## Table of Contents

**Preface** ............................................................................................................................................ ii

**Surface Integrals** .................................................................................................................................. 3  
  Parametric Surfaces .............................................................................................................................. 3  
  Surface Integrals .................................................................................................................................. 14  
  Surface Integrals of Vector Fields ........................................................................................................... 35  
  Stokes’ Theorem .................................................................................................................................. 61  
  Divergence Theorem ................................................................................................................................. 77
Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
Surface Integrals

Parametric Surfaces

1. Write down a set of parametric equations for the plane \( 7x + 3y + 4z = 15 \).

Step 1
There isn’t a whole lot to this problem. There are three different acceptable answers here. To get a set of parametric equations for this plane all we need to do is solve for one of the variables and then write down the parametric equations.

For this problem let’s solve for \( z \) to get,

\[ z = \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y \]

Step 2
The parametric equation for the plane is then,

\[ \mathbf{r}(x, y) = \langle x, y, \frac{15}{4} - \frac{7}{4}x - \frac{3}{4}y \rangle \]

Remember that all we need to do to get the parametric equations is plug in the equation for \( z \) into the \( z \) component of the vector \( \langle x, y, z \rangle \).

Also, as noted in Step 1 we could just have easily done either of the following two forms for the parametric equations for this plane.

\[ \mathbf{r}(x, z) = \langle x, g(x, z), z \rangle \quad \mathbf{r}(y, z) = \langle h(y, z), y, z \rangle \]

where you solve the equation of the plane for \( y \) or \( x \) respectively. All three set of parametric equations are all perfectly valid forms for the answer to this problem.

2. Write down a set of parametric equations for the plane \( 7x + 3y + 4z = 15 \) that lies in the 1st octant.

Step 1
This problem is really just an extension of the previous problem so we’ll redo the set of parametric equations for the plane a little quicker this time.

First we need to solve the equation for any of the three variables. We’ll solve for \( z \) in this case to get,
Calculus II

\[ z = \frac{15}{4} - \frac{7}{4} x - \frac{1}{4} y \]

The parametric equation for this plane is then,

\[ \vec{r}(x, y) = \langle x, y, z \rangle = \langle x, y, \frac{15}{4} - \frac{7}{4} x - \frac{1}{4} y \rangle \]

Remember that all we need to do to get the parametric equations is plug in the equation for \( z \) into the \( z \) component of the vector \( \langle x, y, z \rangle \).

Step 2
Now, the set of parametric equations from above is for the full plane and that isn’t what we want in this problem. In this problem we only want the portion of the plane that is in the 1st octant.

So, we’ll need to restrict \( x \) and \( y \) so that the parametric equation from Step 1 will only give the portion of the plane that is in the 1st octant.

If you recall how to get the region \( D \) for a triple integral then you know how to do this because it is basically the same idea. In this case we need the region \( D \) in the \( xy \)-plane that will give the plane in the 1st octant.

Here is a sketch of this region.

The hypotenuse is just where the plane intersects the \( xy \)-plane and so we can quickly find the equation of the line by setting \( z = 0 \) in the equation of the plane.

We can either solve this for \( x \) or \( y \) to get the ranges for \( x \) and \( y \). It doesn’t really matter which we solve for here so let’s just solve for \( y \) to get the following ranges for \( x \) and \( y \) to describe this triangle.

\[ 0 \leq x \leq \frac{15}{7} \]
\[ 0 \leq y \leq -\frac{7}{3} x + 5 \]

Putting this all together we get the following set of parametric equations for the plane that is in the 1st octant.
3. The cylinder \( x^2 + y^2 = 5 \) for \(-1 \leq z \leq 6\).

Step 1
Because this surface is just a cylinder we just need the cylindrical coordinates conversion formulas with the polar coordinates in the \(xy\)-plane (since the cylinder is given in terms of \(x\) and \(y\)).

The conversion equations are,
\[
\begin{align*}
x &= r \cos \theta \\
y &= r \sin \theta \\
z &= z
\end{align*}
\]

However, recall that we are actually on the surface of the cylinder and so we know that \(r = \sqrt{5}\). The conversion equations are then,
\[
\begin{align*}
x &= \sqrt{5} \cos \theta \\
y &= \sqrt{5} \sin \theta \\
z &= z
\end{align*}
\]

Step 2
We can now write down a set of parametric equations for the cylinder. They are,
\[
\vec{r}(z, \theta) = (x, y, z) = (\sqrt{5} \cos \theta, \sqrt{5} \sin \theta, z)
\]

Remember that all we do is plug the conversion formulas for \(x, y,\) and \(z\) into the \(x, y\) and \(z\) components of the vector \((x, y, z)\) and we have a set of parametric equations. Also note that because the resulting vector equation is an equation in terms of \(z\) and \(\theta\) those will also be the variables for our set of parametric equation.

Step 3
Now, the only issue with the set of parametric equations above is that they are for the full cylinder and we don’t want that. We only want the cylinder in the given range of \(z\) so to finish this problem out all we need to do is add on a set of restrictions or ranges to our variables.

Doing that gives,
\[
\vec{r}(z, \theta) = (\sqrt{5} \cos \theta, \sqrt{5} \sin \theta, z) \quad -1 \leq z \leq 6, \quad 0 \leq \theta \leq 2\pi
\]

Note that the \(z\) range is just the range given in the problem statement and the \(\theta\) range is the full zero to \(2\pi\) range since there was no mention of restricting the portion of the cylinder that we wanted with respect to \(\theta\) (for example, only the top half of the cylinder).
4. The portion of \( y = 4 - x^2 - z^2 \) that is in front of \( y = -6 \).

Step 1
Okay, the basic set of parametric equations in this case is pretty easy since we already have the equation in the form of “\( y = \)”.

The set of parametric equations that will give the full surface is just,

\[
\vec{r}(x,z) = \langle x, y, z \rangle = \langle x, 4 - x^2 - z^2, z \rangle
\]

Remember that all we need to do to get the parametric equations is plug in the equation for \( y \) into the \( y \) component of the vector \( \langle x, y, z \rangle \).

Step 2
Finally, all we need to do is restrict \( x \) and \( z \) to get only the portion of the surface we are looking for. That is pretty simple however since we are given that we only want the portion that is in front of \( y = -6 \).

This is equivalent to requiring that \( y \geq -6 \) and we do have the equation of the surface so all we need to do is plug that into the inequality and do a little rewrite. Doing this gives,

\[
4 - x^2 - z^2 \geq -6 \quad \rightarrow \quad x^2 + z^2 \leq 10
\]

In other words we only want the points \( (x, z) \) that are inside the disk of radius \( \sqrt{10} \).

Putting all of this together gives the following set of parametric equations for the portion of the surface we are after.

\[
\vec{r}(x,z) = \langle x, 4 - x^2 - z^2, z \rangle \quad x^2 + z^2 \leq 10
\]

5. The portion of the sphere of radius 6 with \( x \geq 0 \).

Step 1
Because we have a portion of a sphere we’ll start off with the spherical coordinates conversion formulas.

\[
x = \rho \sin \varphi \cos \theta \quad \quad y = \rho \sin \varphi \sin \theta \quad \quad z = \rho \cos \varphi
\]

However, we are actually on the surface of the sphere and so we know that \( \rho = 6 \). With this the conversion formulas become,

\[
x = 6 \sin \varphi \cos \theta \quad \quad y = 6 \sin \varphi \sin \theta \quad \quad z = 6 \cos \varphi
\]

Step 2
The set of parametric equations that will give the full sphere is then,
\[ \vec{r}(\theta, \phi) = \{x, y, z\} = \{6\sin \phi \cos \theta, 6\sin \phi \sin \theta, 6 \cos \phi\} \]

Remember that all we do is plug the conversion formulas for \( x, y, \) and \( z \) into the \( x, y \) and \( z \) components of the vector \( \{x, y, z\} \) and we have a set of parametric equations. Also note that because the resulting vector equation is an equation in terms of \( \theta \) and \( \phi \) those will also be the variables for our set of parametric equation.

Step 3
Finally, we need to deal with the fact that we don’t actually want the full sphere here. We only want the portion of the sphere for which \( x \geq 0 \).

We can restrict \( x \) to this range if we restrict \( \theta \) to the range \(-\frac{1}{2} \pi \leq \theta \leq \frac{1}{2} \pi \).

