Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
Systems of Equations

Linear Systems with Two Variables

1. Use the Method of Substitution to find the solution to the following system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
7 &- 11y = -11 \\
5x &+ 2y = -18
\end{align*}
\]

Step 1
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course…).

Okay, let’s get started on the solution to this system.

The Method of Substitution tells us that we first need to solve one of the equations for one of the variables. The equation we solve and the variable we solve for technically doesn’t matter as noted above. However, there is often one equation/variable combination that is “easier” than the others. In this case we can quickly solve the first equation for \( x \) without a lot of extra work so let’s do that.

\[
x - 7y = -11 \quad \Rightarrow \quad x = 7y - 11
\]

Step 2
We now take the equation for \( x \) we found above and substitute this into the other equation (the second equation in this case). Doing this gives,

\[
5x + 2y = -18 \\
5(7y - 11) + 2y = -18
\]

Step 3
We can now solve the equation we found in the previous step for \( y \). Doing this gives,

\[
5(7y - 11) + 2y = -18 \\
35y - 55 + 2y = -18 \\
37y = 37 \quad \Rightarrow \quad y = 1
\]

Step 4

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Finally, we can plug the value of \( y \) we found in the previous step into the equation for \( x \) we found in the first step. This gives,

\[
x = 7(1) - 11 = -4
\]

The solution to the system is then: \( x = -4, \ y = 1 \).

2. Use the Method of Substitution to find the solution to the following system or to determine if the system is inconsistent or dependent.

\[
7x - 8y = -12 \\
-4x + 2y = 3
\]

Step 1
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course…).

Okay, let’s get started on the solution to this system.

The Method of Substitution tells us that we first need to solve one of the equations for one of the variables. The equation we solve and the variable we solve for technically doesn’t matter as noted above.

However, there is often one equation/variable combination that is “easier” than the others. In this case we can solve the second equation for \( y \) without a lot of extra work so let’s do that.

\[
-4x + 2y = 3 \\
2y = 4x + 3 \\
\Rightarrow y = 2x + \frac{3}{2}
\]

Note that you will often get fractions showing up at this step and there isn’t going to be a whole lot that you can do about it so don’t worry when they show up!

Step 2
We now take the equation for \( y \) we found above and substitute this into the other equation (the first equation in this case). Doing this gives,

\[
7x - 8y = -12 \\
7x - 8\left(2x + \frac{3}{2}\right) = -12
\]

Step 3
We can now solve the equation we found in the previous step for \( x \). Doing this gives,
College Algebra

\[ 7x - 8\left(2x + \frac{3}{2}\right) = -12 \]
\[ 7x - 16x - 12 = -12 \]
\[ -8x = 0 \quad \rightarrow \quad x = 0 \]

Do not get excited about the zero here! They will be answers occasionally.

Step 4
Finally, we can plug the value of \( x \) we found in the previous step into the equation for \( y \) we found in the first step. This gives,

\[ y = 2(0) + \frac{3}{2} = \frac{3}{2} \]

The solution to the system is then: \( x = 0, \quad y = \frac{3}{2} \).

3. Use the Method of Substitution to find the solution to the following system or to determine if the system is inconsistent or dependent.

\[ 3x + 9y = -6 \]
\[ -4x - 12y = 8 \]

Step 1
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course…).

Okay, let’s get started on the solution to this system.

The Method of Substitution tells us that we first need to solve one of the equations for one of the variables. The equation we solve and the variable we solve for technically doesn’t matter as noted above.

In this case both equations seem equally “easy” to deal with and so let’s solve the second equation for \( x \) since that is a combination we didn’t use in the first couple of problems.

\[ -4x - 12y = 8 \]
\[ -4x = 12y + 8 \quad \Rightarrow \quad x = -3y - 2 \]

Step 2
We now take the equation for \( x \) we found above and substitute this into the other equation (the first equation in this case). Doing this gives,
Step 3
We can now solve the equation we found in the previous step for $y$. Doing this gives,

$$3(-3y - 2) + 9y = -6$$

$$-9y - 6 + 9y = -6$$

$$-6 = -6$$

Step 4
Now, the result from the previous step is true for any value of $y$ or $x$ and so we know that the system is **dependent** and there will be an infinite number of solutions to the system. We can write the “solution” to this system as follows,

$$x = -3t - 2$$

$$y = t$$

$t$ is any number

4. Use the Method of Elimination to find the solution to the following system or to determine if the system is inconsistent or dependent.

$$6x - 5y = 8$$

$$-12x + 2y = 0$$

**Step 1**
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course…).

Okay, let’s get started on the solution to this system.

The Method of Elimination tells us that we first need to multiply one or both of the equations by constants so that one of the variables has the same coefficient but with opposite signs and then add the two equations.

For this system if we multiply the first equation by 2 then the first equation will have an $x$ coefficient of 12 while the second equation will have an $x$ coefficient of -12. This is exactly what we need so we’ll do that and then add the resulting equations.

$$6x - 5y = 8 \quad \times 2 \quad 12x - 10y = 16$$

$$-12x + 2y = 0 \quad \text{same} \quad -12x + 2y = 0$$

$$-8y = 16$$
Step 2
We can now easily solve the result from the above step to see that \( y = -2 \).

Step 3
Finally we can plug the value of \( y \) we found in the previous step in either of the original equations and solve for \( x \). We'll use the first equation for this.

\[
6x - 5(-2) = 8 \\
6x + 10 = 8 \\
6x = -2 \\
\rightarrow \\
x = -\frac{1}{3}
\]

The solution to the system is then: \( x = -\frac{1}{3}, y = -2 \).

5. Use the Method of Elimination to find the solution to the following system or to determine if the system is inconsistent or dependent.

\[
-2x + 10y = 2 \\
5x - 25y = 3
\]

Step 1
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let’s get started on the solution to this system.

The Method of Elimination tells us that we first need to multiply one or both of the equations by constants so that one of the variables has the same coefficient but with opposite signs and then add the two equations.

