \) shifted right by 3 and up by 6.

The graph of this function is shown below. The blue dashed line is the “base” function, \( f(x) \), and the red solid line is the graph for this part, \( g(x) \).
Logarithm Functions

1. Write $7^5 = 16807$ in logarithmic form.

Solution
There really isn’t all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the logarithmic form for this expression.

$$\log_7 16807 = 5$$

2. Write $16^{\frac{3}{4}} = 8$ in logarithmic form.

Solution
There really isn’t all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the logarithmic form for this expression.

$$\log_{16} 8 = \frac{3}{4}$$
3. Write $\left(\frac{1}{3}\right)^{-2} = 9$ in logarithmic form.

Solution
There really isn’t all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the logarithmic form for this expression.

$$\log_3 9 = -2$$

4. Write $\log_2 32 = 5$ in exponential form.

Solution
There really isn’t all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the exponential form for this expression.

$$2^5 = 32$$

5. Write $\log_4 \frac{1}{625} = 4$ in exponential form.

Solution
There really isn’t all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the exponential form for this expression.

$$\left(\frac{1}{5}\right)^4 = \frac{1}{625}$$

6. Write $\log_9 \frac{1}{81} = -2$ in exponential form.

Solution
There really isn’t all that much to do here other than refer to the definition of the logarithm function given in the notes for this section.

Here is the exponential form for this expression.
7. Without using a calculator determine the exact value of \( \log_3 81 \).

Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1
Converting the logarithm to exponential form gives,

\[
\log_3 81 = \, ? 
\Rightarrow 
3^\, ? = 81
\]

Step 2
From this we can quickly see that \( 3^4 = 81 \) and so we must have,

\[
\log_3 81 = 4
\]

8. Without using a calculator determine the exact value of \( \log_5 125 \).

Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1
Converting the logarithm to exponential form gives,

\[
\log_5 125 = \, ? 
\Rightarrow 
5^\, ? = 125
\]

Step 2
From this we can quickly see that \( 5^3 = 125 \) and so we must have,

\[
\log_5 125 = 3
\]

9. Without using a calculator determine the exact value of \( \log_2 \frac{1}{8} \).

Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1
Converting the logarithm to exponential form gives,
\[
\log_2 \frac{1}{8} = ? \quad \Rightarrow \quad 2^\frac{1}{8} = \frac{1}{8}
\]

Step 2
Now, we know that if we raise an integer to a negative exponent we’ll get a fraction and so we must have a negative exponent and then we know that \(2^3 = 8\). Therefore we can see that \(2^{-3} = \frac{1}{8}\) and so we must have,

\[
\log_2 \frac{1}{8} = -3
\]

10. Without using a calculator determine the exact value of \(\log_{\frac{1}{4}} 16\).

Hint : Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms.

Step 1
Converting the logarithm to exponential form gives,

\[
\log_{\frac{1}{4}} 16 = ? \quad \Rightarrow \quad \left(\frac{1}{4}\right)^? = 16
\]

Step 2
Now, we know that if we raise a fraction to a power and get an integer out we must have had a negative exponent. Now, we also know that \(4^2 = 16\). Therefore we can see that \(\left(\frac{1}{4}\right)^{-2} = \left(\frac{4}{1}\right)^2 = 16\) and so we must have,

\[
\log_{\frac{1}{4}} 16 = -2
\]

11. Without using a calculator determine the exact value of \(\ln e^4\).

Hint : Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a natural logarithm.

Step 1
Recalling that the base for a natural logarithm is \(e\) and converting the logarithm to exponential form gives,

\[
\ln e^4 = \log_e e^4 = ? \quad \Rightarrow \quad e^7 = e^4
\]

Step 2
From this we can quickly see that \(e^4 = e^4\) and so we must have,
4 \ln 4 = e

Note that an easier method of determining the value of this logarithm would have been to recall the properties of logarithm. In particular the property that states,
\[ \log_b b^x = x \]

Using this we can also very quickly see what the value of the logarithm is.

12. Without using a calculator determine the exact value of \( \log \frac{1}{100} \).

Hint: Recall that converting a logarithm to exponential form can often help to evaluate these kinds of logarithms. Also recall what the base is for a common logarithm.

Step 1
Recalling that the base for a natural logarithm is 10 and converting the logarithm to exponential form gives,
\[ \log \frac{1}{100} = \log_{10} \frac{1}{100} = ? \quad \Rightarrow \quad 10^? = \frac{1}{100} \]

Step 2
Now, we know that if we raise an integer to a negative exponent we’ll get a fraction and so we must have a negative exponent and then we know that \( 10^2 = 100 \). Therefore we can see that \( 10^{-2} = \frac{1}{100} \) and so we must have,
\[ \log \frac{1}{100} = -2 \]

13. Write \( \log \left( 3x^4 y^{-7} \right) \) in terms of simpler logarithms.

Step 1
So, we’re being asked here to use as many of the properties as we can to reduce this down into simpler logarithms.

First we can use Property 5 to break up the product into individual logarithms. Note that just because the property only has two terms in it does not mean that it won’t work for three (or more) terms. Here is the application of Property 5.
\[ \log \left( 3x^4 y^{-7} \right) = \log \left( 3 \right) + \log \left( x^4 \right) + \log \left( y^{-7} \right) \]

Step 2
Finally we need to use Property 7 on the last two logarithms to bring the exponents out of the logarithms. Here is that work.

\[
\log(3x^4y^{-7}) = \log(3) + 4\log(x) - 7\log(y)
\]

Remember that we can only bring an exponent out of a logarithm if it is on the whole argument of the logarithm. In other words, we couldn’t bring any of the exponents out of the logarithms until we had dealt with the product.

14. Write \( \ln\left(\sqrt[4]{y^2 + z^2}\right) \) in terms of simpler logarithms.

Step 1
So, we’re being asked here to use as many of the properties as we can to reduce this down into simpler logarithms.

First we can use Property 5 to break up the product into individual logarithms. Here is that work.

\[
\ln\left(\sqrt[4]{y^2 + z^2}\right) = \ln(x) + \ln\left((y^2 + z^2)^{\frac{1}{4}}\right)
\]

Note that we converted to root to a fractional exponent at the same time to help with the next step.

Step 2
Finally we need to use Property 7 on the last logarithm to bring the root exponent out of the logarithm. Here is that work.

\[
\ln\left(\sqrt[4]{y^2 + z^2}\right) = \ln(x) + \frac{1}{2}\ln\left(y^2 + z^2\right)
\]

Remember that we can only bring an exponent out of a logarithm if it is on the whole argument of the logarithm. In other words, we couldn’t bring any of the exponents out of the logarithms until we had dealt with the product. Also, in the second logarithm while each term is squared the whole argument is not squared, \(i.e.\) it’s not \((x + y)^2\) and so we can’t bring those 2’s out of the logarithm.