We’ve not put any restrictions on \( z \) and so that means that we’ll take the full range of possible \( \phi \) or \( 0 \leq \phi \leq \pi \). Recall that \( \phi \) is the angle a point in spherical coordinates makes with the positive \( z \)-axis and so that is the quantity we’d need to restrict if we’d wanted to restrict \( z \) (for example \( z \leq 0 \)).

Putting all of this together gives the following set of parametric equations for the portion of the surface we are after.

\[
\vec{r}(\theta, \phi) = \{6\sin \phi \cos \theta, 6\sin \phi \sin \theta, 6 \cos \phi\} \quad -\frac{1}{2} \pi \leq \theta \leq \frac{1}{2} \pi, \quad 0 \leq \phi \leq \pi
\]

6. The tangent plane to the surface given by the following parametric equation at the point \((8,14,2)\).

\[ \vec{r}(u, v) = (u^2 + 2u) \hat{i} + (3v - 2u) \hat{j} + (6v - 10) \hat{k} \]

Step 1
In order to write down the equation of a plane we need a point, which we have, \((8,14,2)\), and a normal vector, which we don’t have yet.

However, recall that \( \vec{r}_u \times \vec{r}_v \) will be normal to the surface. So, let’s compute that.

\[
\vec{r}_u = (2u + 2) \hat{i} - 2 \hat{j} \quad \vec{r}_v = 3 \hat{j} + 6 \hat{k}
\]

\[
\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u + 2 & -2 & 0 \\ 0 & 3 & 6 \end{vmatrix} = -12 \hat{i} - 6(2u + 2) \hat{j} + 3(2u + 2) \hat{k}
\]

Step 2
Calculus II

Now having \( \vec{r}_u \times \vec{r}_v \) is all well and good but it is really only useful if we also know the point, \((u, v)\) for which we are at \((8,14, 2)\) so we next need to set the \(x\), \(y\) and \(z\) coordinates of our point equal to the \(x\), \(y\) and \(z\) components of our parametric equation to determine the value of \(u\) and \(v\) we need.

Here are the equations we get if we do that.

\[
\begin{align*}
8 &= u^2 + 2u & 0 &= u^2 + 2u - 8 = (u + 4)(u - 2) \\
14 &= 3v - 2u & \Rightarrow & 14 &= 3v - 2u \\
2 &= 6v - 10 & 12 &= 6v
\end{align*}
\]

Step 3
From the third equation above we can see that we must have \(v = 2\) and from the first equation we can see that we must have either \(u = -4\) or \(u = 2\).

Plugging our only choice for \(v\) and both choices for \(u\) into the second equation we can see that we must have \(u = -4\).

Step 4
Plugging \(u = -4\) and \(v = 2\) into the equation for \(\vec{r}_u \times \vec{r}_v\) we will arrive at the following normal vector to the surface at \((8,14, 2)\).

\[
\vec{n} = (\vec{r}_u \times \vec{r}_v)_{u=-4, v=2} = -12\hat{i} + 36\hat{j} - 18\hat{k}
\]

Note that, in this case, the normal vector didn’t actually depend on the value of \(v\). That won’t happen in general, but as we’ve seen here that kind of thing can happen on occasion so don’t get excited about it when it does.

The equation of the tangent plane to the surface at \((8,14, 2)\) with normal vector \(\vec{n} = -12\hat{i} + 36\hat{j} - 18\hat{k}\) is,

\[-12(x - 8) + 36(y - 14) - 18(z - 2) = 0 \Rightarrow -12x + 36y - 18z = 372\]

Step 5
To get a set of parametric equations for the tangent plane all we need to do is solve the equation for \(z\) to get,

\[z = -\frac{6z}{3} + \frac{1}{2}x - 2y\]

We can then plug this into the vector \(\langle x, y, z \rangle\) to get the following set of parametric equations for the tangent plane.

\[
\vec{r}(x, y) = \langle x, y, -\frac{6z}{3} + \frac{1}{2}x - 2y \rangle
\]
Note that there will be no restrictions on $x$ and $y$ because we wanted the full tangent plane.

7. Determine the surface area of the portion of $2x + 3y + 6z = 9$ that is inside the cylinder $x^2 + y^2 = 7$.

Step 1
We first need to parameterize the surface. Because we are wanting the portion that is inside the cylinder centered on the $z$-axis it makes sense to first solve the equation of the plane for $z$ to get,

$$z = \frac{3}{2} - \frac{1}{3}x - \frac{1}{2}y$$

The parameterization for the full plane is then,

$$\vec{r}(x, y) = \left\langle x, y, \frac{3}{2} - \frac{1}{3}x - \frac{1}{2}y \right\rangle$$

We only want the portion that is inside the cylinder given in the problem statement so we’ll also need to restrict $x$ and $y$ to those in the disk $x^2 + y^2 \leq 7$. This will now give only the portion of the plane that is inside the cylinder.

Step 2
Next we need to compute $\vec{r}_x \times \vec{r}_y$. Here is that work.

$$\vec{r}_x = \left\langle 1, 0, -\frac{1}{3} \right\rangle \quad \vec{r}_y = \left\langle 0, 1, -\frac{1}{2} \right\rangle$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{2} \end{vmatrix} = \frac{1}{2} \hat{i} + \frac{3}{2} \hat{j} + \hat{k}$$

Now, we what we really need is,

$$\|\vec{r}_x \times \vec{r}_y\| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + 1} = \sqrt{\frac{52}{36}} = \frac{7}{6}$$

Step 3
The integral for the surface area is then,

$$A = \iint_D \frac{7}{6} \, dA$$

In this case $D$ is just the restriction on $x$ and $y$ that we noted in Step 1. So, $D$ is just the disk $x^2 + y^2 \leq 7$.

Step 4
Computing the integral in this case is very simple. All we need to do is take advantage of the fact that,
\[ \iint_D dA = \text{Area of } D \]

So, the surface area is simply,

\[
A = \iint_D \frac{3}{8} dA = \frac{3}{6} \iint_D dA = \frac{3}{6} \left[ \text{Area of } D \right] = \frac{3}{6} \left[ \pi \left( \sqrt[3]{7} \right)^2 \right] = \frac{9}{6} \pi
\]

8. Determine the surface area of the portion of \( x^2 + y^2 + z^2 = 25 \) with \( z \leq 0 \).

Step 1
We first need to parameterize the sphere and we’ve already done a sphere in this problem set so we won’t go into great detail with the parameterization here.

The parameterization for the full sphere is,

\[
\vec{r}(\theta, \phi) = \langle 5 \sin \phi \cos \theta, 5 \sin \phi \sin \theta, 5 \cos \phi \rangle
\]

We don’t want the full sphere of course. We only want the lower half of the sphere, \( i.e. \) the portion with \( z \leq 0 \). This means that we’ll need to restrict \( \phi \) to \( \frac{1}{2} \pi \leq \phi \leq \pi \). Recall that \( \phi \) is the angle points make with the positive \( z \)-axis and because we only want points below the \( xy \)-plane we’ll need the range of \( \frac{1}{2} \pi \leq \phi \leq \pi \).

We want the full lower half and so we’ll use \( 0 \leq \theta \leq 2\pi \) for our \( \theta \) range.

Step 2
Next we need to compute \( \vec{r}_\theta \times \vec{r}_\phi \). Here is that work.

\[
\vec{r}_\theta = \langle -5 \sin \phi \sin \theta, 5 \sin \phi \cos \theta, 0 \rangle \quad \vec{r}_\phi = \langle 5 \cos \phi \cos \theta, 5 \cos \phi \sin \theta, -5 \sin \phi \rangle
\]

\[
\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-5 \sin \phi \sin \theta & 5 \sin \phi \cos \theta & 0 \\
5 \cos \phi \cos \theta & 5 \cos \phi \sin \theta & -5 \sin \phi
\end{vmatrix}
\]

\[
= -25 \sin^2 \phi \cos \theta \hat{i} - 25 \sin \phi \cos \phi \sin^2 \theta \hat{k} - 25 \sin \phi \cos \phi \cos^2 \theta \hat{k} - 25 \sin^2 \phi \sin \theta \hat{j}
\]

Now, we what we really need is,
Note that we can drop the absolute value bars on the sine because we know that sine will be positive in $\frac{1}{2}\pi \leq \varphi \leq \pi$.