For this system if we multiply the first equation by 5 and the second equation by 2 then the first equation will have an \( x \) coefficient of -10 while the second equation will have an \( x \) coefficient of 10. This is exactly what we need so we'll do that and then add the resulting equations.

\[
\begin{align*}
-2x + 10y &= 2 \\
5x - 25y &= 3 \\
\times 5 &\quad \times 2 \\
-10x + 50y &= 10 \\
10x - 50y &= 6 \\
\rightarrow \\
0 &= 16
\end{align*}
\]

Step 2
The result above is clearly not true and so this system is inconsistent and has no solution.
6. Use the Method of Elimination to find the solution to the following system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
2x + 3y &= 20 \\
7x + 2y &= 53
\end{align*}
\]

**Step 1**
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course…).

Okay, let’s get started on the solution to this system.

The Method of Elimination tells us that we first need to multiply one or both of the equations by constants so that one of the variables has the same coefficient but with opposite signs and then add the two equations.

For this system if we multiply the first equation by 2 and the second equation by -3 then the first equation will have a \( y \) coefficient of 6 while the second equation will have a \( y \) coefficient of -6. This is exactly what we need so we’ll do that and then add the resulting equations.

\[
\begin{align*}
2x + 3y &= 20 \\
7x + 2y &= 53
\end{align*}
\]
\[
\begin{align*}
\times 2 \quad & 4x + 6y = 40 \\
\times -3 \quad & -21x - 6y = -159
\end{align*}
\]
\[
\begin{align*}
x + y &= 7
\end{align*}
\]

**Step 2**
We can now easily solve the result from the above step to see that \( x = 7 \).

**Step 3**
Finally we can plug the value of \( y \) we found in the previous step in either of the original equations and solve for \( y \). We’ll use the first equation for this.

\[
\begin{align*}
7(7) + 2y &= 53 \\
49 + 2y &= 53 \\
2y &= 4 \\
\rightarrow & \quad y = 2
\end{align*}
\]

The solution to the system is then: \( x = 7, \ y = 2 \).
**Linear Systems with Three Variables**

1. Find the solution to the following system of equations.

\[
\begin{align*}
2x + 5y + 2z &= -38 \\
3x - 2y + 4z &= 17 \\
-6x + y - 7z &= -12
\end{align*}
\]

Step 1
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course…).

Okay, let’s get started on the solution to this system.

For this system it looks like if we multiply the first equation by 3 and the second equation by 2 both of these equations will have \(x\) coefficients of 6 which we can then eliminate if we add the third equation to each of them.

So, let’s first do the multiplication.

\[
\begin{align*}
2x + 5y + 2z &= -38 \\
3x - 2y + 4z &= 17 \\
-6x + y - 7z &= -12
\end{align*}
\]

\[
\begin{align*}
2x + 5y + 2z &= -38 \\
3x - 2y + 4z &= 17 \\
-6x + y - 7z &= -12
\end{align*}
\]

\[
\begin{align*}
6x + 15y + 6z &= -114 \\
6x - 4y + 8z &= 34 \\
-6x + y - 7z &= -12
\end{align*}
\]

Step 2
Okay, we’ll now replace the first equation with the sum of the first and third equation and we’ll replace the second equation with the sum of the second and third equation. Here is the result from doing those operations.

\[
\begin{align*}
16y - z &= -126 \\
-3y + z &= 22 \\
-6x + y - 7z &= -12
\end{align*}
\]

Step 3
Next notice that we can eliminate \(z\) from the first equation simply by replacing it with the sum of the first and second equation. Here is the result from that operation.

\[
\begin{align*}
13y &= -104 \\
-3y + z &= 22 \\
-6x + y - 7z &= -12
\end{align*}
\]
Step 4
Okay, form the first equation we can see that we must have \( y = -8 \).

Step 5
We can plug \( y = -8 \) into the second equation and solve that for \( z \).

\[
-3(-8) + z = 22 \quad \rightarrow \quad z = -2
\]

Step 6
Finally, plug \( y = -8 \) and \( z = -2 \) into the third equation and solve for \( x \).

\[
-6x + (-8) - 7(-2) = -12
\]
\[
-6x + 6 = -12 \quad \rightarrow \quad x = 3
\]

The solution to the system is then: \( x = 3, y = -8, z = -2 \).

2. Find the solution to the following system of equations.

\[
\begin{align*}
3x - 9z &= 33 \\
7x - 4y - z &= -15 \\
4x + 6y + 5z &= -6
\end{align*}
\]

Step 1
Before we get started with the solution process for this system we need to make it clear that there is no “one correct solution path”. There are lots of solution paths that we can take to find the solution to this system. All are correct and all will end up with the same solution to the system (provided the work has been done correctly of course...).

Okay, let’s get started on the solution to this system.

For this system it looks like we can easily solve the first equation for \( x \) and get an equation involving only \( z \) which we can in turn plug in the second and third equation.

Here is the first equation solved for \( x \).

\[
\begin{align*}
3x - 9z &= 33 \\
3x &= 9z + 33 \\
&\quad \rightarrow \quad x = 3z + 11
\end{align*}
\]

Step 2
Plugging the equation we found above into the second and third equations and doing some simplification gives,
\[
\begin{align*}
7(3z + 11) - 4y - z &= -15 & \Rightarrow & \quad -4y + 20z = -92 \\
4(3z + 11) + 6y + 5z &= -6 & \Rightarrow & \quad 6y + 17z = -50
\end{align*}
\]

Step 3
Now, notice that if we multiply the first equation above by 3 and the second equation above by 2 we can cancel the \( y \)'s when we add the results. Here is that work.

\[
\begin{align*}
-4y + 20z &= -92 & \times 3 & \Rightarrow & \quad -12y + 60z &= -276 \\
6y + 17z &= -50 & \times 2 & \Rightarrow & \quad 12y + 34z &= -100
\end{align*}
\]

\[94z = -376\]

Step 4
From the equation above we can see that we must have \( z = -4 \).

Step 5
We can plug \( z = -4 \) into either of the equations we got in Step 2 and solve for \( y \). We’ll use the second equation for this propose.

\[6y + 17(-4) = -50 \quad \Rightarrow \quad y = 3\]

Step 6
Finally, plug \( z = -4 \) we got in Step 1 to determine the value of \( x \).

\[x = 3(-4) + 11 = -1\]

The solution to the system is then: \( x = -1, y = 3, z = -4 \).