15. Write \( \log_4\left(\frac{x - 4}{y^{2\sqrt{2}}z}\right) \) in terms of simpler logarithms.

Step 1
So, we’re being asked here to use as many of the properties as we can to reduce this down into simpler logarithms.
First we can use Property 6 to break up the quotient into two logarithms. Here is that work.

\[
\log_4 \left( \frac{x - 4}{y^2 \sqrt[3]{z}} \right) = \log_4 (x - 4) - \log_4 \left( y^2 z^{\frac{1}{5}} \right)
\]

Step 2
Next we need to use Property 5 to break up the product in the second logarithm into two logarithms.

\[
\log_4 \left( \frac{x - 4}{y^2 \sqrt[3]{z}} \right) = \log_4 (x - 4) - \left( \log_4 (y^2) + \log_4 \left( \frac{z}{5} \right) \right)
\]

Be careful with the minus sign that was in front of the second logarithm from Step 1! Because of that we need to have parenthesis on the product once we use Property 5. The sum of the two “smaller” logarithms is the same as the product logarithm from Step 1 and so because we have a minus sign in front of the product logarithm we also need to have a minus sign in front of the two logarithms after using Property 5. The only way to make sure of this is to use the parenthesis as shown.

Step 3
Finally we’ll distribute the minus sign through the parenthesis and then use Property 7 on the last two logarithms to bring the exponents out of the logarithms. Here is that work.

\[
\log_4 \left( \frac{x - 4}{y^2 \sqrt[3]{z}} \right) = \log_4 (x - 4) - \log_4 (y) - \frac{1}{5} \log_4 (z)
\]

Remember that we can only bring an exponent out of a logarithm if is on the whole argument of the logarithm. In other words, we couldn’t bring any of the exponents out of the logarithms until we had dealt with the quotient and product. Recall as well that we can’t split up a sum/difference in a logarithm. Finally, make sure that you are careful in dealing with the minus sign we get from breaking up the quotient when dealing with the product in the denominator.

16. Combine \(2 \log_4 x + 5 \log_4 y - \frac{1}{2} \log_4 z\) into a single logarithm with a coefficient of one.

Hint : The properties that we use to break up logarithms can be used in reverse as well.

Step 1
To convert this into a single logarithm we’ll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

\[
\log_b (x^r) = r \log_b x
\]

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,
Step 2
Now, there are several ways to proceed from this point. We can use either of the two properties.

\[ \log_b(xy) = \log_b x + \log_b y \quad \text{and} \quad \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \]

and in fact we’ll need to use both in the end. We can use the product property on the first two logarithms (because they are a sum of logarithms) or the quotient property on the last two logarithms (because they are a difference of logarithms).

Which we use first does not matter as we’ll end up with the same result in the end. For this problems we’ll first use the product property on the first two logarithms to get,

\[ 2 \log_4 x + 5 \log_4 y - \frac{1}{2} \log_4 z = \log_4 \left( x^2 y^5 \right) - \log_4 \left( \sqrt{z} \right) \]

Step 3
Finally, we can see that we have a difference of two logarithms left and so we’ll use the quotient property to combine these to get,

\[ 2 \log_4 x + 5 \log_4 y - \frac{1}{2} \log_4 z = \log_4 \left( \frac{x^2 y^5}{\sqrt{z}} \right) \]

Note that the only reason we converted the fractional exponent to a root was to make the final answer a little nicer.

17. Combine \(3 \ln (t + 5) - 4 \ln t - 2 \ln (s - 1)\) into a single logarithm with a coefficient of one.

Hint: The properties that we use to break up logarithms can be used in reverse as well.

Step 1
To convert this into a single logarithm we’ll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

\[ \log_b \left( x^r \right) = r \log_b x \]

to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

\[ \ln (t + 5)^3 - \ln (t^4) - \ln (s - 1)^2 \]

Step 2
Now, there are several ways to proceed from this point. We can use either of the two properties.
\[
\log_b(xy) = \log_b x + \log_b y \quad \quad \quad \quad \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y
\]
and in fact we’ll need to use both in the end.

We should also be careful with the fact that there are two minus signs in here as that sometimes adds confusion to the problem. They are easy to deal with however if we just factor a minus sign out of the last two terms to get,

\[
3 \ln(t + 5) - 4 \ln t - 2 \ln(s - 1) = \ln(t + 5)^3 - \left( \ln(t^4) + \ln(s - 1)^2 \right)
\]

Written in this form we can see that the last two logarithms are a sum and so we can use the product property to combine them to get,

\[
3 \ln(t + 5) - 4 \ln t - 2 \ln(s - 1) = \ln(t + 5)^3 - \ln \left( t^4 (s - 1)^2 \right)
\]

Step 3
We now have a difference of two logarithms and we can use the quotient property to combine them to get,

\[
3 \ln(t + 5) - 4 \ln t - 2 \ln(s - 1) = \ln \left[ \frac{(t + 5)^3}{t^4 (s - 1)^2} \right]
\]

18. Combine \(\frac{1}{3} \log a - 6 \log b + 2\) into a single logarithm with a coefficient of one.

Hint : The properties that we use to break up logarithms can be used in reverse as well. For the constant see if you figure out a way to write that as a logarithm.

Step 1
To convert this into a single logarithm we’ll be using the properties that we used to break up logarithms in reverse. The first step in this process is to use the property,

\[
\log_b \left( x^r \right) = r \log_b x
\]
to make sure that all the logarithms have coefficients of one. This needs to be done first because all the properties that allow us to combine sums/differences of logarithms require coefficients of one on individual logarithms. So, using this property gives,

\[
\log \left( \frac{1}{3} \right) - \log(b^6) + 2
\]

Step 2
Now, for the 2 let’s notice that we can write this in terms of a logarithm as,

\[
2 = \log 10^2 = \log 100
\]
Note that this is really just using the property,
\[ \log_b b^x = x \]

So, we now have,
\[ \log \left( a^{\frac{1}{3}} \right) - \log (b^6) + \log 100 \]

Step 3
Now, there are several ways to proceed from this point. We can use either of the two properties.
\[ \log_b (xy) = \log_b x + \log_b y \]
\[ \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \]
and in fact we'll need to use both in the end.

The first two logarithms are a difference so let's use the quotient property to first combine those to get,
\[ \log \left( a^{\frac{1}{3}} \right) - \log (b^6) + \log 10^2 = \log \left( \frac{\sqrt[3]{a}}{b^6} \right) + \log 100 \]

We converted the fractional exponent in the first term to a root to make the answer a little nicer but doesn't really need to be done in general.