Step 3
The integral for the surface area is then,

$$A = \iint_D 25 \sin \varphi \, dA = \int_0^{2\pi} \int_{\frac{\pi}{2}}^\pi 25 \sin \varphi \, d\varphi \, d\theta$$

As noted in the integral above $D$ is just the ranges of $\theta$ and $\varphi$ we found in Step 1.

Step 4
Now we just need to evaluate the integral to get the surface area.

$$A = \int_0^{2\pi} \int_{\frac{\pi}{2}}^\pi 25 \sin \varphi \, d\varphi \, d\theta = \int_0^{2\pi} -25 \cos \varphi \bigg|_{\frac{\pi}{2}}^\pi \, d\theta = \int_0^{2\pi} 25 \, d\theta = 50\pi$$

9. Determine the surface area of the portion of $z = 3 + 2y + \frac{1}{4}x^4$ that is above the region in the $xy$-plane bounded by $y = x^5$, $x = 1$ and the $y$-axis.

Step 1
Parameterizing this surface is pretty simple. We have the equation of the surface in the form $z = f(x, y)$ and so the parameterization of the surface is,

$$\vec{r}(x, y) = \left\langle x, y, 3 + 2y + \frac{1}{4}x^4 \right\rangle$$

Now, this is the parameterization of the full surface and we only want the portion that lies over the following region.
So, to get only the portion of the surface we’ll need to restrict \( x \) and \( y \) to the following ranges,

\[
0 \leq x \leq 1 \\
0 \leq y \leq x^5
\]

On a side note we can see that we are in the 1st quadrant here and so we know that \( x \geq 0 \) and \( y \geq 0 \). Therefore we can see that the surface in the 1st quadrant is always above the \( xy \)-plane and so will in fact always be above the region above as suggested in the problem statement.

Step 2
Next we need to compute \( \vec{r}_x \times \vec{r}_y \). Here is that work.

\[
\vec{r}_x = \langle 1, 0, x^3 \rangle \\
\vec{r}_y = \langle 0, 1, 2 \rangle
\]

\[
\vec{r}_x \times \vec{r}_y = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & x^3 \\
0 & 1 & 2
\end{vmatrix} = -x^3\hat{i} - 2\hat{j} + \hat{k}
\]

Now, what we really need is,

\[
\left\| \vec{r}_x \times \vec{r}_y \right\| = \sqrt{(-x^3)^2 + (-2)^2 + (1)^2} = \sqrt{x^6 + 5}
\]

Step 3
The integral for the surface area is then,

\[
A = \int_D \sqrt{x^6 + 5} \, dA = \int_0^1 \int_0^{x^5} \sqrt{x^6 + 5} \, dy \, dx
\]

As noted in the integral above \( D \) is just the ranges of \( x \) and \( y \) we found in Step 1.

Step 4
Now we just need to evaluate the integral to get the surface area.
10. Determine the surface area of the portion of the surface given by the following parametric equation that lies inside the cylinder \( u^2 + v^2 = 4 \).

\[ \mathbf{r}(u, v) = \langle 2u, vu, 1 - 2v \rangle \]

**Step 1**
We’ve already been given the parameterization of the surface in the problem statement so we don’t need to worry about that for this problem. All we really need to do yet is to acknowledge that we’ll need to restrict \( u \) and \( v \) to the disk \( u^2 + v^2 \leq 4 \).

**Step 2**
Next we need to compute \( \mathbf{r}_u \times \mathbf{r}_v \). Here is that work.

\[
\mathbf{r}_u = \langle 2, v, 0 \rangle \quad \quad \mathbf{r}_v = \langle 0, u, -2 \rangle
\]

\[
\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
2 & v & 0 \\
0 & u & -2
\end{vmatrix} = -2vi + 4v \mathbf{j} + 2u \mathbf{k}
\]

Now, we what we really need is,

\[
\left\| \mathbf{r}_u \times \mathbf{r}_v \right\| = \sqrt{(-2v)^2 + (4v)^2 + (2u)^2} = \sqrt{4u^2 + 4v^2 + 16} = 2\sqrt{u^2 + v^2 + 4}
\]

**Step 3**
The integral for the surface area is then,

\[
A = \iint_D 2\sqrt{u^2 + v^2 + 4} \, dA
\]

Where \( D \) is the disk \( u^2 + v^2 \leq 4 \).

**Step 4**
Because \( D \) is a disk the best bet for this integral is to use the following “version” of polar coordinates.

\[
u = r \cos \theta \quad v = r \sin \theta \quad u^2 + v^2 = r^2 \quad dA = r \, dr \, d\theta
\]
The polar coordinate limits for this $D$ is,

\[
0 \leq \theta \leq 2\pi \\
0 \leq r \leq 2 
\]

So, the integral to converting to polar coordinates gives,

\[
A = \iint_D 2\sqrt{u^2 + v^2 + 4} \, dA = \int_0^{2\pi} \int_0^2 2r \sqrt{r^2 + 4} \, dr \, d\theta 
\]

Step 5
Now we just need to evaluate the integral to get the surface area.

\[
A = \int_0^{2\pi} \int_0^2 2r \sqrt{r^2 + 4} \, dr \, d\theta = \int_0^{2\pi} \left( \frac{2}{3} (r^2 + 4)^{3/2} \right) \bigg|_0^2 \, d\theta \\
= \int_0^{2\pi} \left( \frac{2}{3} \left( 8^{3/2} - 4^{3/2} \right) \right) \, d\theta \\
= \frac{2\pi}{3} \left( 8^{3/2} - 4^{3/2} \right) \bigg|_0^{2\pi} = \frac{2\pi}{3} (\sqrt[3]{8} - 1) = 61.2712 
\]

---

**Surface Integrals**

1. Evaluate $\iint_S z + 3y - x^2 \, dS$ where $S$ is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the $xy$-plane with vertices $(0, 0)$, $(2, 0)$ and $(2, -4)$.

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface.

The orange surface is the sketch of \( z = 2 - 3y + x^2 \) that we are working with in this problem. The greenish triangle below the surface is the triangle referenced in the problem statement that lies below the surface. This triangle will be the region \( D \) for this problem.

Here is a quick sketch of \( D \) just to get a better view of it than the mostly obscured view in the sketch above.
Calculus II

We could use either of the following sets of limits to describe \( D \).

\[
0 \leq x \leq 2 \quad \quad -4 \leq y \leq 0 \\
-2x \leq y \leq 0 \quad \quad -\frac{1}{2}y \leq x \leq 2
\]

We’ll decide which set to use in the integral once we get that set up.

Step 2
Let’s get the integral set up now. In this case the surface is in the form,

\[
z = g(x, y) = 2 - 3y + x^2
\]

so we’ll use the following formula for the surface integral.

\[
\iint_S f(x, y, z) \, dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} + 1 \, dA
\]

The integral is then,

\[
\iint_S z + 3y - x^2 \, dS = \iint_D \left[ (2 - 3y + x^2) + 3y - x^2 \right] \sqrt{(2x)^2 + (-3)^2} + 1 \, dA
\]

\[
= \iint_D 2\sqrt{4x^2 + 10} \, dA
\]

Don’t forget to plug the equation of the surface into \( z \) in the integrand and recall that \( D \) is the triangle sketched in Step 1.

Step 3
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

First note that from the integrand it should be pretty clear that we’ll want to integrate with respect to \( y \) first (unless you want to do a trig substitution of course…). So, the integral becomes,
\[ \iint_S z + 3y - x^2 \, dS = \iint_D 2\sqrt{4x^2 + 10} \, dA \]
\[ = \int_0^2 \int_{-2x}^0 2\sqrt{4x^2 + 10} \, dy \, dx \]
\[ = \int_0^2 \left[ 2\sqrt{4x^2 + 10} \right]_{-2x}^0 \, dx \]
\[ = \int_0^2 4x\sqrt{4x^2 + 10} \, dx \]
\[ = \frac{1}{3} (4x^2 + 10)^{3/2} \bigg|_0^2 = \frac{1}{3} \left(26^{3/2} + 10^{3/2}\right) = 33.6506 \]

2. Evaluate \( \iint_S 40y \, dS \) where \( S \) is the portion of \( y = 3x^2 + 3z^2 \) that lies behind \( y = 6 \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
Note that the surface in this problem is only the elliptic paraboloid and does not include the “cap” at \( y = 6 \). We would only include the “cap” if the problem had specified that in some manner to make it clear.