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**Augmented Matrices**

1. For the following augmented matrix perform the indicated elementary row operations.

\[
\begin{bmatrix}
4 & -1 & 3 & | & 5 \\
0 & 2 & 5 & | & 9 \\
-6 & 1 & -3 & | & 10
\end{bmatrix}
\]

(a) \( 8R_1 \) \quad (b) \( R_2 \leftrightarrow R_3 \) \quad (c) \( R_2 + 3R_1 \rightarrow R_2 \)
(a) $8R_1$
This operation is telling us to multiply all the entries in Row 1 of the augmented matrix by 8 so let’s do that.

$$
\begin{bmatrix}
4 & -1 & 3 & 5 \\
0 & 2 & 5 & 9 \\
-6 & 1 & -3 & 10 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
32 & -8 & 24 & 40 \\
0 & 2 & 5 & 9 \\
-6 & 1 & -3 & 10 \\
\end{bmatrix}
$$

(b) $R_2 \leftrightarrow R_3$
This operation is telling us to interchange Row 2 and Row 3 of the augmented matrix. Here is that work.

$$
\begin{bmatrix}
4 & -1 & 3 & 5 \\
0 & 2 & 5 & 9 \\
-6 & 1 & -3 & 10 \\
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
4 & -1 & 3 & 5 \\
-6 & 1 & -3 & 10 \\
0 & 2 & 5 & 9 \\
\end{bmatrix}
$$

(c) $R_2 + 3R_1 \rightarrow R_2$
For this operation we are going to replace Row 2 with the results of taking the original entries from Row 2 and add to them 3 times the entries in Row 1.

$$
\begin{bmatrix}
4 & -1 & 3 & 5 \\
0 & 2 & 5 & 9 \\
-6 & 1 & -3 & 10 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & -1 & 3 & 5 \\
12 & -1 & 14 & 24 \\
-6 & 1 & -3 & 10 \\
\end{bmatrix}
$$

Here are the individual computations for this operation.

- Column 1 : $0 + 3(4) = 12$
- Column 2 : $2 + 3(-1) = -1$
- Column 3 : $5 + 3(3) = 14$
- Column 4 : $9 + 3(5) = 24$

2. For the following augmented matrix perform the indicated elementary row operations.

$$
\begin{bmatrix}
1 & -6 & 2 & 0 \\
2 & -8 & 10 & 4 \\
3 & -4 & -1 & 2 \\
\end{bmatrix}
$$

(a) $\frac{1}{2}R_2$  (b) $R_1 \leftrightarrow R_3$  (c) $R_1 - 6R_3 \rightarrow R_1$
(a) \( \frac{1}{2} R_2 \)

This operation is telling us to multiply all the entries in Row 2 of the augmented matrix by \( \frac{1}{2} \) so let’s do that.

\[
\begin{bmatrix}
1 & -6 & 2 & 0 \\
2 & -8 & 10 & 4 \\
3 & -4 & -1 & 2
\end{bmatrix}
\overset{\frac{1}{2} R_2}{\rightarrow}
\begin{bmatrix}
1 & -6 & 2 & 0 \\
1 & -4 & 5 & 2 \\
3 & -4 & -1 & 2
\end{bmatrix}
\]

(b) \( R_1 \leftrightarrow R_3 \)

This operation is telling us to interchange Row 1 and Row 3 of the augmented matrix. Here is that work.

\[
\begin{bmatrix}
1 & -6 & 2 & 0 \\
2 & -8 & 10 & 4 \\
3 & -4 & -1 & 2
\end{bmatrix}
\overset{R_1 \leftrightarrow R_3}{\rightarrow}
\begin{bmatrix}
3 & -4 & -1 & 2 \\
2 & -8 & 10 & 4 \\
1 & -6 & 2 & 0
\end{bmatrix}
\]

(c) \( R_1 - 6R_3 \rightarrow R_1 \)

For this operation we are going to replace Row 1 with the results of taking the original entries from Row 1 and subtract from them 6 times the entries in Row 3.

\[
\begin{bmatrix}
1 & -6 & 2 & 0 \\
2 & -8 & 10 & 4 \\
3 & -4 & -1 & 2
\end{bmatrix}
\overset{R_1 - 6R_3 \rightarrow R_1}{\rightarrow}
\begin{bmatrix}
-17 & 18 & 8 & -12 \\
2 & -8 & 10 & 4 \\
3 & -4 & -1 & 2
\end{bmatrix}
\]

Here are the individual computations for this operation.

Column 1 : \( 1 - 6(3) = -17 \)

Column 2 : \( -6 - 6(-4) = 18 \)

Column 3 : \( 2 - 6(-1) = 8 \)

Column 4 : \( 0 - 6(2) = -12 \)

3. For the following augmented matrix perform the indicated elementary row operations.

\[
\begin{bmatrix}
10 & -1 & -5 & 1 \\
4 & 0 & 7 & -1 \\
0 & 7 & -2 & 3
\end{bmatrix}
\]
(a) $-9R_3$

This operation is telling us to multiply all the entries in Row 3 of the augmented matrix by -9 so let’s do that.

\[
\begin{bmatrix}
10 & -1 & -5 & 1 \\
4 & 0 & 7 & -1 \\
0 & 7 & -2 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
10 & -1 & -5 & 1 \\
4 & 0 & 7 & -1 \\
0 & -63 & 18 & -27
\end{bmatrix}
\]

(b) $R_1 \leftrightarrow R_2$

This operation is telling us to interchange Row 1 and Row 2 of the augmented matrix. Here is that work.

\[
\begin{bmatrix}
10 & -1 & -5 & 1 \\
4 & 0 & 7 & -1 \\
0 & 7 & -2 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & 0 & 7 & -1 \\
10 & -1 & -5 & 1 \\
0 & 7 & -2 & 3
\end{bmatrix}
\]

(c) $R_3 - R_1 \rightarrow R_3$

For this operation we are going to replace Row 3 with the results of taking the original entries from Row 3 and subtract from them the entries in Row 1.

\[
\begin{bmatrix}
10 & -1 & -5 & 1 \\
4 & 0 & 7 & -1 \\
0 & 7 & -2 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
10 & -1 & -5 & 1 \\
4 & 0 & 7 & -1 \\
-10 & 8 & 3 & 2
\end{bmatrix}
\]

Here are the individual computations for this operation.

\[
\begin{align*}
\text{Column 1} & : 0 - (10) = -10 \\
\text{Column 2} & : 7 - (-1) = 8 \\
\text{Column 3} & : -2 - (-5) = 3 \\
\text{Column 4} & : 3 - (1) = 2
\end{align*}
\]

More on the Augmented Matrix
1. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
x - 7y &= -11 \\
5x + 2y &= -18
\end{align*}
\]

Step 1  
The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
1 & -7 & | & -11 \\
5 & 2 & | & -18
\end{bmatrix}
\]

Step 2  
We need to make the number in the upper left corner a one. In this case it already is and so there really isn’t anything to do in this step for this particular problem.