Step 4
Finally, note that we now have a sum of two logarithms and we can use the product property to combine those to get,
\[ \log \left( a^{\frac{1}{3}} \right) - \log (b^6) + \log 100 = \log \left( \frac{100 \sqrt[3]{a}}{b^6} \right) \]

19. Use the change of base formula and a calculator to find the value of \( \log_{12} 35 \).

Solution
We can use either the natural logarithm or the common logarithm to do this so we'll do both.
\[ \log_{12} 35 = \frac{\ln 35}{\ln 12} = \frac{3.55534806}{2.48490665} = 1.43077731 \]
\[ \log_{12} 35 = \frac{\log 35}{\log 12} = \frac{1.54406804}{1.07918125} = 1.43077731 \]

So, as we noted at the start it doesn’t matter which logarithm we use we’ll get the same answer in the end.
20. Use the change of base formula and a calculator to find the value of \( \log_{\frac{2}{3}} 53 \).

Solution
We can use either the natural logarithm or the common logarithm to do this so we’ll do both.

\[
\log_{\frac{2}{3}} 53 = \frac{\ln 53}{\ln \frac{2}{3}} = \frac{3.97029191}{-0.40546511} = -9.79194469
\]

\[
\log_{\frac{2}{3}} 53 = \frac{\log 53}{\log \frac{2}{3}} = \frac{1.72427587}{-0.17609126} = -9.79194469
\]

So, as we noted at the start it doesn’t matter which logarithm we use we’ll get the same answer in the end.

21. Sketch the graph of \( g(x) = -\ln(x) \).

Solution
For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the “base” function of \( f(x) = \ln(x) \) the function for this part can be written as,

\[ g(x) = -\ln(x) = -f(x) \]

Therefore, the graph for this part is just the graph of \( f(x) \) reflected about the \( x \)-axis.

The graph of this function is shown below. The blue dashed line is the “base” function, \( f(x) \), and the red solid line is the graph for this part, \( g(x) \).
22. Sketch the graph of \( g(x) = \ln(x + 5) \).

Solution
For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the “base” function of \( f(x) = \ln(x) \) the function for this part can be written as,

\[
g(x) = \ln(x + 5) = f(x + 5)
\]

Therefore, the graph for this part is just the graph of \( f(x) \) shifted left by 5.

The graph of this function is shown below. The blue dashed line is the “base” function, \( f(x) \), and the red solid line is the graph for this part, \( g(x) \).

Do not get excited about the fact that we plugged negative values of \( x \) into the function! The problem with negative values is not the values we plug into a logarithm. Instead the problem with negative values is when we go to evaluate the logarithm.

It is perfectly fine to plug negative values into a logarithm as long as we don’t end up actually evaluating a negative number. So, in this case we can see that as long as we require \( x > -5 \) then \( x + 5 > 0 \) and so those are acceptable values of \( x \) to plug in since we aren’t going to evaluate negative number in the logarithm.

Note however that we do have avoid \( x < -5 \) since that would mean evaluating logarithms at negative numbers.

23. Sketch the graph of \( g(x) = \ln(x) - 4 \).
Solution
For this problem all we need to do is recall the Transformations section from a couple of chapters ago. Using the “base” function of \( f(x) = \ln(x) \) the function for this part can be written as,

\[
g(x) = \ln(x) - 4 = f(x) - 4
\]

Therefore, the graph for this part is just the graph of \( f(x) \) shifted down by 4.

The graph of this function is shown below. The blue dashed line is the “base” function, \( f(x) \), and the red solid line is the graph for this part, \( g(x) \).

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**Solving Exponential Equations**

1. Solve the following equation.

\[
6^{2x} = 6^{1-3x}
\]

Step 1
Recall the property that says if \( b^x = b^y \) then \( x = y \). Since each exponential has the same base, 6 in this case, we can use this property to just set the exponents equal. Doing this gives,

\[
2x = 1 - 3x
\]

Step 2
Now all we need to do is solve the equation from Step 1 and that is a simple linear equation. Here is the solution work.
2. Solve the following equation.

\[ 5^{1-x} = 25 \]

Step 1
Recall the property that says if \( b^x = b^y \) then \( x = y \). In this case it looks like we can’t use this property. However, recall that \( 25 = 5^2 \) and if we write the right side of our equation using this we get,

\[ 5^{1-x} = 5^2 \]

Now each exponential has the same base, 5 to be exact, so we can use this property to just set the exponents equal. Doing this gives,

\[ 1 - x = 2 \]

Step 2
Now all we need to do is solve the equation from Step 1 and that is a simple linear equation. Here is the solution work.

\[
\begin{align*}
1 - x &= 2 \\
-x &= 1 \\
\Rightarrow x &= -1
\end{align*}
\]

So, the solution to the equation is then : \( x = -1 \).

3. Solve the following equation.

\[ 8^{x^2} = 8^{3x+10} \]

Step 1
Recall the property that says if \( b^x = b^y \) then \( x = y \). Since each exponential has the same base, 8 in this case, we can use this property to just set the exponents equal. Doing this gives,

\[ x^2 = 3x + 10 \]

Step 2
Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we should be able to quickly solve. Here is the solution work.
\[ x^2 = 3x + 10 \]
\[ x^2 - 3x - 10 = 0 \]
\[ (x - 5)(x + 2) = 0 \quad \rightarrow \quad x = -2, \quad x = 5 \]

So, the solutions to the equation are then: \( x = -2 \) and \( x = 5 \).

4. Solve the following equation.
\[ 7^{4-x} = 7^{4x} \]

Step 1
Recall the property that says if \( b^x = b^y \) then \( x = y \). Since each exponential has the same base, 7 in this case, we can use this property to just set the exponents equal. Doing this gives,

\[ 4 - x = 4x \]

Step 2
Now all we need to do is solve the equation from Step 1 and that is a simple linear equation. Here is the solution work.

\[ 4 - x = 4x \]
\[ 4 = 5x \quad \rightarrow \quad x = \frac{4}{5} \]

So, the solution to the equation is then: \( x = \frac{4}{5} \).

5. Solve the following equation.
\[ 2^{3x} = 10 \]

Step 1
For this equation there is no way to easily get both sides with the same base. Therefore, we’ll need to take the logarithm of both sides.

We can use any logarithm and the natural logarithm and common logarithm are usually good choices since most calculators can handle them. Because one of the bases in this equation is a 10 the common logarithm will probably be the better choice (although we can use the natural logarithm if we wanted to).

Taking the logarithm (using the common logarithm) of both sides gives,

\[ \log 2^{3x} = \log 10 \]
Step 2
Now we can easily compute the right side (which is also why we chose the common logarithm for this case) and we can use the logarithm property that says,

$$\log_b x^r = r \log_b x$$

to move the $3x$ out of the exponent from the logarithm on the left. Doing this gives,

$$3x(\log 2) = x(3\log 2) = 1$$

We did a little rearranging of the left side to put all the numbers together in order to make the next step a little easier.

Step 3
Finally, all we need to do is solve for $x$. Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equation at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations.

In other words, all we need to do is divide both sides by the coefficient of the $x$ and then use our calculators to get a decimal answer.