In this case \( D \) will be the circle/disk we get by setting the two equations equal or,

\[
6 = 3x^2 + 3z^2 \quad \Rightarrow \quad x^2 + z^2 = 2
\]

So, \( D \) will be the disk \( x^2 + z^2 \leq 2 \).

**Step 2**

Let’s get the integral set up now. In this case the surface is in the form,

\[
y = g(x, z) = 3x^2 + 3z^2
\]

so we’ll use the following formula for the surface integral.

\[
\iint_S f(x, y, z) \, dS = \iint_D f(x, g(x, z), z) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial z}\right)^2} + 1} \, dA
\]

The integral is then,
\[
\iint_{S} 40y\,dS = \iiint_{D} 40\left(3x^2 + 3z^2\right)\sqrt{\left(6x\right)^2 + 1 + \left(6z\right)^2}\,dA
\]
\[
= \iiint_{D} 120\left(x^2 + z^2\right)\sqrt{36\left(x^2 + z^2\right) + 1}\,dA
\]

Don’t forget to plug the equation of the surface into \( y \) in the integrand and recall that \( D \) is the disk we found in Step 1.

Step 3
Now, for this problem it should be pretty clear that we’ll want to use polar coordinates to do the integral. We’ll use the following set of polar coordinates.

\[
x = r \cos \theta \quad z = r \sin \theta \quad x^2 + z^2 = r^2
\]

Also, because \( D \) is the disk \( x^2 + z^2 \leq 2 \) the limits for the integral will be,

\[
0 \leq \theta \leq 2\pi \\
0 \leq r \leq \sqrt{2}
\]

Converting the integral to polar coordinates gives,

\[
\iiint_{S} 40y\,dS = \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} \int_{0}^{\sqrt{36r^2 + 1}} 120r^2\sqrt{36r^2 + 1}\left(r\right)dr\,d\theta
\]
\[
= \int_{0}^{2\pi} \int_{0}^{\sqrt{2}} 120r^3\sqrt{36r^2 + 1}\,dr\,d\theta
\]

Don’t forget to pick up the extra \( r \) when converting the \( dA \) into polar coordinates.

Step 4
Now all that we need to do is evaluate the double integral and this one can be a little tricky unless you’ve seen this kind of integral done before.

We’ll use the following substitution to do the integral.

\[
u = 36r^2 + 1 \quad \rightarrow \quad du = 72r\,dr \quad \rightarrow \quad \frac{1}{72}du = r\,dr
\]

The problem is that this doesn’t seem to work at first glance because the differential will only get rid of one of the three \( r \)’s in front of the root. However, we can also solve the substitution for \( r^2 \) to get,

\[
r^2 = \frac{1}{36}\left(u - 1\right)
\]

and we can now convert the remaining two \( r \)’s into \( u \)’s.

So, using the substitution the integral becomes,
\[
\iiint_S 40\, y\, dS = \int_0^{2\pi} \int_1^{73/2} 120 \left( \frac{1}{36} \right) (u - 1) u^{\frac{1}{3}} \, du \, d\theta
\]
\[
= \int_0^{2\pi} \int_1^{73/2} \frac{5}{108} \left( u^{\frac{1}{3}} - u^{\frac{1}{2}} \right) \, du \, d\theta
\]

Note that we also converted the \( r \) limits in the original integral into \( u \) limits simply by plugging the “old” \( r \) limits into the substitution to get “new” \( u \) limits.

We can now easily finish evaluating the integral.

\[
\iiint_S 40\, y\, dS = \int_0^{2\pi} \int_1^{73/2} \left( \frac{5}{108} \right) \left( \frac{2}{3} u^{\frac{5}{3}} - \frac{1}{3} u^{\frac{1}{3}} \right) \, du \, d\theta
\]
\[
= \int_0^{2\pi} \left[ \frac{5}{108} \left( \frac{2}{3} \left( \frac{73^{\frac{5}{3}}}{3} \right) - \frac{1}{3} \left( \frac{73^{\frac{1}{3}}}{3} \right) \right) - \left( -\frac{4}{15} \right) \right] d\theta
\]
\[
= \frac{5\pi}{44} \left[ \frac{2}{3} \left( \frac{73^{\frac{5}{3}}}{3} \right) - \frac{1}{3} \left( \frac{73^{\frac{1}{3}}}{3} \right) + \frac{4}{15} \right] = 5176.8958
\]

Kind of messy integral with a messy answer but that will happen on occasion so we shouldn’t get too excited about it when that does happen.

3. Evaluate \( \iiint_S 2\, y\, dS \) where \( S \) is the portion of \( y^2 + z^2 = 4 \) between \( x = 0 \) and \( x = 3 - z \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
Note that the surface in this problem is only the cylinder itself. The “caps” of the cylinder are not part of this surface despite the red “cap” in the sketch. That was included in the sketch to make the front edge of the cylinder clear in the sketch. We would only include the “caps” if the problem had specified that in some manner to make it clear.

Step 2
Now because our surface is a cylinder we’ll need to parameterize it and use the following formula for the surface integral.

\[
\mathbf{r}' \left( u, v \right) = \frac{\nabla \mathbf{r} \times \mathbf{r}' \left( u, v \right)}{\left\| \nabla \mathbf{r} \right\|} = \mathbf{A} \,
\]

where \( u \) and \( v \) will be chosen as needed when doing the parameterization.

We saw how to parameterize a cylinder in the previous section so we won’t go into detail for the parameterization. The parameterization is,

\[
\mathbf{r} \left( x, \theta \right) = \left( x, 2 \sin \theta, 2 \cos \theta \right) \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq x \leq 3 - z = 3 - 2 \cos \theta
\]

We’ll use the full range of \( \theta \) since we are allowing it to rotate all the way around the \( x \)-axis. The \( x \) limits come from the two planes that “bound” the cylinder and we’ll need to convert the upper limit using the parameterization.

Next, we’ll need to compute the cross product.

\[
\mathbf{r}_x = \left< 1, 0, 0 \right> \quad \mathbf{r}_\theta = \left< 0, 2 \cos \theta, -2 \sin \theta \right>
\]
Calculus II

\[ \vec{r}_z \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 2\cos\theta & -2\sin\theta \end{vmatrix} = 2\sin\theta \vec{j} + 2\cos\theta \vec{k} \]

The magnitude of the cross product is,

\[ \| \vec{r}_z \times \vec{r}_\theta \| = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2 \]

The integral is then,

\[ \iint_S 2y \, dS = \iint_D (2\sin\theta)(2) \, dA = \iint_D 8\sin\theta \, dA \]

Don’t forget to plug the \( y \) component of the surface parametrization into the integrand and \( D \) is just the limits on \( x \) and \( \theta \) we noted above in the parameterization.

Step 3
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

The integral is then,

\[ \iint_S 2y \, dS = \int_0^{2\pi} \int_0^{3-\cos\theta} 8\sin\theta \, dx \, d\theta \]

\[ = \int_0^{2\pi} (8x \sin\theta) \bigg|_0^{1-\cos\theta} \, d\theta \]

\[ = \int_0^{2\pi} 8(3-\cos\theta)\sin\theta \, d\theta \]

\[ = \int_0^{2\pi} 24\sin\theta - 16\sin\theta \cos\theta \, d\theta \]

\[ = \int_0^{2\pi} 24\sin\theta - 8\sin(2\theta) \, d\theta \]

\[ = (\sin\theta - 4\cos(2\theta)) \bigg|_0^{2\pi} = 0 \]

4. Evaluate \( \iint_S xz \, dS \) where \( S \) is the portion of the sphere of radius 3 with \( x \leq 0 \), \( y \geq 0 \) and \( z \geq 0 \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
Note that the surface in this problem is only the part of the sphere itself. The “edges” (the greenish portions on the right/left) are not part of this surface despite the fact that they are in the sketch. They
Calculus II

were included in the sketch to try and make the surface a little clearer in the sketch. We would only include the “edges” if the problem had specified that in some manner to make it clear.

Step 2
Now because our surface is a sphere we’ll need to parameterize it and use the following formula for the surface integral.

$$\iint_S f(x, y, z) dS = \iint_D f(\vec{r}(u, v)) \left\| \vec{r}_u \times \vec{r}_v \right\| dA$$

where \( u \) and \( v \) will be chosen as needed when doing the parameterization.