Step 3  
Next, we need to convert the 5 below the 1 into a zero and we can do that with the following elementary row operation.

\[
\begin{bmatrix}
1 & -7 & | & -11 \\
5 & 2 & | & -18
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -7 & | & -11 \\
0 & 37 & | & 37
\end{bmatrix}
\]

Step 4  
The next step is to turn the number at the bottom of the second column (37 in this case) into a one. The following elementary row operation will do that for us.

\[
\begin{bmatrix}
1 & -7 & | & -11 \\
0 & 37 & | & 37
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & | & -4 \\
0 & 1 & | & 1
\end{bmatrix}
\]

Step 5  
Finally we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

\[
\begin{bmatrix}
1 & -7 & | & -11 \\
0 & 1 & | & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & | & -4 \\
0 & 1 & | & 1
\end{bmatrix}
\]

Step 6  
From the final augmented matrix we found in Step 5 we get the solution to the system is: \[x = -4, \ y = 1\].

2. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
7x - 8y &= -12 \\
-4x + 2y &= 3
\end{align*}
\]

Step 1
The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
7 & -8 & -12 \\
-4 & 2 & 3
\end{bmatrix}
\]

Step 2
We need to make the number in the upper left corner a one. There are several ways to do this. One way would be to use the elementary row operation \( \frac{1}{7} R_1 \). However, this would put fractions into the other two entries in the first row and it might be nice to avoid them.

So, instead let’s do the following elementary row operation.

\[
\begin{bmatrix}
7 & -8 & -12 \\
-4 & 2 & 3
\end{bmatrix}
→
\begin{bmatrix}
1 & -4 & -6 \\
-4 & 2 & 3
\end{bmatrix}
\]

Now, this isn’t quite what we want since the number in the upper left is a minus one and not a positive one. However, we can easily fix that by multiplying the first row by -1.

\[
\begin{bmatrix}
-1 & -4 & -6 \\
-4 & 2 & 3
\end{bmatrix}
→
\begin{bmatrix}
1 & 4 & 6 \\
-4 & 2 & 3
\end{bmatrix}
\]

Note that as this step has shown there are several different paths to do these problems. Some will result in “messier” intermediate steps, but the solution we get in the end will be the same regardless of the path we chose to follow in the solution process.

Step 3
Next, we need to convert the -4 below the 1 into a zero and we can do that with the following elementary row operation.

\[
\begin{bmatrix}
1 & 4 & 6 \\
-4 & 2 & 3
\end{bmatrix}
→
\begin{bmatrix}
1 & 4 & 6 \\
0 & 18 & 27
\end{bmatrix}
\]

Step 4
The next step is to turn the number at the bottom of the second column (18 in this case) into a one. The following elementary row operation will do that for us.
In the first step we chose to avoid the step that put fractions into the augmented matrix, but sometimes, as in this step, they can’t be avoided.

Step 5
Finally we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

\[
\begin{bmatrix}
1 & 4 & 6 \\
0 & 1 & \frac{27}{18}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 4 & 6 \\
0 & 1 & \frac{3}{2}
\end{bmatrix}
\]

Step 6
From the final augmented matrix we found in Step 5 we get the solution to the system is: \(x = 0, y = \frac{3}{2}\).

3. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[
3x + 9y = -6
\]
\[
-4x -12y = 8
\]

Step 1
The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
3 & 9 & -6 \\
-4 & -12 & 8
\end{bmatrix}
\]

Step 2
We need to make the number in the upper left corner a one. In this case we can quickly do that by dividing the top row by 3.

\[
\begin{bmatrix}
3 & 9 & -6 \\
-4 & -12 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & -2 \\
0 & 1 & 2
\end{bmatrix}
\]

Step 3
Next, we need to convert the -4 below the 1 into a zero and we can do that with the following elementary row operation.

\[
\begin{bmatrix}
1 & 3 & -2 \\
-4 & -12 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & -2 \\
0 & 0 & 0
\end{bmatrix}
\]
Step 4

The minute we see the bottom row of all zeroes we know that the system is dependent. We can convert the top row into an equation and solve for \( x \) as follows,

\[
x + 3y = -2 \quad \rightarrow \quad x = -3y - 2
\]

From this we can write the solution as,

\[
x = -3t - 2 \quad \quad y = t \quad \quad t \text{ is any number}
\]

4. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
6x - 5y & = 8 \\
-12x + 2y & = 0
\end{align*}
\]

Step 1

The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
6 & -5 & 8 \\
-12 & 2 & 0
\end{bmatrix}
\]

Step 2

We need to make the number in the upper left corner a one. There are several ways to do this. One way would be to use the elementary row operation \( \frac{1}{6} R_1 \). However, this would put fractions into the other two entries in the first row.

We’re not going to be able to avoid fractions after this step and the above idea would do what we need but it would lead to two fractions. Note however that if we interchange the two rows we get,

\[
\begin{bmatrix}
6 & -5 & 8 \\
-12 & 2 & 0
\end{bmatrix}
\]

We could now do the elementary row operation \( -\frac{1}{12} R_1 \) and we’ll only end up with one fraction in the first row instead of two so let’s do that.

\[
\begin{bmatrix}
-12 & 2 & 0 \\
6 & -5 & 8
\end{bmatrix}
\]
Note that as this step has shown there are several different paths to do these problems. Some will result in “messier” intermediate steps, but the solution we get in the end will be the same regardless of the path we chose to follow in the solution process.