Here is the rest of the work for this problem.

$$x(3\log 2) = 1 \quad \rightarrow \quad x = \frac{1}{3\log 2} = \frac{1}{3(0.301029996)} = 1.10730936$$

6. Solve the following equation.

$$7^{1-x} = 4^{3x+1}$$

Step 1
For this equation there is no way to easily get both sides with the same base. Therefore, we’ll need to take the logarithm of both sides.

We can use any logarithm and the natural logarithm and common logarithm are usually good choices since most calculators can handle them. In this case there really isn’t any reason to use one or the other so we’ll use the natural logarithm (it’s easier to write two letters – $\ln$ versus three letters – $\log$ afterall…).

Taking the logarithm (using the natural logarithm) of both sides gives,

$$\ln 7^{1-x} = \ln 4^{3x+1}$$

Step 2
Now we can use the logarithm property that says,

$$\log_b x^r = r \log_b x$$
to move the exponents out of each of the logarithms. Doing this gives,

\[(1-x)\ln 7 = (3x+1)\ln 4\]

Step 3
Finally, all we need to do is solve for \(x\). Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

\[
\begin{align*}
(1-x)\ln 7 &= (3x+1)\ln 4 \\
\ln 7 - x\ln 7 &= 3x\ln 4 + \ln 4 \\
\ln 7 - \ln 4 &= 3x\ln 4 + x\ln 7 \\
\ln 7 - \ln 4 &= (3\ln 4 + \ln 7)x \\
x &= \frac{\ln 7 - \ln 4}{3\ln 4 + \ln 7} = \frac{1.945910149 - 1.386294361}{3(1.386294361) + 1.945910149} = 0.091668262
\end{align*}
\]

Again, the work is messier than we are used to but it is not really different from work we’ve done previously in solving equations. The answer is also going to be “messier” in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don’t worry about that.

7. Solve the following equation.

\[9 = 10^{4+6x}\]

Step 1
For this equation there is no way to easily get both sides with the same base. Therefore, we’ll need to take the logarithm of both sides.

We can use any logarithm and the natural logarithm and common logarithm are usually good choices since most calculators can handle them. In this case one of the bases is a 10 and so the common logarithm is probably the better choice.

Taking the logarithm (using the common logarithm) of both sides gives,

\[\log 9 = \log 10^{4+6x}\]

Step 2
Now we can use the logarithm property that says,

\[\log a^b = b \log a\]

\[\log 10^{f(x)} = f(x)\]
to simplify the right side of the equation. Doing this gives,

\[ \log 9 = 4 + 6x \]

Step 3
Finally, all we need to do is solve for \( x \). Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

\[
\begin{align*}
\log 9 &= 4 + 6x \\
\log 9 - 4 &= 6x \\
x &= \frac{\log 9 - 4}{6} = \frac{0.9542425094 - 4}{6} = -0.5076262484
\end{align*}
\]

Again, the work is messier than we are used to but it is not really different from work we’ve done previously in solving equations. The answer is also going to be “messier” in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don’t worry about that.

Also, be careful when evaluating the numerator in the final answer. The 4 was outside of the logarithm and so cannot be moved into the logarithm. We probably should have been a little more careful with parenthesis and written the answer as,

\[ x = \frac{\log(9) - 4}{6} \]

which makes it a little more clear that the 4 isn’t inside the logarithm. However, we typically don’t put the parenthesis on the logarithm when it is just a number.

8. Solve the following equation.

\[ e^{7+2x} - 3 = 0 \]

Step 1
Before we put any logarithms into this problem we first need to get the exponential on one side by itself so let’s do that first.

\[ e^{7+2x} = 3 \]

Step 2
Now we can take the logarithm of both sides and because we have a base of \( e \) in this problem the natural logarithm is probably the best choice. So, taking the logarithm (using the natural logarithm) of both sides gives,
\[ \ln e^{7+2x} = \ln 3 \]

Step 3
Now we can use the logarithm property that says,
\[ \ln e^{f(x)} = f(x) \]

This simplifies the left side of the equation. Doing this gives,
\[ 7 + 2x = \ln 3 \]

Step 4
Finally, all we need to do is solve for \( x \). Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

\[
\begin{align*}
7 + 2x &= \ln 3 \\
2x &= \ln 3 - 7 \\
x &= \frac{\ln 3 - 7}{2} = \frac{1.098612289 - 7}{2} = -2.950693856
\end{align*}
\]

Again, the work is messier than we are used to but it is not really different from work we’ve done previously in solving equations. The answer is also going to be “messier” in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don’t worry about that.

Also, be careful when evaluating the numerator in the final answer. The 7 was outside of the logarithm and so cannot be moved into the logarithm. We probably should have been a little more careful with parenthesis and written the answer as,

\[
x = \frac{\ln (3) - 7}{2}
\]

which makes it a little more clear that the 7 isn’t inside the logarithm. However, we typically don’t put the parenthesis on the logarithm when it is just a number.

9. Solve the following equation.
\[ e^{4-7x} + 11 = 20 \]

Step 1
Before we put any logarithms into this problem we first need to get the exponential on one side by itself so let’s do that first.
\[ e^{4-7x} = 9 \]

Step 2
Now we can take the logarithm of both sides and because we have a base of $e$ in this problem the natural logarithm is probably the best choice. So, taking the logarithm (using the natural logarithm) of both sides gives,

\[ \ln e^{4-7x} = \ln 9 \]

Step 3
Now we can use the logarithm property that says,

\[ \ln e^{f(x)} = f(x) \]

to simplify the left side of the equation. Doing this gives,

\[ 4 - 7x = \ln 9 \]

Step 4
Finally, all we need to do is solve for $x$. Recall that the equations at this step tend to look messier than we are used to dealing with. However, the logarithms in the equations at this point are just numbers and so we treat them as we treat all numbers with these kinds of equations. The work will be messier than we are used to but just keep in mind that the logarithms are just numbers!

Here is the rest of the work for this problem.

\[ 4 - 7x = \ln 9 \]
\[ -7x = \ln 9 - 4 \]
\[ x = \frac{\ln 9 - 4}{-7} = \frac{2.197224577 - 4}{-7} = \frac{-1.802775423}{-7} = 0.2575393461 \]

Again, the work is messier than we are used to but it is not really different from work we’ve done previously in solving equations. The answer is also going to be “messier” in the sense that it is a decimal and is liable to almost always be a decimal for most of these types of problems so don’t worry about that.

Also, be careful when evaluating the numerator in the final answer. The 4 was outside of the logarithm and so cannot be moved into the logarithm. We probably should have been a little more careful with parenthesis and written the answer as,

\[ x = \frac{\ln (9) - 4}{-7} \]

which makes it a little more clear that the 4 isn’t inside the logarithm. However, we typically don’t put the parenthesis on the logarithm when it is just a number.
1. Solve the following equation.

\[ \log_4 (x^2 - 2x) = \log_4 (5x - 12) \]

Hint: We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base! Also, don’t forget that the values with get when we are done solving logarithm equations don’t always correspond to actual solutions to the equation so be careful!