We saw how to parameterize a sphere in the previous section so we won’t go into detail for the parameterization. The parameterization is,

$$\vec{r}(\theta, \varphi) = \left\langle 3 \sin \varphi \cos \theta, 3 \sin \varphi \sin \theta, 3 \cos \varphi \right\rangle \quad \frac{\pi}{2} \leq \theta \leq \pi, \quad 0 \leq \varphi \leq \frac{1}{2} \pi$$

We needed the restriction on \( \varphi \) to make sure that we only get a portion of the upper half of the sphere (i.e. \( z \geq 0 \)). Likewise the restriction on \( \theta \) was needed to get only the portion that was in the 2nd quadrant of the \( xy \)-plane (i.e. \( x \leq 0 \) and \( y \geq 0 \)).

Next, we’ll need to compute the cross product.

$$\vec{r}_\theta = \left\langle -3 \sin \varphi \sin \theta, 3 \sin \varphi \cos \theta, 0 \right\rangle \quad \vec{r}_\varphi = \left\langle 3 \cos \varphi \cos \theta, 3 \cos \varphi \sin \theta, -3 \sin \varphi \right\rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \left| \begin{array}{ccc}
\vec{i} & \vec{j} & \vec{k} \\
-3 \sin \varphi \sin \theta & 3 \sin \varphi \cos \theta & 0 \\
3 \cos \varphi \cos \theta & 3 \cos \varphi \sin \theta & -3 \sin \varphi \\
\end{array} \right|$$

$$= -9 \sin^2 \varphi \cos \theta \vec{i} - 9 \sin \varphi \cos \varphi \sin^2 \theta \vec{k} - 9 \sin \varphi \cos \varphi \cos^2 \theta \vec{j} - 9 \sin^2 \varphi \sin \theta \vec{j}$$

$$= -9 \sin^2 \varphi \cos \theta \vec{i} - 9 \sin^2 \varphi \sin \theta \vec{j} - 9 \sin \varphi \cos \varphi \left( \sin^2 \theta + \cos^2 \theta \right) \vec{k}$$

$$= -9 \sin^2 \varphi \cos \theta \vec{i} - 9 \sin^2 \varphi \sin \theta \vec{j} - 9 \sin \varphi \cos \varphi \vec{k}$$

The magnitude of the cross product is,

$$\left\| \vec{r}_\theta \times \vec{r}_\varphi \right\| = \sqrt{\left( -9 \sin^2 \varphi \cos \theta \right)^2 + \left( -9 \sin^2 \varphi \sin \theta \right)^2 + \left( -9 \sin \varphi \cos \varphi \right)^2}$$

$$= \sqrt{81 \sin^4 \varphi \left( \cos^2 \theta + \sin^2 \theta \right) + 81 \sin^2 \varphi \cos^2 \varphi}$$

$$= \sqrt{81 \sin^2 \varphi \left( \sin^2 \varphi + \cos^2 \varphi \right)}$$

$$= 9 \left| \sin \varphi \right|$$

$$= 9 \sin \varphi$$
The integral is then,

\[ \iiint_S xz \, dS = \iiint_D (3 \sin \varphi \cos \theta)(3 \cos \varphi)(9 \sin \varphi) \, dA = \iiint_D 81 \cos \varphi \sin^2 \varphi \cos \theta \, dA \]

Don’t forget to plug the \( x \) and \( z \) component of the surface parameterization into the integrand and \( D \) is just the limits on \( \theta \) and \( \varphi \) we noted above in the parameterization.

Step 3
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

The integral is then,

\[ \iiint_S xz \, dS = \iiint_D 81 \cos \varphi \sin^2 \varphi \cos \theta \, dA \]

\[ = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} 81 \cos \varphi \sin^2 \varphi \cos \theta \, d\varphi \, d\theta \]

\[ = \int_0^{\frac{\pi}{2}} \left( 27 \sin^3 \varphi \cos \theta \, d\varphi \right) \bigg|_{0}^{\frac{\pi}{2}} \, d\theta \]

\[ = \int_0^{\frac{\pi}{2}} 27 \cos \theta \, d\theta \]

\[ = \left( 27 \sin \theta \right) \bigg|_{\frac{\pi}{2}}^{\frac{\pi}{2}} = -27 \]

5. Evaluate \[ \iiint_S yz + 4xy \, dS \] where \( S \) is the surface of the solid bounded by \( 4x + 2y + z = 8 \), \( z = 0 \), \( y = 0 \) and \( x = 0 \). Note that all four surfaces of this solid are included in \( S \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
Okay, as noted in the problem statement all four surfaces in the sketch (two not shown) are part of $S$ so let’s define each of them as follows.

$S_1 : \text{Plane given by } 4x + 2y + z = 8 \ (\text{i.e the top of the solid)}$

$S_2 : \text{Plane given by } y = 0 \ (\text{i.e the triangle on right side of the solid)}$

$S_3 : \text{Plane given by } x = 0 \ (\text{i.e the triangle at back of the solid - not shown in sketch)}$

$S_4 : \text{Plane given by } z = 0 \ (\text{i.e the triangle on bottom of the solid - not shown in sketch)}$

As noted in the definitions above the first two surfaces are shown in the sketch but the last two are not actually shown due to the orientation of the solid. Below are sketches of each of the three surfaces that correspond to the coordinates planes.
With each of the sketches we gave limits on the variables for each of them since we’ll eventually need that when we start doing the surface integral along each surface.

Now we need to go through and do the integral for each of these surfaces and we’re going to go through these a little quicker than we did for the first few problems in this section.

Step 2
Let’s start with $S_1$. In this case the surface can easily be solved for $z$ to get,

$$z = 8 - 4x - 2y$$

With the equation of the surface written in this manner the region $D$ will be in the $xy$-plane and if you think about it you’ll see that in fact $D$ is nothing more than $S_4$!

The integral in this case is,
\[
\int_S yz + 4xy \, dS = \int_D \left[ y(8 - 4x - 2y) + 4xy\right]\sqrt{(-4)^2 + (-2)^2 + 1} \, dA \\
= \sqrt{21} \int_D 8y - 2y^2 \, dA
\]

Don’t forget to plug the equation of the surface into \( z \) in the integrand and don’t forget to use the equation of the surface in the computation of the root!

Now, as noted above \( D \) for this surface is nothing more than \( S_4 \) and so we can use the limits from the sketch of \( S_4 \) in Step 1.

Now let’s compute the integral for this surface.

\[
\int_S yz + 4xy \, dS = \sqrt{21} \int_0^2 \int_0^{4-2x} 8y - 2y^2 \, dy \, dx
\]

\[
= \sqrt{21} \int_0^2 \left[ 4y^2 - \frac{2}{3} y^3 \right]_0^{4-2x} \, dx
\]

\[
= \sqrt{21} \int_0^2 4(4 - 2x)^2 - \frac{2}{3}(4 - 2x)^3 \, dx
\]

\[
= \sqrt{21} \left[ -\frac{2}{3}(4 - 2x)^3 + \frac{1}{12}(4 - 2x)^4 \right]_0^2 = \frac{64\sqrt{21}}{3} = 97.7616
\]

Step 3
Next we’ll take care of \( S_2 \). In this case the equation for the surface is simply \( y = 0 \) and \( D \) is given in the sketch of \( S_2 \) in Step 1.

The integral in this case is,

\[
\int_S yz + 4xy \, dS = \int_D \sqrt{(0)^2 + (0)^2} \, dA = \int_D 0 \, dA = 0
\]

So, in this case we didn’t need to actually compute the integral. Sometimes we’ll get lucky like this, although it probably won’t happen all that often.

Step 4
Now we can take care of \( S_3 \). In this case the equation for the surface is simply \( x = 0 \) and \( D \) is given in the sketch of \( S_3 \) in Step 1.

The integral in this case is,

\[
\int_S yz + 4xy \, dS = \int_D \left[ yz + 4(0) \right] \sqrt{1 + (0)^2 + (0)^2} \, dA = \int_D yz \, dA
\]
Don’t forget to plug the equation of the surface into \( x \) in the integrand and don’t forget to use the equation of the surface in the computation of the root (although in this case the root just evaluates to one)!