Step 3
Next, we need to convert the 6 below the 1 into a zero and we can do that with the following elementary row operation.

\[
\begin{bmatrix}
1 & -\frac{1}{6} & 0 \\
6 & -5 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{1}{6} & 0 \\
0 & -4 & 8
\end{bmatrix}
\]

Step 4
The next step is to turn the number at the bottom of the second column (-4 in this case) into a one. The following elementary row operation will do that for us.

\[
\begin{bmatrix}
1 & -\frac{1}{6} & 0 \\
0 & -4 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{1}{6} & 0 \\
0 & 1 & -2
\end{bmatrix}
\]

Step 5
Finally we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

\[
\begin{bmatrix}
1 & -\frac{1}{6} & 0 \\
0 & 1 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -\frac{1}{3} \\
0 & 1 & -2
\end{bmatrix}
\]

Step 6
From the final augmented matrix we found in Step 5 we get the solution to the system is:

\[x = -\frac{1}{3}, \ y = -2\]

5. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
5x - 25y &= 3 \\
-2x + 10y &= 2
\end{align*}
\]

Step 1
The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
5 & -25 & | & 3 \\
-2 & 10 & | & 2
\end{bmatrix}
\]

Step 2
We need to make the number in the upper left corner a one. In this case we can do this with the following elementary row operation.

\[
\begin{bmatrix}
5 & -25 & 3 \\
-2 & 10 & 2
\end{bmatrix} \rightarrow \begin{bmatrix}
0 & 0 & 8 \\
-2 & 10 & 2
\end{bmatrix}
\]

**Step 3**
Okay let’s step back for a second and convert the first row back to an equation. Doing this gives,

\[0 = 8\]

That is clearly not true and we’ve done all our work correctly and so this system is **inconsistent** and there is **no solution** to the system.

---

6. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[\begin{align*}
2x + 3y &= 20 \\
7x + 2y &= 53
\end{align*}\]

**Step 1**
The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
2 & 3 & | & 20 \\
7 & 2 & | & 53
\end{bmatrix}
\]

**Step 2**
We need to make the number in the upper left corner a one. There are several ways to do this. One way would be to use the elementary row operation \(\frac{1}{2}R_1\). However, this would put fractions into the other two entries in the first row and it might be nice to avoid them.

While this may seem to not be of any use let’s take a look at the following elementary row operation.

\[
\begin{bmatrix}
2 & 3 & | & 20 \\
7 & 2 & | & 53
\end{bmatrix} \rightarrow \begin{bmatrix}
2 & 3 & | & 20 \\
1 & -7 & | & -7
\end{bmatrix}
\]

This operation worked on the second row instead of the first row that we need to work on. Note however, that we did put a 1 in the lower number of the first column. We need a 1 in the upper number of the first column and we can do that now simply by switching rows as follows,

\[
\begin{bmatrix}
2 & 3 & | & 20 \\
1 & -7 & | & -7
\end{bmatrix} \leftrightarrow \begin{bmatrix}
1 & -7 & | & -7 \\
2 & 3 & | & 20
\end{bmatrix}
\]
Note that as this step has shown there are several different paths to do these problems. Some will result in “messier” intermediate steps, but the solution we get in the end will be the same regardless of the path we chose to follow in the solution process.

Step 3
Next, we need to convert the 2 below the 1 into a zero and we can do that with the following elementary row operation.

\[
\begin{bmatrix}
1 & -7 & | & -7 \\
2 & 3 & | & 20
\end{bmatrix}
\xrightarrow{\text{}}
\begin{bmatrix}
1 & -7 & | & -7 \\
0 & 17 & | & 34
\end{bmatrix}
\]

Step 4
The next step is to turn the number at the bottom of the second column (17 in this case) into a one. The following elementary row operation will do that for us.

\[
\begin{bmatrix}
1 & -7 & | & -7 \\
0 & 17 & | & 34
\end{bmatrix}
\xrightarrow{\frac{1}{17}R_2}
\begin{bmatrix}
1 & -7 & | & -7 \\
0 & 1 & | & 2
\end{bmatrix}
\]

Step 5
Finally we need to convert the number above the one we got in Step 4 into a zero. To do that we can use the following elementary row operation.

\[
\begin{bmatrix}
1 & -7 & | & -7 \\
0 & 1 & | & 2
\end{bmatrix}
\xrightarrow{\text{}}
\begin{bmatrix}
1 & 0 & | & 7 \\
0 & 1 & | & 2
\end{bmatrix}
\]

Step 6
From the final augmented matrix we found in Step 5 we get the solution to the system is: \[x = 7, \ y = 2\].

7. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
2x + 5y + 2z &= -38 \\
3x - 2y + 4z &= 17 \\
-6x + y - 7z &= -12
\end{align*}
\]

Step 1
The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
2 & 5 & 2 & | & -38 \\
3 & -2 & 4 & | & 17 \\
-6 & 1 & -7 & | & -12
\end{bmatrix}
\]
Step 2
We need to make the number in the upper left corner a one. Much like with the previous problems (i.e. solving systems with two variables) we can quickly do it with the elementary row operation \( \frac{1}{2} R_1 \) but that will put fractions into the augmented matrix and they would probably be around for quite a few steps and it would be really nice to avoid them for as long as possible when the augmented matrix starts getting this size.

So, let’s start with the following elementary row operation.

\[
\begin{bmatrix}
2 & 5 & 2 & | & -38 \\
3 & -2 & 4 & | & 17 \\
-6 & 1 & -7 & | & -12
\end{bmatrix}
\xrightarrow{R_1 - R_2 \rightarrow R_1}
\begin{bmatrix}
-1 & 7 & -2 & | & -55 \\
3 & -2 & 4 & | & 17 \\
-6 & 1 & -7 & | & -12
\end{bmatrix}
\]

With this operation we got a negative one in the spot where we needed a plus one, but we can easily fix that with the next elementary row operation.

\[
\begin{bmatrix}
-1 & 7 & -2 & | & -55 \\
3 & -2 & 4 & | & 17 \\
-6 & 1 & -7 & | & -12
\end{bmatrix}
\xrightarrow{-R_1}
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
3 & -2 & 4 & | & 17 \\
-6 & 1 & -7 & | & -12
\end{bmatrix}
\]

Now, a quick note before we really jump into the rest of this problem. Using augmented matrices to solve systems with three variables can be a very tedious process and there are a great number of possible paths to take in the solution process so your solution may well vary from this solution depending on the path you took. The final answers however will the same regardless of the path we take provided we did all the arithmetic correctly.