Step 1
Recall the property that says if \( \log_b x = \log_b y \) then \( x = y \). Since each logarithm is on opposite sides of the equal sign and each has the same base, 4 in this case, we can use this property to just set the arguments of each equal. Doing this gives,

\[ x^2 - 2x = 5x - 12 \]

Step 2
Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we know how to solve. Here is the solution work.

\[ x^2 - 2x = 5x - 12 \]
\[ x^2 - 7x + 12 = 0 \]
\[ (x - 3)(x - 4) = 0 \]

\[ \rightarrow \quad x = 3, \quad x = 4 \]

Step 3
As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

\( x = 3 \): 

\[ \log_4 (3^2 - 2(3)) = \log_4 (5(3) - 12) \]
\[ \log_4 (3) = \log_4 (3) \quad \text{OKAY} \]

\( x = 4 \): 

\[ \log_4 (4^2 - 2(4)) = \log_4 (5(4) - 12) \]
\[ \log_4 (8) = \log_4 (8) \quad \text{OKAY} \]

In this case, both numbers do not produce negative numbers in the logarithms and so they are in fact both solutions (won’t happen with every problem so don’t always expect this to happen!).

Therefore, the solutions to the equation are then: $x = 3$ and $x = 4$.

2. Solve the following equation.

$$\log(6x) - \log(4 - x) = \log(3)$$

Hint: We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base! Also, don’t forget that the values with get when we are done solving logarithm equations don’t always correspond to actual solutions to the equation so be careful!

Step 1
Recall the property that says if $\log_b x = \log_b y$ then $x = y$. That doesn’t appear to have any use here since there are three logarithms in the equation. However, recall that we can combine a difference of logarithms (provide the coefficient of each is a one of course…) as follows,

$$\log \left( \frac{6x}{4 - x} \right) = \log(3)$$

We now have only two logarithms and each logarithm is on opposite sides of the equal sign and each has the same base, 10 in this case. Therefore, we can use this property to just set the arguments of each equal. Doing this gives,

$$\frac{6x}{4 - x} = 3$$

Step 2
Now all we need to do is solve the equation from Step 1 and that is an equation that we know how to solve. Here is the solution work.

$$\frac{6x}{4 - x} = 3$$

$$6x = 3(4 - x) = 12 - 3x$$

$$9x = 12 \quad \rightarrow \quad x = \frac{12}{9} = \frac{4}{3}$$

Step 3
As the final step we need to take the number from the above step and plug it into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for the number.

$$x = \frac{4}{3}$$
\[
\log\left(6\left(\frac{4}{3}\right)\right) - \log\left(4 - \frac{4}{3}\right) = \log(3) \\
\log(8) - \log\left(\frac{4}{3}\right) = \log(3) \quad \text{OKAY}
\]

In this case, the number did not produce negative numbers in the logarithms so it is in fact a solution (won’t happen with every problem so don’t always expect this to happen!).

Therefore, the solution to the equation is then: \[x = \frac{4}{3}.\]

3. Solve the following equation.

\[\ln(x) + \ln(x + 3) = \ln(20 - 5x)\]

Hint: We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base! Also, don’t forget that the values with get when we are done solving logarithm equations don’t always correspond to actual solutions to the equation so be careful!

Step 1

Recall the property that says if \(\log_b x = \log_b y\) then \(x = y\). That doesn’t appear to have any use here since there are three logarithms in the equation. However, recall that we can combine a sum of logarithms (provide the coefficient of each is a one of course…) as follows,

\[\ln(x + 3) + \ln(20 - 5x) = \ln\left(x\left(x + 3\right)\right) = \ln\left(20 - 5x\right)\]

We now have only two logarithms and each logarithm is on opposite sides of the equal sign and each has the same base, \(e\) in this case. Therefore, we can use this property to just set the arguments of each equal. Doing this gives,

\[x(x + 3) = 20 - 5x\]

Step 2

Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we know how to solve. Here is the solution work.

\[x(x + 3) = 20 - 5x\]
\[x^2 + 3x = 20 - 5x\]
\[x^2 + 8x - 20 = 0\]
\[(x + 10)(x - 2) = 0 \quad \rightarrow \quad x = -10, \quad x = 2\]

Step 3

As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!
Here is the checking work for each of the numbers.

\[ x = -10: \]
\[ \ln(-10) + \ln(-10 + 3) = \ln(20 - 5(-10)) \]
\[ \ln(-10) + \ln(-7) = \ln(70) \quad \text{NOT OKAY} \]

\[ x = 2: \]
\[ \ln(2) + \ln(2 + 3) = \ln(20 - 5(2)) \]
\[ \ln(2) + \ln(5) = \ln(10) \quad \text{OKAY} \]

In this case, the only one number did not produce negative numbers in the logarithms so that is the only number that will be a solution. The number that produced negative numbers in the logarithm is not a solution.

Therefore, the only solution to the equation is then: \( x = 2 \).

Note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that \( x = -10 \) would be a solution to the equation. However, that is only because we were checking in a “modified” equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

4. Solve the following equation.

\[ \log_3(25 - x^2) = 2 \]

Hint: We had a very nice property from the notes on how to solve equations that contained exactly two logarithms with the same base and yes we can use that property here! Also, don’t forget that the values with get when we are done solving logarithm equations don’t always correspond to actual solutions to the equation so be careful!

Step 1
Recall the property that says if \( \log_b x = \log_b y \) then \( x = y \). That doesn’t appear to have any use here since there is only one logarithm in the equation. Note however that we could write the right side, \( i.e. \) the 2, as,

\[ 2 = \log_3(3^2) \]
Doing this means we can write the equation as,

\[ \log_3 \left( 25 - x^2 \right) = \log_3 \left( 3^2 \right) = \log_3 \left( 9 \right) \]

We now have two logarithms and each logarithm is on opposite sides of the equal sign and each has the same base, 3 in this case. Therefore, we can use this property to just set the arguments of each equal. Doing this gives,

\[ 25 - x^2 = 9 \]

Step 2
Now all we need to do is solve the equation from Step 1 and that is a quadratic equation that we know how to solve. Here is the solution work.

\[
25 - x^2 = 9 \\
16 = x^2 \quad \Rightarrow \quad x = \pm \sqrt{16} = \pm 4
\]

Step 3
As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

\[
x = -4: \\
\log_3 \left( 25 - (-4)^2 \right) = 2 \\
\log_3 (9) = 2 \quad \text{OKAY}
\]

\[
x = 4: \\
\log_3 \left( 25 - (4)^2 \right) = 2 \\
\log_3 (9) = 2 \quad \text{OKAY}
\]

In this case, both numbers do not produce negative numbers in the logarithms and so they are in fact both solutions (won’t happen with every problem so don’t always expect this to happen!).

Therefore, the solutions to the equation are then: \[ x = -4 \] and \[ x = 4 \].