Using the limits for \( D \) from the sketch in Step 1 we can quickly evaluate the integral for this surface.

\[
\iint_{S_1} yz + 4xy \, dS = \int_0^4 \int_0^{y/2} yz \, dz \, dy \\
= \int_0^4 \left[ \frac{1}{2} yz^2 \right]_0^{y/2} \, dy \\
= \int_0^4 32y - 16y^2 + 2y^3 \, dy \\
= \left[ 16y^2 - \frac{16}{3} y^3 + \frac{1}{2} y^4 \right]_0^4 = \frac{128}{3} = 42.6667
\]

Step 5
Finally let's take care of \( S_4 \). In this case the equation for the surface is simply \( z = 0 \) and \( D \) is given in the sketch of \( S_4 \) in Step 1.

The integral in this case is,

\[
\iint_{S_4} yz + 4xy \, dS = \iint_D [y(0) + 4xy] \sqrt{(0)^2 + (0)^2 + 1} \, dA = \iint_D 4xy \, dA
\]

Don’t forget to plug the equation of the surface into \( z \) in the integrand and don’t forget to use the equation of the surface in the computation of the root (although in this case the root just evaluates to one)!

Using the limits for \( D \) from the sketch in Step 1 we can quickly evaluate the integral for this surface.

\[
\iint_{S_4} yz + 4xy \, dS = \int_0^2 \int_0^{4-2x} 4xy \, dy \, dx \\
= \int_0^2 \left[ 2xy^2 \right]_0^{4-2x} \, dx \\
= \int_0^2 32x - 32x^2 + 8x^3 \, dx \\
= \left[ 16x^2 - \frac{32}{3} x^3 + 2x^4 \right]_0^2 = \frac{32}{3} = 10.6667
\]

Step 6
Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the four surfaces above. Doing this gives,

\[
\iint_S yz + 4xy \, dS = \left( \frac{64\sqrt{2}}{3} \right) + (0) + \left( \frac{128}{3} \right) + \left( \frac{32}{3} \right) = \frac{64\sqrt{2}}{3} + \frac{160}{3} = 151.0949
\]

We put parenthesis around each of the individual integral values just to indicate where each came from. In general these aren’t needed of course.
6. Evaluate \( \int_S (x - z) \, dS \) where \( S \) is the surface of the solid bounded by \( x^2 + y^2 = 4 \), \( z = x - 3 \), and \( z = x + 2 \). Note that all three surfaces of this solid are included in \( S \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
As noted in the problem statement there are three surfaces here. The “top” of the cylinder is a little hard to see. We made the walls of the cylinder slightly transparent and the top of the cylinder can be seen as a darker ellipse along the top of the surface.

To help visualize the relationship between the top and bottom of the cylinder here is a different view of the surface.
From this view we can see that the top and bottom planes that “cap” the cylinder are parallel.

Let’s define the three surfaces in the sketch as follows.

\( S_1 \): Cylinder given by \( x^2 + y^2 = 4 \) (i.e. the walls of the solid)
\( S_2 \): Plane given by \( z = x + 2 \) (i.e. the top "cap" of the cylinder)
\( S_3 \): Plane given by \( z = x - 3 \) (i.e. the bottom "cap" of the cylinder)

Now we need to go through and do the integral for each of these surfaces and we’re going to go through these a little quicker than we did for the first few problems in this section.

Step 2
Let’s start with \( S_1 \). The surface in this case is a cylinder and so we’ll need to parameterize it.

The parameterization of the surface is,

\[ \vec{r}(z, \theta) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle \]

The limits on \( z \) and \( \theta \) are,
With the z limits we’ll need to make sure that we convert the x’s into their parameterized form.

In order to evaluate the integral in this case we’ll need the cross product \( \vec{r}_z \times \vec{r}_\theta \) so here is that work.

\[
\vec{r}_z = \langle 0, 0, 1 \rangle \quad \vec{r}_\theta = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle
\]

\[
\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix}
i & j & k \\
0 & 0 & 1 \\
-2 \sin \theta & 2 \cos \theta & 0
\end{vmatrix} = -2 \cos \theta \hat{i} - 2 \sin \theta \hat{j}
\]

Next we’ll need the magnitude of the cross product so here is that.

\[
\| \vec{r}_z \times \vec{r}_\theta \| = \sqrt{4 \cos^2 \theta + 4 \sin^2 \theta} = 2
\]

The integral in this case is,

\[
\iint_{S_1} x - z \, dS = \iint_{D} [2 \cos \theta - z] (2) \, dA = \iint_{D} [4 \cos \theta - 2z] \, dA
\]

Don’t forget to plug the parameterization of the surface into the integrand and don’t forget to add in the magnitude of the cross product!

Now, \( D \) for this surface is nothing more than the limits on \( z \) and \( \theta \) we gave above.

Now let’s compute the integral for this surface.

\[
\iint_{S_1} x - z \, dS = \int_0^{2\pi} \int_{\frac{2 \cos \theta + 2}{2 \cos \theta - 3}}^{\frac{2 \cos \theta + 2}{2 \cos \theta - 3}} 4 \cos \theta - 2z \, dz \, d\theta
\]

\[
= \int_0^{2\pi} \left( 4 \cos \theta - z^2 \right) \bigg|_{\frac{2 \cos \theta + 2}{2 \cos \theta - 3}}^{\frac{2 \cos \theta + 2}{2 \cos \theta - 3}} \, d\theta
\]

\[
= \int_0^{2\pi} 4 \cos \theta \left[ 2 \cos \theta + 2 \right] - \left[ (2 \cos \theta + 2)^2 - (2 \cos \theta - 3)^2 \right] \, d\theta
\]

\[
= \int_0^{2\pi} \left[ 8 \cos \theta - 5 \right] \, d\theta = 10\pi
\]

Do not forget to simplify! As we saw with this problem after the \( z \) integration the integrand looked really messy but after some pretty simple simplification it reduced down to an incredibly simple integrand.

Step 3
Next we’ll take care of \( S_2 \). In this case the equation for the surface is simply \( z = x + 2 \) and \( D \) is the disk \( x^2 + y^2 \leq 4 \).
The integral in this case is,
\[
\iint_{S_2} x - z \, dS = \iint_{D} \left[ x - (x + 2) \right] \sqrt{1^2 + (0)^2 + 1} \, dA = \iint_{D} -2\sqrt{2} \, dA = -2\sqrt{2} \iint_{D} dA
\]

Okay, in this case we don’t need to actually do the evaluation of the integral because we know that,
\[
\iint_{D} dA = \text{Area of } D
\]

and in this case \(D\) is just a disk and we can quickly determine its area without any evaluation.

So, the integral for this surface is then just,
\[
\iint_{S_2} x - z \, dS = -2\sqrt{2} \left( \text{Area of } D \right) = -2\sqrt{2} \left[ (2)^2 \pi \right] = -8\sqrt{2}\pi
\]

Step 4
Finally, let’s integrate over \(S_3\). In this case the equation for the surface is simply \(z = x - 3\) and \(D\) is the disk \(x^2 + y^2 \leq 4\).

The integral in this case is,
\[
\iint_{S_3} x - z \, dS = \iint_{D} \left[ x - (x - 3) \right] \sqrt{1^2 + (0)^2 + 1} \, dA
\]
\[
= \iint_{D} 3\sqrt{2} \, dA = 3\sqrt{2} \iint_{D} dA = 3\sqrt{2} \left( 4\pi \right) = 12\sqrt{2}\pi
\]

So, the integral in this case ended up being every similar to the integral in Step 3 and so we didn’t put in any of the explanation here.

Step 5
Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the three surfaces above. Doing this gives,
\[
\iint_{S} x - z \, dS = \left( 10\pi \right) + \left( -8\sqrt{2}\pi \right) + \left( 12\sqrt{2}\pi \right) = \left( 10 + 4\sqrt{2} \right)\pi = 49.1875
\]

We put parenthesis around each of the individual integral values just to indicate where each came from. In general these aren’t needed of course.
1. Evaluate $\mathbf{F} \cdot d\mathbf{S}$ where $\mathbf{F} = 3x \mathbf{i} + 2z \mathbf{j} + (1 - y^2) \mathbf{k}$ and $S$ is the portion of $z = 2 - 3y + x^2$ that lies over the triangle in the $xy$-plane with vertices $(0,0), (2,0)$ and $(2,-4)$ oriented in the negative $z$-axis direction.

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.

We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface.
The orange surface is the sketch of \( z = 2 - 3y + x^2 \) that we are working with in this problem. The greenish triangle below the surface is the triangle referenced in the problem statement that lies below the surface. This triangle will be the region \( D \) for this problem.