Step 3
Next, we need to convert the 3 and the -6 below the 1 in the first column into zeroes and we can do that with the following elementary row operations.

\[
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
3 & -2 & 4 & | & 17 \\
-6 & 1 & -7 & | & -12
\end{bmatrix}
\xrightarrow{R_2 - 3R_1 \rightarrow R_2}
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 19 & -2 & | & -148 \\
-6 & 1 & -7 & | & -12
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 19 & -2 & | & -148 \\
0 & -41 & 5 & | & 318
\end{bmatrix}
\xrightarrow{R_3 + 6R_1 \rightarrow R_3}
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 1 & -\frac{2}{19} & | & -\frac{148}{19} \\
0 & -41 & 5 & | & 318
\end{bmatrix}
\]

Step 4
We now need to turn the 19 in the second row into a one and it seems like the only easy way to do that is the following elementary row operation.

\[
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 19 & -2 & | & -148 \\
0 & -41 & 5 & | & 318
\end{bmatrix}
\xrightarrow{\frac{1}{19} R_2}
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 1 & -\frac{2}{19} & | & -\frac{148}{19} \\
0 & -41 & 5 & | & 318
\end{bmatrix}
\]
In the first step we chose to avoid the step that put fractions into the augmented matrix, but sometimes, as in this step, they can’t be avoided. With augmented matrices for systems with three variables fractions will almost inevitably show up and they will often be “messy” when they do.

This is just something we’ll need to deal with when solving these systems. We try to avoid them for as long as possible but except it when they show up and continue with the solution process.

Step 5
Next we need to turn the \(-41\) in the third row into a zero. The following elementary row operation will do that for us.

\[
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 1 & -\frac{2}{19} & | & -\frac{148}{19} \\
0 & -41 & 5 & | & 318
\end{bmatrix} \quad R_3 + 41R_2 \rightarrow R_3 \quad \Rightarrow \quad \begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 1 & -\frac{2}{19} & | & -\frac{148}{19} \\
0 & 0 & \frac{13}{19} & | & -\frac{26}{19}
\end{bmatrix}
\]

Again, we had to put more fraction into the augmented matrix. This is just a fact of life with these types of problems. However, as we’ll see in the next step they do often disappear as well.

Step 6
Okay, we need to turn the \(\frac{13}{19}\) in the third row into a one and we can do that as follows,

\[
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 1 & -\frac{2}{19} & | & -\frac{148}{19} \\
0 & 0 & \frac{13}{19} & | & -\frac{26}{19}
\end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 1 & -\frac{2}{19} & | & -\frac{148}{19} \\
0 & 0 & 1 & | & -2
\end{bmatrix}
\]

Step 7
Next we need to turn the \(-\frac{2}{19}\) and the 2 in the third column into zeroes. The following elementary row operations will do that for us.

\[
\begin{bmatrix}
1 & -7 & 2 & | & 55 \\
0 & 1 & -\frac{2}{19} & | & -\frac{148}{19} \\
0 & 0 & 1 & | & -2
\end{bmatrix} \quad R_1 - 2R_3 \rightarrow R_1 \quad \Rightarrow \quad \begin{bmatrix}
1 & -7 & 0 & | & 59 \\
0 & 1 & 0 & | & -8 \\
0 & 0 & 1 & | & -2
\end{bmatrix}
\]

Note that the fractions are now completely gone! This won’t always happen but it also will happen fairly regularly that fractions get introduced in intermediate steps and then go away in later steps.

Step 8
For the final operation we need to turn the -7 in the second column into a zero and we can do that as follows,

\[
\begin{bmatrix}
1 & -7 & 0 & | & 59 \\
0 & 1 & 0 & | & -8 \\
0 & 0 & 1 & | & -2
\end{bmatrix} \quad R_1 + 7R_2 \rightarrow R_1 \quad \Rightarrow \quad \begin{bmatrix}
1 & 0 & 0 & | & 3 \\
0 & 1 & 0 & | & -8 \\
0 & 0 & 1 & | & -2
\end{bmatrix}
\]
8. For the following system of equations convert the system into an augmented matrix and use the augmented matrix techniques to determine the solution to the system or to determine if the system is inconsistent or dependent.

\[
\begin{align*}
3x - 9z &= 33 \\
7x - 4y - z &= -15 \\
4x + 6y + 5z &= -6
\end{align*}
\]

Step 1
The first step is to write down the augmented matrix for the system of equations.

\[
\begin{bmatrix}
3 & 0 & -9 & \mid & 33 \\
7 & -4 & -1 & \mid & -15 \\
4 & 6 & 5 & \mid & -6
\end{bmatrix}
\]

Note the zero in the second column of the first row. Recall that the second column corresponds to the coefficients of the \(y\)'s in each equation and because there is no \(y\) in the first equation that coefficient must be zero.

Step 2
We need to make the number in the upper left corner a one. We can easily do that with the following elementary row operation.

\[
\begin{bmatrix}
3 & 0 & -9 & \mid & 33 \\
7 & -4 & -1 & \mid & -15 \\
4 & 6 & 5 & \mid & -6
\end{bmatrix}
\xrightarrow{\frac{1}{3}R_1}
\begin{bmatrix}
1 & 0 & -3 & \mid & 11 \\
7 & -4 & -1 & \mid & -15 \\
4 & 6 & 5 & \mid & -6
\end{bmatrix}
\]

Step 3
Next, we need to convert the 7 and the 4 below the 1 in the first column into zeroes and we can do that with the following elementary row operations.

\[
\begin{bmatrix}
1 & 0 & -3 & \mid & 11 \\
7 & -4 & -1 & \mid & -15 \\
4 & 6 & 5 & \mid & -6
\end{bmatrix}
\xrightarrow{R_2 - 7R_1 \rightarrow R_2}
\begin{bmatrix}
1 & 0 & -3 & \mid & 11 \\
0 & -4 & 20 & \mid & -59 \\
4 & 6 & 5 & \mid & -6
\end{bmatrix}
\xrightarrow{R_3 - 4R_1 \rightarrow R_3}
\begin{bmatrix}
1 & 0 & -3 & \mid & 11 \\
0 & -4 & 20 & \mid & -59 \\
0 & 6 & 17 & \mid & -50
\end{bmatrix}
\]

Step 4
We now need to turn the -4 in the second row into a one and that can be done with the following elementary row operation.
Step 5
Next we need to turn the 6 in the third row into a zero. The following elementary row operation will do that for us.