Be careful to not make the mistake of assuming that just because a value of \( x \) is negative that it will automatically not be a solution to the equation. As we’ve shown here, even though \( x = -4 \) is negative it did not produce any negative values in the logarithms and so is perfectly acceptable as a solution.
5. Solve the following equation.

\[ \log_2 (x+1) - \log_2 (2 - x) = 3 \]

Hint: If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don’t forget that the values with get when we are done solving don’t always correspond to actual solutions so be careful!

Step 1
First let’s notice that we can combine the two logarithms on the left side to get,

\[ \log_2 \left( \frac{x+1}{2-x} \right) = 3 \]

Step 2
Now, we can easily convert this to exponential form.

\[ \frac{x+1}{2-x} = 2^3 = 8 \]

Step 3
Now all we need to do is solve the equation from Step 2 and that is an equation that we know how to solve. Here is the solution work.

\[ \frac{x+1}{2-x} = 8 \]
\[ x + 1 = 8(2 - x) = 16 - 8x \]
\[ 9x = 15 \quad \rightarrow \quad x = \frac{15}{9} = \frac{5}{3} \]

Step 4
As the final step we need to take the number from the above step and plug it into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for the number.

\[ x = \frac{5}{3}: \]

\[ \log_2 \left( \frac{5}{3} + 1 \right) - \log_2 \left( 2 - \frac{5}{3} \right) = 3 \]
\[ \log_2 \left( \frac{8}{3} \right) - \log_2 \left( \frac{1}{3} \right) = 3 \quad \text{OKAY} \]

In this case, the number did not produce negative numbers in the logarithms so it is in fact a solution (won’t happen with every problem so don’t always expect this to happen!).

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Therefore, the solution to the equation is then: \[ x = \frac{5}{3}. \]

6. Solve the following equation.

\[ \log_4 (-x) + \log_4 (6 - x) = 2 \]

Hint: If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don’t forget that the values with get when we are done solving don’t always correspond to actual solutions so be careful!

Step 1
First let’s notice that we can combine the two logarithms on the left side to get,

\[ \log_4 (-x(6-x)) = 2 \]

Step 2
Now, we can easily convert this to exponential form.

\[ -x(6-x) = 4^2 = 16 \]

Step 3
Now all we need to do is solve the equation from Step 2 and that is a quadratic equation that we know how to solve. Here is the solution work.

\[ -x(6-x) = 16 \]
\[ x^2 - 6x - 16 = 0 \]
\[ (x - 8)(x + 2) = 0 \quad \rightarrow \quad x = 8, \quad x = -2 \]

Step 4
As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

\[ x = 8: \]
\[ \log_4 (-8) + \log_4 (6 - 8) = 2 \]
\[ \log_4 (-8) + \log_4 (-2) = 2 \quad \text{NOT OKAY} \]

\[ x = -2: \]
\[ \log_4 (-(-2)) + \log_4 (6 - (-2)) = 2 \]
\[ \log_4 (2) + \log_4 (8) = 2 \quad \text{OKAY} \]
In this case, the only one number did not produce negative numbers in the logarithms so that is the only number that will be a solution. The number that produced negative numbers in the logarithm is not a solution.

Therefore, the only solution to the equation is then: \( x = -2 \).

Be careful to not make the mistake of assuming that just because a value of \( x \) is negative that it will automatically not be a solution to the equation and just because a value of \( x \) is positive it will automatically be a solution. As we’ve shown here we can have negative values of \( x \) that are solutions and positive values of \( x \) that are not solutions.

Also note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that \( x = 8 \) would be a solution to the equation. However, that is only because we were checking in a “modified” equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

7. Solve the following equation.

\[
\log(x) = 2 - \log(x - 21)
\]

Hint: If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don’t forget that the values with get when we are done solving don’t always correspond to actual solutions so be careful!

Step 1
First let’s notice that if we move the logarithm on the right side to the left side we can combine the two logarithms on the left side to get,

\[
\log(x) + \log(x - 21) = 2
\]

\[
\log(x(x - 21)) = 2
\]

Step 2
Now, we can easily convert this to exponential form (recall that because there is no base given it is assumed to be 10!).

\[
x(x - 21) = 10^2 = 100
\]

Step 3
Now all we need to do is solve the equation from Step 2 and that is a quadratic equation that we know how to solve. Here is the solution work.

\[ x(x - 21) = 100 \]
\[ x^2 - 21x - 100 = 0 \]
\[ (x - 25)(x + 4) = 0 \quad \rightarrow \quad x = 25, \quad x = -4 \]

**Step 4**
As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

\[ x = 25: \]
\[ \log(25) = 2 - \log(25 - 21) \]
\[ \log(25) = 2 - \log(4) \quad \text{OKAY} \]

\[ x = -4: \]
\[ \log(-4) = 2 - \log(-4 - 21) \]
\[ \log(-4) = 2 - \log(-25) \quad \text{NOT OKAY} \]

In this case, the only one number did not produce negative numbers in the logarithms so that is the only number that will be a solution. The number that produced negative numbers in the logarithm is not a solution.

Therefore, the only solution to the equation is then: \[ x = 25 \].

Note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that \( x = -4 \) would be a solution to the equation. However, that is only because we were checking in a “modified” equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

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8. Solve the following equation.

\[ \ln(x - 1) = 1 + \ln(3x + 2) \]
Hint: If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don’t forget that the values with get when we are done solving don’t always correspond to actual solutions so be careful!

Step 1
First let’s notice that if we move the logarithm on the right side to the left side we can combine the two logarithms on the left side to get,

\[
\ln (x - 1) = 1 + \ln (3x + 2)
\]

\[
\ln (x - 1) - \ln (3x + 2) = 1
\]

\[
\log \left( \frac{x - 1}{3x + 2} \right) = 1
\]

Step 2
Now, we can easily convert this to exponential form (recall that because we are working with the natural logarithm the base is \(e\)).

\[
\frac{x - 1}{3x + 2} = e^1 = e
\]

Step 3
Now all we need to do is solve the equation from Step 2 and that is an equation that we know how to solve. Here is the solution work.

\[
\frac{x - 1}{3x + 2} = e
\]

\[
x - 1 = e(3x + 2)
\]

\[
x - 1 = 3e x + 2e
\]

\[
x - 3e x = 1 + 2e
\]

\[
(1 - 3e) x = 1 + 2e \quad \rightarrow \quad x = \frac{1 + 2e}{1 - 3e} = \frac{1 + 2(0.89961)}{1 - 3(0.89961)} = -0.89961
\]

Do not get excited about the \(e\) in the equation. It works the same as if it was just a 4 or 5 or any other number. The only real difference is that the answer is a little messier that we usually get with these kinds of problems. Also, for the next step it is probably best to convert these kinds of numbers into decimal form.