Here is a quick sketch of \( D \) just to get a better view of it than the mostly obscured view in the sketch above.

We could use either of the following sets of limits to describe \( D \).

\[
\begin{align*}
0 \leq x \leq 2 & \quad -4 \leq y \leq 0 \\
-2x \leq y \leq 0 & \quad -\frac{1}{2} y \leq x \leq 2
\end{align*}
\]

We’ll decide which set to use in the integral once we get that set up.

Step 2
Let’s get the integral set up now. In this case the we can write the equation of the surface as follows,

\[ f(x, y, z) = 2 - 3y + x^2 - z = 0 \]

A unit normal vector for the surface is then,

\[ \mathbf{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 2x, -3, -1 \rangle}{\|\nabla f\|} \]

We didn’t compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that, in this case, the normal vector we computed above has the correct orientation. We were told in the problem statement that the orientation was in the negative \( z \)-axis direction and this means that the normal vector should always have a downwards direction (\( i.e. \) a negative \( z \) component) and this one does.

Step 3
Next, we’ll need to compute the following dot product.
\( \vec{F}(x, y, 2 - 3y + x^2) \cdot \vec{n} = \langle 3x, 2(2 - 3y + x^2), 1 - y^2 \rangle \cdot \frac{\langle 2x, -3, -1 \rangle}{\|
abla f\|} \)

\[
= \frac{1}{\|
abla f\|} \left( 6x^2 - 6(2 - 3y + x^2) - (1 - y^2) \right) \\
= \frac{1}{\|
abla f\|} \left( y^2 + 18y - 13 \right)
\]

Remember that we needed to plug in the equation of the surface, \( z = 2 - 3y + x^2 \), into \( z \) in the vector field!

The integral is then,

\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_D \frac{1}{\|
abla f\|} \left( y^2 + 18y - 13 \right) dS \\
= \iint_D \frac{1}{\|
abla f\|} \left( y^2 + 18y - 13 \right) \|
abla f\| dA \\
= \iint_D y^2 + 18y - 13 dA
\]

As noted above we didn’t need to compute the magnitude of the gradient since it would just cancel out when we converted the surface integral into a “normal” double integral.

Also, recall that \( D \) was given in Step 1. We had two sets of limits to use here but it seems like the first set is probably just as easy to use so we’ll use that one in the integral.

Step 4
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

Here is the integral,

\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_D y^2 + 18y - 13 dA \\
= \int_0^2 \int_{-2x}^0 y^2 + 18y - 13 dy dx \\
= \int_0^2 \left( \frac{1}{3}y^3 + 9y^2 - 13y \right)_{-2x}^0 dx \\
= \int_0^2 \left( \frac{2}{3}x^3 - 36x^2 - 26x \right) dx \\
= \left( \frac{2}{3}x^4 - 12x^3 - 13x^2 \right) \bigg|_0^{4/3} = \left[ \frac{412}{81} \right]
\]
2. Evaluate \( \int_{S} F \cdot d\vec{S} \) where \( \vec{F} = -x \vec{i} + 2y \vec{j} - z \vec{k} \) and \( S \) is the portion of \( y = 3x^2 + 3z^2 \) that lies behind \( y = 6 \) oriented in the positive \( y \)-axis direction.

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
Note that the surface in this problem is only the elliptic paraboloid and does not include the “cap” at $y = 6$. We would only include the “cap” if the problem had specified that in some manner to make it clear.

In this case $D$ will be the circle/disk we get by setting the two equations equal or,

$$6 = 3x^2 + 3z^2 \quad \Rightarrow \quad x^2 + z^2 = 2$$

So, $D$ will be the disk $x^2 + z^2 \leq 2$.

**Step 2**

Let’s get the integral set up now. In this case the we can write the equation of the surface as follows,

$$f(x, y, z) = 3x^2 + 3z^2 - y = 0$$

A unit normal vector for the surface is then,

$$\mathbf{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 6x, -1, 6z \rangle}{\|\nabla f\|}$$

We didn’t compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that, in this case, the normal vector we computed above does not have the correct orientation. We were told in the problem statement that the orientation was in the positive $y$-axis direction and this means that the normal vector should always point in the general direction of the positive $y$-axis (i.e. a positive $y$ component) and this one does not.

That is easy to fix however. All we need to do is multiply the above normal vector by minus one and we’ll get what we need. So, here is the normal vector we need for this problem.

$$\mathbf{n} = -\frac{\nabla f}{\|\nabla f\|} = \frac{-6x, 1, -6z}{\|\nabla f\|}$$

As we can see this normal vector does in fact have a positive $y$ component as we need.

**Step 3**

Next, we’ll need to compute the following dot product.

$$\mathbf{F}(x, 3x^2 + 3z^2, z) \cdot \mathbf{n} = \langle -x, 2(3x^2 + 3z^2), -z \rangle \cdot \frac{-6x, 1, -6z}{\|\nabla f\|}$$

$$= \frac{1}{\|\nabla f\|} \left[ 6x^2 + 2(3x^2 + 3z^2) + 6z^2 \right]$$

$$= \frac{1}{\|\nabla f\|} \left[ 12(x^2 + z^2) \right]$$
Remember that we needed to plug in the equation of the surface, \( y = 3x^2 + 3z^2 \), into \( y \) in the vector field!

The integral is then,

\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_D \left[ \frac{1}{\sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2}} \cdot 12 \left( x^2 + z^2 \right) \right] dS
\]

\[
= \iint_D \left[ \frac{1}{\sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2}} \right] \left( \nabla f \right) \cdot dA
\]

\[
= \iint_D 12 \left( x^2 + z^2 \right) dA
\]

As noted above we didn’t need to compute the magnitude of the gradient since it would just cancel out when we converted the surface integral into a “normal” double integral.

Also, recall that \( D \) was given in Step 1 and is just the disk \( x^2 + z^2 \leq 2 \)

Step 4
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

Note as well that we’ll want to use polar coordinates in the double integral. We’ll use the following set of polar coordinates.

\[
x = r \cos \theta \quad z = r \sin \theta \quad x^2 + z^2 = r^2
\]

The polar limits for \( D \) are,

\[
0 \leq \theta \leq 2\pi \\
0 \leq r \leq \sqrt{2}
\]

The integral is then,

\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_D 12 \left( x^2 + z^2 \right) dA
\]

\[
= \int_0^{2\pi} \int_0^{\sqrt{2}} 12r^3 \, dr \, d\theta
\]

\[
= \int_0^{2\pi} 3r^4 \bigg|_0^{\sqrt{2}} \, d\theta
\]

\[
= \int_0^{2\pi} 12 \, d\theta = 24\pi
\]

Don’t forget that we pick up an extra \( r \) from the \( dA \) when converting to polar coordinates.
3. Evaluate $\int \int_S \vec{F} \cdot d\vec{S}$ where $\vec{F} = x^2 \vec{i} + 2z \vec{j} - 3y \vec{k}$ and $S$ is the portion of $y^2 + z^2 = 4$ between $x = 0$ and $x = 3 - z$ oriented outwards (i.e. away from the $x$-axis).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.

Note that the surface in this problem is only the cylinder itself. The “caps” of the cylinder are not part of this surface despite the red “cap” in the sketch. That was included in the sketch to make the front edge of the cylinder clear in the sketch. We would only include the “caps” if the problem had specified that in some manner to make it clear.
Step 2
Let’s get the integral set up now. In this case the we are integrating over a cylinder and so we’ll need to set up a parameterization for the surface.

We saw how to parameterize a cylinder in the first section of this chapter so we won’t go into detail for the parameterization. The parameterization is,

\[
\vec{r}(x, \theta) = \langle x, 2\sin \theta, 2\cos \theta \rangle \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq x \leq 3 - z = 3 - 2\cos \theta
\]

We’ll use the full range of \( \theta \) since we are allowing it to rotate all the way around the \( x \)-axis. The \( x \) limits come from the two planes that “bound” the cylinder and we’ll need to convert the upper limit using the parameterization.