\[
\begin{bmatrix}
1 & 0 & -3 & & 11 \\
0 & 1 & -5 & & 23 \\
0 & 6 & 17 & & -50
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -3 & & 11 \\
0 & 1 & -5 & & 23 \\
0 & 0 & 47 & & -188
\end{bmatrix}
\]

Step 6
Okay, we need to turn the 47 in the third row into a one and we can do that as follows,

\[
\begin{bmatrix}
1 & 0 & -3 & & 11 \\
0 & 1 & -5 & & 23 \\
0 & 0 & 47 & & -188
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -3 & & 11 \\
0 & 1 & -5 & & 23 \\
0 & 0 & 1 & & -4
\end{bmatrix}
\]

Step 7
Next we need to turn the -5 and the -3 in the third column into zeroes. The following elementary row operations will do that for us.

\[
\begin{bmatrix}
1 & 0 & -3 & & 11 \\
0 & 1 & -5 & & 23 \\
0 & 0 & 1 & & -4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 0 & & -1 \\
0 & 1 & 0 & & 3 \\
0 & 0 & 1 & & -4
\end{bmatrix}
\]

Step 8
Normally we would have another step to do. We would need to turn the number in the first row and second column into a zero. However, in this case there is already a zero there and so there is no work to do in this step.

The final form of the augmented matrix is then,

\[
\begin{bmatrix}
1 & 0 & 0 & & -1 \\
0 & 1 & 0 & & 3 \\
0 & 0 & 1 & & -4
\end{bmatrix}
\]

As this step has shown we occasionally will get a number “for free”. In other words, the work we put into an intermediate step will give us not only the number we were looking for in that step but will also put in a number that we need in a later step. Or, as in this case, the number we needed was actually there from the start.
Step 9
From the final augmented matrix we found in Step 8 we get the solution to the system is:
\[ x = -1, \ y = 3, \ z = -4. \]

---

**Non-Linear Systems**

1. Find the solution to the following system of equation.

\[
y = x^2 + 6x - 8
\]

\[
y = 4x + 7
\]

Step 1
Before we get too far into the solution we first should mention that there is no one correct solution path to
these. Many of these types of problems will have multiple paths that we can take to find the solution.
However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case we can notice that both of the equations are in the form “\( y = \)”. This
means that we can “substitute” \( y \) from one of the equations into the other. In these kinds of problems this
is often called “setting the equations equal”.

So, setting the equations equal gives,

\[
x^2 + 6x - 8 = 4x + 7
\]

Step 2
Now, this is just a quadratic equation and by this point we should be able to solve that so here is the
solution work for the quadratic.

\[
x^2 + 6x - 8 = 4x + 7
\]

\[
x^2 + 2x - 15 = 0
\]

\[
(x - 3)(x + 5) = 0 \quad \rightarrow \quad x = -5, \ x = 3
\]

Step 3
We now have two values of \( x \) and so all we need to do is plug into either of the original equations (the
line would be easier) to determine the corresponding values of \( y \) for each \( x \).

\[
x = -5: \ y = 4(-5) + 7 = -13 \quad \Rightarrow \quad (-5, -13)
\]

\[
x = 3: \ y = 4(3) + 7 = 19 \quad \Rightarrow \quad (3, 19)
\]
So, for this system of equations we have two solutions: \((-5, -13)\) and \((3, 19)\).

2. Find the solution to the following system of equation.

\[
\begin{align*}
\frac{x^2}{4} + y^2 &= 1 \\
y &= 1 - 3x \\
\end{align*}
\]

Step 1
Before we get too far into the solution we first should mention that there is no one correct solution path to these. Many of these types of problems will have multiple paths that we can take to find the solution. However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case the first equation is in the form “\(y = \)” and so we can just plug this directly into the second equation. Doing this gives,

\[
\frac{x^2}{4} + (1 - 3x)^2 = 1
\]

Be careful with the parenthesis when plugging the first equation in. We had \(y^2\) and so we need to make sure and square the whole \(y\) form the first equation when we plugged that in. In other words, we need the parenthesis in there to make sure we deal with the exponent properly.

Step 2
Now, this is just a quadratic equation (which admittedly needs some simplification) and by this point we should be able to solve that so here is the solution work for the quadratic.

\[
\begin{align*}
\frac{x^2}{4} + (1 - 3x)^2 &= 1 \\
\frac{x^2}{4} + 1 - 6x + 9x^2 - 1 &= 0 \\
\frac{37}{4}x^2 - 6x &= 0 \\
x \left(\frac{37}{4}x - 6\right) &= 0 \\
x = 0, & \quad \frac{37}{4}x - 6 = 0
\end{align*}
\]

From this we see we have two values of \(x\) for our solution: \(x = 0\) and \(x = \frac{24}{37}\).

Step 3
We now have two values of \(x\) and so all we need to do is plug into either of the original equations (the line would be easier) to determine the corresponding values of \(y\) for each \(x\).
Note that with this system we have also run into a potential problem. We found corresponding \( y \)'s by plugging our \( x \)'s into the line. What if we had plugged them into the ellipse? Let's use \( x = 0 \) as an example since it will be a little easier to do the work for. Plugging this into the equation for the ellipse gives,

\[
\left( \frac{0}{4} \right)^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1
\]

This implies that there should be two solutions corresponding to \( x = 0 \) and not the single solution we found above! So, which is correct? Well recall that whatever solution we get must satisfy both equations and only one of those values of \( y \) will satisfy the line when using \( x = 0 \). Therefore the only solution is the one we got from the line. We would have a similar issue with the second value of \( x \).

The problem arose when we plugged the line into the ellipse and squared it. The process of squaring it introduced potentially “bad” solutions. We saw similar issues when we solved equations with radicals several chapters back and the problem arose there for the same reason, we squared something.

The nice thing about these problems however is that if we use the equation we plugged in (the line in this case) to find the second values we don’t need to worry about the “bad” solutions since they only arise from the equation that we plugged into (the ellipse in this case).

This in fact is the real reason we used the line to find the corresponding \( y \), although it was also the easier of the two equations to use.

So, for this system of equations we have two solutions: \((0,1)\) and \(\left(\frac{24}{37}, -\frac{35}{37}\right)\).