Step 4
As the final step we need to take the number from the above step and plug it into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

\[
x = -0.89961:
\]

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\[
\ln(-0.89961 - 1) = 1 + \ln\left(3(-0.89961) + 2\right) \\
\ln(-1.89961) = 1 + \ln(-0.69883) \quad \text{NOT OKAY}
\]

So, in this case the only number we got from Step 3 produced negative numbers in the logarithms and so can’t be a solution. What this means for us is that there is **no solution** to this equation. This happens on occasion and we shouldn’t worry about it when it does.

Note that it is vitally important that you do the check in the original equation. In the first step (where we combined two of the logarithms) we changed the equation and in the process introduced a number that is not in fact a solution.

Had we checked in any other equation in the solution work it would appear that \( x = 8 \) would be a solution to the equation. However, that is only because we were checking in a “modified” equation and not the original equation which is what we were being asked to solve.

This is always the danger of modifying equations during the solution process. Unfortunately, with many logarithm equations that is our only solution path and so is something that we need to be prepared to deal with.

9. Solve the following equation.

\[
2 \log(x) - \log(7x - 1) = 0
\]

Hint : If we can reduce all the logarithms to a single logarithm it would be quite easy to convert to exponential form. Also, don’t forget that the values with get when we are done solving don’t always correspond to actual solutions so be careful!

**Step 1**

First let’s notice that we can move the 2 in front of the first logarithm into the logarithm as follows,

\[
\log(x^2) - \log(7x - 1) = 0
\]

We can now combine the two logarithms to get,

\[
\log\left(\frac{x^2}{7x - 1}\right) = 0
\]

**Step 2**

Now, we can easily convert this to exponential form (recall that because there is no base given it is assumed to be 10!).

\[
\frac{x^2}{7x - 1} = 10^0 = 1
\]

**Step 3**
Now all we need to do is solve the equation from Step 2 and that is a quadratic equation that we know how to solve. Here is the solution work.

\[
\frac{x^2}{7x-1} = 1
\]

\[
x^2 = 7x - 1
\]

\[
x^2 - 7x + 1 = 0
\]

We can’t factor this but we can use the quadratic formula on it. Doing that gives the following two numbers.

\[
x = \frac{7 \pm \sqrt{7^2 - 4(1)(1)}}{2(1)} = \frac{7 \pm \sqrt{45}}{2} \rightarrow x = 0.1459, \quad x = 6.8541
\]

Don’t worry about the fact that we needed to use the quadratic formula to solve this. This will happen on occasion and we need to be able to deal with it when it happens.

Step 4
As the final step we need to take each of the numbers from the above step and plug them into the original equation from the problem statement to make sure we don’t end up taking the logarithm of zero or negative numbers!

Here is the checking work for each of the numbers.

\[ x = 0.1459: \]

\[ 2\log(0.1459) - \log(7(0.1459) - 1) = 0 \]

\[ 2\log(0.1459) - \log(0.0213) = 0 \quad \text{OKAY} \]

\[ x = 6.8541: \]

\[ 2\log(6.8541) - \log(7(6.8541) - 1) = 0 \]

\[ 2\log(6.8541) - \log(46.9787) = 0 \quad \text{OKAY} \]

In this case, both numbers do not produce negative numbers in the logarithms and so they are in fact both solutions (won’t happen with every problem so don’t always expect this to happen!).

Therefore, the solutions to the equation are then: \[ x = 0.1459 \] and \[ x = 6.8541 \].

---

**Applications**
1. We have $10,000 to invest for 44 months. How much money will we have if we put the money into an account that has an annual interest rate of 5.5% and interest is compounded,
(a) quarterly    (b) monthly   (c) continuously

(a) quarterly
From the problem statement we can see that,
\[ P = 10000 \]
\[ r = \frac{5.5}{100} = 0.055 \]
\[ t = \frac{44}{12} = \frac{11}{3} \]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also remember that \( t \) must be in years and so we’ll need to convert the months we are given to years.

For this part we are compounding interest rate quarterly and that means it will compound 4 times per year and so we also then know that,
\[ m = 4 \]

At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we’ll have.
\[ A = 10000 \left( 1 + \frac{0.055}{4} \right)^{11/3} = 10000 \left( 1.01375 \right)^{44/3} = 10000 \left( 1.221760422 \right) = 12217.60 \]

So, we’ll have $12,217.60 in the account after 44 months.

(b) monthly
From the problem statement we can see that,
\[ P = 10000 \]
\[ r = \frac{5.5}{100} = 0.055 \]
\[ t = \frac{44}{12} = \frac{11}{3} \]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also remember that \( t \) must be in years and so we’ll need to convert the months we are given to years.

For this part we are compounding interest rate monthly and that means it will compound 12 times per year and so we also then know that,
\[ m = 12 \]

At this point all that we need to do is plug into the equation and run the numbers through a calculator to compute the amount of money that we’ll have.
\[ A = 10000 \left( 1 + \frac{0.055}{12} \right)^{11/12} = 10000 \left( 1.00453333 \right)^{44} = 10000 \left( 1.222876562 \right) = 12228.77 \]

So, we’ll have $12,228.77 in the account after 44 months.

(c) continuously
From the problem statement we can see that,
\[ P = 10000 \]
\[ r = \frac{5.5}{100} = 0.055 \]
\[ t = \frac{44}{12} = \frac{11}{3} \]
Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also remember that \( t \) must be in years and so we’ll need to convert the months we are given to years.

For this part we are compounding continuously and so we won’t have an \( m \) and will be using the other equation and all we have all we need to do the computation so,

\[
A = 10000e^{\left(0.055 \cdot \frac{11}{12}\right)} = 10000e^{0.2016666667} = 10000 \left(1.223440127 \right) = 12234.40
\]

So, we’ll have \$12,234.40 in the account after 44 months.

2. We are starting with \$5000 and we’re going to put it into an account that earns an annual interest rate of 12%. How long should we leave the money in the account in order to double our money if interest is compounded,

(a) quarterly \hspace{1cm} (b) monthly \hspace{1cm} (c) continuously

(a) quarterly
From the problem statement we can see that,

\[
A = 10000 \hspace{1cm} P = 5000 \hspace{1cm} r = \frac{12}{100} = 0.12
\]

Remember that the value of \( r \) must be given as a decimal, \( i.e. \) the percentage divided by 100. Also, for this part we are compounding interest rate quarterly and that means it will compound 4 times per year and so we also then know that,

\[ m = 4 \]

Plugging into the equation gives us,

\[
10000 = 5000 \left(1 + \frac{0.12}{4}\right)^{4t} = 5000 \left(1.03\right)^{4t}
\]

Using the techniques from the Solve Exponential Equations section we can solve for \( t \).

\[
2 = 1.03^{4t}
\]

\[
\ln \left(2\right) = \ln \left(1.03^{4t}\right)
\]

\[
\ln \left(2\right) = 4t \ln \left(1.03\right)
\]

\[
t = \frac{\ln \left(2\right)}{4 \ln \left(1.03\right)} = 5.8624
\]

So, we’ll double our money in approximately 5.8624 years.

(b) monthly
From the problem statement we can see that,

\[
A = 10000 \hspace{1cm} P = 5000 \hspace{1cm} r = \frac{12}{100} = 0.12
\]
Remember that the value of $r$ must be given as a decimal, i.e. the percentage divided by 100. Also, for this part we are compounding interest rate monthly and that means it will compound 12 times per year and so we also then know that,

$$m = 12$$

Plugging into the equation gives us,

$$10000 = 5000 \left(1 + \frac{0.12}{12}\right)^{12t} = 5000 \left(1.01\right)^{12t}$$

Using the techniques from the Solve Exponential Equations section we can solve for $t$.

$$2 = 1.01^{12t}$$

$$\ln(2) = \ln\left(1.01^{12t}\right)$$

$$\ln(2) = 12t \ln(1.01)$$

$$t = \frac{\ln(2)}{12 \ln(1.01)} = 5.8051$$

So, we’ll double our money in approximately 5.8051 years.

(c) **continuously**

From the problem statement we can see that,

$$A = 10000 \quad P = 5000 \quad r = \frac{12}{100} = 0.12$$

Remember that the value of $r$ must be given as a decimal, i.e. the percentage divided by 100. For this part we are compounding continuously and so we won’t have an $m$ and will be using the other equation.

Plugging into the continuously compounding interest equation gives,

$$10000 = 5000e^{0.12t}$$

Now, solving this using the techniques from the Solve Exponential Equations section gives,

$$2 = e^{0.12t}$$

$$\ln(2) = \ln\left(e^{0.12t}\right)$$

$$\ln(2) = 0.12t$$

$$t = \frac{\ln(2)}{0.12} = 5.7762$$

So, we’ll double our money in approximately 5.7762 years.

3. A population of bacteria initially has 250 present and in 5 days there will be 1600 bacteria present.
(a) Determine the exponential growth equation for this population.
(b) How long will it take for the population to grow from its initial population of 250 to a population of 2000?

(a) Determine the exponential growth equation for this population.

We can start off here by acknowledging that we know the initial number of bacteria is 250 and so $Q_0 = 250$. Therefore the equation is then,

$$Q(t) = 250e^{kt}$$

Now, we also know that $Q(5) = 1600$ and plugging this into the equation above gives,

$$1600 = Q(5) = 250e^{5k}$$

We can use techniques from the Solve Logarithm Equations section to determine the value of $k$.

$$1600 = 250e^{5k}$$
$$\frac{1600}{250} = e^{5k}$$
$$\ln\left(\frac{32}{5}\right) = 5k$$

$$k = \frac{1}{5}\ln\left(\frac{32}{5}\right) = 0.3712596$$

Depending upon your preferences we can use either the exact value or the decimal value. Note however that because $k$ is in the exponent of an exponential function we’ll need to use quite a few decimal places to avoid potentially large differences in the value that we’d get if we rounded off too much.

Putting all of this together the exponential growth equation for this population is,

$$Q = 250e^{\frac{1}{5}\ln\left(\frac{32}{5}\right)t}$$

(b) How long will it take for the population to grow from its initial population of 250 to a population of 2000?

What we’re really being asked to do here is to solve the equation,

$$2000 = Q(t) = 250e^{\frac{1}{5}\ln\left(\frac{32}{5}\right)t}$$

and we know from the Solve Logarithm Equations section how to do that. Here is the solution work for this part.
\[
\frac{2000}{250} = e^{\frac{1}{5} \ln \left( \frac{32}{5} \right) t}
\]
\[
\ln(8) = \frac{1}{5} \ln \left( \frac{32}{5} \right) t
\]
\[
t = \frac{5 \ln(8)}{\ln \left( \frac{32}{5} \right)} = 5.6010
\]

It will take 5.601 days for the population to reach 2000.

4. We initially have 100 grams of a radioactive element and in 1250 years there will be 80 grams left.
   (a) Determine the exponential decay equation for this element.
   (b) How long will it take for half of the element to decay?
   (c) How long will it take until there is only 1 gram of the element left?

(a) **Determine the exponential decay equation for this element.**

We can start off here by acknowledging that we know the initial amount of the radioactive element is 100 and so \( Q_0 = 100 \). Therefore the equation is then,

\[
Q(t) = 100e^{kt}
\]

Now, we also know that \( Q(1250) = 80 \) and plugging this into the equation above gives,

\[
80 = Q(1250) = 100e^{1250k}
\]

We can use techniques from the [Solve Logarithm Equations](http://tutorial.math.lamar.edu/terms.aspx) section to determine the value of \( k \).

\[
\frac{80}{100} = e^{1250k}
\]

\[
\ln \left( \frac{4}{5} \right) = 1250k
\]

\[
k = \frac{1}{1250} \ln \left( \frac{4}{5} \right) = -0.000178515
\]

Depending upon your preferences we can use either the exact value or the decimal value. Note however that because \( k \) is in the exponent of an exponential function we’ll need to use quite a few decimal places to avoid potentially large differences in the value that we’d get if we rounded off too much.

Putting all of this together the exponential decay equation for this population is,
\[ Q = 100e^{\frac{1}{1250} \ln\left(\frac{4}{5}\right) t} \]

(b) **How long will it take for half of the element to decay?**

What we’re really being asked to do here is to solve the equation,

\[ 50 = Q(t) = 100e^{\frac{1}{1250} \ln\left(\frac{4}{5}\right) t} \]

and we know from the [Solve Logarithm Equations](http://tutorial.math.lamar.edu/terms.aspx) section how to do that. Here is the solution work for this part.

\[
\frac{50}{100} = e^{\frac{1}{1250} \ln\left(\frac{4}{5}\right) t}
\]

\[
\ln\left(\frac{1}{2}\right) = \frac{1}{1250} \ln\left(\frac{4}{5}\right) t
\]

\[
t = \frac{1250 \ln\left(\frac{1}{2}\right)}{\ln\left(\frac{4}{5}\right)} = 3882.8546
\]

It will take 3882.8546 years for half of the element to decay. On a side note this time is called the **half-life** of the element.

(c) **How long will it take until there is only 1 gram of the element left?**

In this part we’re being asked to solve the equation,

\[ 1 = Q(t) = 100e^{\frac{1}{1250} \ln\left(\frac{4}{5}\right) t} \]

The solution process for this part is the same as that for the previous part. Here is the solution work for this part.

\[
\frac{1}{100} = e^{\frac{1}{1250} \ln\left(\frac{4}{5}\right) t}
\]

\[
\ln\left(\frac{1}{100}\right) = \frac{1}{1250} \ln\left(\frac{4}{5}\right) t
\]

\[
t = \frac{1250 \ln\left(\frac{1}{100}\right)}{\ln\left(\frac{4}{5}\right)} = 25797.1279
\]

There will only be 1 gram of the element left after 25,797.1279 years.