3. Find the solution to the following system of equations.

\[
\begin{align*}
xy &= 4 \\
\frac{x^2}{4} + \frac{y^2}{25} &= 1
\end{align*}
\]

Step 1
Before we get too far into the solution we first should mention that there is no one correct solution path to these. Many of these types of problems will have multiple paths that we can take to find the solution. However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case we can solve the first equation for either \( x \) or \( y \) and plug this into the second equation. For no reason other than we had equations in \( x \) for the first two practice problems for
this section we’ll solve the first equation for \( x \) and plug this into the second equation. The result will be an equation involving only \( y \)'s.

Here is that work.

\[
\begin{align*}
x &= \frac{4}{y} \\
\frac{16}{y^2} + \frac{y^2}{25} &= 1 \\
\frac{4}{y^2} + \frac{y^2}{25} &= 1
\end{align*}
\]

Step 2

Now, let’s multiply both sides of this by \( 25y^2 \) to clear denominators.

\[
\begin{align*}
100 + y^4 &= 25y^2 \\
y^4 - 25y^2 + 100 &= 0
\end{align*}
\]

Step 3

This is quadratic in form so we can define \( u = y^2 \) (and so \( u^2 = (y^2)^2 = y^4 \)). Using this substitution the equation becomes,

\[
\begin{align*}
u^2 - 25u + 100 &= 0 \\
(u - 5)(u - 20) &= 0 \\
\rightarrow \quad u &= 5, \quad u = 20
\end{align*}
\]

Step 4

So, we got two values of \( u \) and each of these correspond to the following equation in terms of \( y \) (i.e. using the substitution above).

\[
\begin{align*}
u = 5 : \quad y^2 &= 5 \quad \rightarrow \quad y = \pm\sqrt{5} \\
u = 20 : \quad y^2 &= 20 \quad \rightarrow \quad y = \pm\sqrt{20} = \pm2\sqrt{5}
\end{align*}
\]

Step 5

We have four values of \( y \) that we need to find corresponding values of \( x \) for. We’ll plug these into the first equation (much easier to plug these into that equation).
So, for this system of equations we have the four solutions listed above.

4. Find the solution to the following system of equation.

\[
\begin{align*}
y &= 1 - 2x^2 \\
x^2 - \frac{y^2}{9} &= 1
\end{align*}
\]

Step 1
Before we get too far into the solution we first should mention that there is no one correct solution path to these. Many of these types of problems will have multiple paths that we can take to find the solution. However, regardless of the path we take the solution to the system will be the same.

Okay on to the problem. In this case the first equation is in the form “\(y = \)” and so we can just plug this directly into the second equation. Doing this gives,

\[
x^2 - \left(1 - 2x^2\right)^2 = 1
\]

Be careful with the parenthesis when plugging the first equation in. We had \(y^2\) and so we need to make sure and square the whole \(y\) form the first equation when we plugged that in. In other words, we need the parenthesis in there to make sure we deal with the exponent properly.

Step 2
Now, this is just a quadratic equation (which admittedly needs some simplification) and by this point we should be able to solve that so here is the solution work for the quadratic.
In the last step we multiplied by -9 to clear out the denominators and to eliminate the minus sign on the $x^4$ term.

Step 3
This is quadratic in form so we can define $u = x^2$ (and so $u^2 = (x^2)^2 = x^4$). Using this substitution the equation becomes,

$$4u^2 - 13u + 10 = 0$$

$$(4u - 5)(u - 2) = 0 \quad \rightarrow \quad u = \frac{5}{4}, \quad u = 2$$

Step 4
So, we got two values of $u$ and each of these correspond to the following equation in terms of $y$ (i.e. using the substitution above).

$$u = \frac{5}{4} : \quad x^2 = \frac{5}{4} \quad \rightarrow \quad x = \pm\sqrt{\frac{5}{2}}$$
$$u = 2 : \quad x^2 = 2 \quad \rightarrow \quad x = \pm\sqrt{2}$$

Step 5
We have four values of $x$ that we need to find corresponding values of $y$ for. We’ll plug these into the first equation (much easier to plug these into that equation).

$$x = \frac{\sqrt{5}}{2} : \quad y = 1 - 2\left(\frac{\sqrt{5}}{2}\right)^2 = -\frac{3}{2} \quad \Rightarrow \quad \left(\frac{\sqrt{5}}{2}, -\frac{3}{2}\right)$$
$$x = -\frac{\sqrt{5}}{2} : \quad y = 1 - 2\left(-\frac{\sqrt{5}}{2}\right)^2 = -\frac{3}{2} \quad \Rightarrow \quad \left(-\frac{\sqrt{5}}{2}, -\frac{3}{2}\right)$$
$$x = \sqrt{2} : \quad y = 1 - 2\left(\sqrt{2}\right)^2 = -3 \quad \Rightarrow \quad (\sqrt{2}, -3)$$
$$x = -\sqrt{2} : \quad y = 1 - 2\left(-\sqrt{2}\right)^2 = -3 \quad \Rightarrow \quad (-\sqrt{2}, -3)$$

So, for this system of equations we have the four solutions listed above.
Note that with this system we have also run into a potential problem. We found corresponding $y$’s by plugging our $x$’s into the parabola. What if we had plugged them into the hyperbola?

Let’s use $x = \sqrt{2}$ as an example since it will be a little easier to do the work for. Plugging this into the equation for the hyperbola gives,

$$
\left(\sqrt{2}\right)^2 - \frac{y^2}{9} = 1 \quad \Rightarrow \quad 2 - \frac{y^2}{9} = 1 \quad \Rightarrow \quad y^2 = 9 \quad \Rightarrow \quad y = \pm 3
$$

This implies that there should be two solutions corresponding to $x = \sqrt{2}$ and not the single solution we found above! So, which is correct? Well recall that whatever solution we get must satisfy both equations and only one of those values of $y$ will satisfy the parabola when using $x = \sqrt{2}$. Therefore the only solution is the one we got from the parabola. We would have a similar issue with the other values of $x$.

The problem arose when we plugged the parabola into the hyperbola and squared it. The process of squaring it introduced potentially “bad” solutions. We saw similar issues when we solved equations with radicals several chapters back and the problem arose there for the same reason, we squared something.

The nice thing about these problems however is that if we use the equation we plugged in (the parabola in this case) to find the second values we don’t need to worry about the “bad” solutions since they only arise from the equation that we plugged into (the hyperbola in this case).

This in fact is the real reason we used the parabola to find the corresponding $y$, although it was also the easier of the two equations to use.