Next, we’ll need to compute the cross product.

\[
\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 2\cos \theta & -2\sin \theta \end{vmatrix} = 2\sin \theta \hat{j} + 2\cos \theta \hat{k}
\]

A unit normal vector for the surface is then,

\[
\vec{n} = \frac{\vec{r}_x \times \vec{r}_\theta}{\left\| \vec{r}_x \times \vec{r}_\theta \right\|} = \frac{\langle 0, 2\sin \theta, 2\cos \theta \rangle}{\left\| \vec{r}_x \times \vec{r}_\theta \right\|}
\]

We didn’t compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

Now we need to determine if this vector has the correct orientation. First let’s look at the cylinder from in front of the cylinder and directly along the \( x \)-axis. This is what we’d see.
In this sketch the $x$ axis will be coming straight out of the sketch at the origin. Plugging in a few value of $\theta$ into the parameterization we can see that we’ll be at the points listed above.

Now, in the range $0 \leq \theta \leq \frac{1}{2} \pi$ we know that sine and cosine are both positive and so in the normal vector both the $y$ and $z$ components will be positive. This means that in the 1st quadrant above the normal vector would need to be pointing out away from the origin. This is exactly what we need to see since the orientation was given as pointing away from the $x$-axis and recall that the $x$-axis is coming straight out of the sketch from the origin.

Next, if we look at $\frac{1}{2} \pi \leq \theta \leq \pi$ (so we’re in the 4th quadrant of the graph above….) we know that in this range sine is still positive but cosine is now negative. From our unit vector above this means that the $y$ component is positive (so pointing in positive $y$ direction) and the $z$ component is negative (so pointing in negative $z$ direction). Together this again means that we have to be pointing away from the origin in the 4th quadrant which is again the orientation we want.

We could continue in this fashion looking at the remaining two quadrants but once we’ve done a couple and gotten the correct orientation we know we’ll continue to get the correct orientation for the rest.

Step 3
Next, we’ll need to compute the following dot product.

$$\vec{F}(\vec{r}(x, \theta)) \cdot \vec{n} = \left\langle x^2, 2(2 \cos \theta), -3(2 \sin \theta) \right\rangle \cdot \left\langle 0, 2 \sin \theta, 2 \cos \theta \right\rangle \left\| \vec{F} \times \vec{e}_\theta \right\|$$

$$= \frac{1}{\left\| \vec{F} \times \vec{e}_\theta \right\|} (-4 \sin \theta \cos \theta)$$

Remember that we needed to plug in the parameterization for the surface into the vector field!

The integral is then,
\[ \iint_S \vec{F} \cdot d\vec{S} = \iint_D \frac{\vec{r}_\phi \times \vec{r}_\phi}{\vec{r}_\phi} \cdot \left( -4 \sin \theta \cos \theta \right) dS \]
\[ = \iint_D \frac{\vec{r}_\phi \times \vec{r}_\phi}{\vec{r}_\phi} \cdot \left( -4 \sin \theta \cos \theta \right) \left\| \vec{r}_\phi \times \vec{r}_\phi \right\| dA \]
\[ = \iint_D -4 \sin \theta \cos \theta dA \]

As noted above we didn’t need to compute the magnitude of the cross product since it would just cancel out when we converted the surface integral into a “normal” double integral.

Also, recall that \( D \) is given by the limits on \( x \) and \( \theta \) we found at the start of Step 2.

Step 4
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

The integral is then,
\[ \iint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_{-2\cos \theta}^{2\cos \theta} -4 \sin \theta \cos \theta \, dx \, d\theta \]
\[ = \left[ -4x \sin \theta \cos \theta \right]_0^{1-2\cos \theta} 
\[ = \int_0^{2\pi} -4 \sin \theta \cos \theta \, d\theta \]
\[ = \int_0^{2\pi} -12 \sin \theta \cos \theta + 8 \sin \theta \cos^2 \theta \, d\theta \]
\[ = \int_0^{2\pi} -6 \sin (2\theta) + 8 \sin \theta \cos^2 \theta \, d\theta \]
\[ = \left[ -3 \cos (2\theta) - \frac{8}{3} \cos^3 \theta \right]_0^{2\pi} = 0 \]

4. Evaluate \( \iint_S \vec{F} \cdot d\vec{S} \) where \( \vec{F} = \vec{i} + z \vec{j} + 6x \vec{k} \) and \( S \) is the portion of the sphere of radius 3 with \( x \leq 0 \), \( y \geq 0 \) and \( z \geq 0 \) oriented inward (i.e. towards the origin).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
Note that the surface in this problem is only the part of the sphere itself. The “edges” (the greenish portions on the right/left) are not part of this surface despite the fact that they are in the sketch. They
were included in the sketch to try and make the surface a little clearer in the sketch. We would only include the “edges” if the problem had specified that in some manner to make it clear.

Step 2
Let’s get the integral set up now. In this case the we are integrating over a sphere and so we’ll need to set up a parameterization for the surface.

We saw how to parameterize a sphere in the first section of this chapter so we won’t go into detail for the parameterization. The parameterization is,

\[
\vec{r}(\theta, \varphi) = \left( 3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi \right) \quad \frac{\pi}{2} \leq \theta \leq \pi, \quad 0 \leq \varphi \leq \frac{\pi}{2}
\]

We needed the restriction on \( \varphi \) to make sure that we only get a portion of the upper half of the sphere \((i.e. \ z \geq 0)\). Likewise the restriction on \( \theta \) was needed to get only the portion that was in the 2nd quadrant of the \( xy \)-plane \((i.e. \ x \leq 0 \ and \ y \geq 0)\).

Next, we’ll need to compute the cross product.

\[
\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3\sin\varphi\sin\theta & 3\sin\varphi\cos\theta & 0 \\ 3\cos\varphi\cos\theta & 3\cos\varphi\sin\theta & -3\sin\varphi \end{vmatrix}
= -9\sin^2\varphi\cos\theta\hat{i} - 9\sin\varphi\cos\varphi\sin^2\theta\hat{k} - 9\sin\varphi\cos\varphi\cos^2\theta\hat{k} - 9\sin^2\varphi\sin\theta\hat{j}
= -9\sin^2\varphi\cos\theta\hat{i} - 9\sin^2\varphi\sin\theta\hat{j} - 9\sin\varphi\cos\varphi\left(\sin^2\theta + \cos^2\theta\right)\hat{k}
= -9\sin^2\varphi\cos\theta\hat{i} - 9\sin^2\varphi\sin\theta\hat{j} - 9\sin\varphi\cos\varphi\hat{k}
\]

A unit normal vector for the surface is then,

\[
\vec{n} = \frac{\vec{r}_\theta \times \vec{r}_\varphi}{\| \vec{r}_\theta \times \vec{r}_\varphi \|} = \left\langle -9\sin^2\varphi\cos\theta, -9\sin^2\varphi\sin\theta, -9\sin\varphi\cos\varphi \right\rangle
\]

We didn’t compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

Now we need to determine if this vector has the correct orientation. We know that the normal vector needs to point in towards the origin. Let’s think about what that would mean for a normal vector on the upper half of a sphere and it won’t matter which quadrant in the \( xy \)-plane we are in.

If we are on the upper half of a sphere and the normal vectors must point towards the origin then we know that they will all need to point downwards. They could point in the positive or negative \( x \) (or \( y \)) direction depending on which quadrant from the \( xy \)-plane we are on but they will have to all point downwards. Or in other words, the \( z \) component must be negative.
So, the $z$ component of the normal vector above is $-9 \sin \varphi \cos \varphi$ and we know that we are restricted to $0 \leq \varphi \leq \frac{1}{2} \pi$ for the portion of the sphere we are working on in this problem. In this range of $\varphi$ we know that both sine and cosine are positive and so the $z$ component must always be negative. This means that the normal vector above has the correct orientation for this problem.

Note that if we were on the lower half of a sphere (not relevant for this problem but useful to think about anyway) and the normal vector would be pointing towards the origin and so they would have to all be pointing upwards.

Also note that if the normal vectors were all pointing out away from the origin then we’d just need to multiply the normal vector above by minus one to get the normal vector we’d need.

Step 3
Next, we’ll need to compute the following dot product.

$$\vec{F}(\vec{r}(\theta, \varphi)) \cdot \vec{n} = \left< 1, 3 \cos \varphi, 18 \sin \varphi \cos \theta \right> \cdot \left< -9 \sin^2 \varphi \cos \theta, -9 \sin^2 \varphi \sin \theta, -9 \sin \varphi \cos \varphi \right> \left\| \vec{r}_\theta \times \vec{r}_\varphi \right\|$$

$$= \frac{1}{\|\vec{r}_\theta \times \vec{r}_\varphi\|} \left( -9 \sin^2 \varphi \cos \theta - 27 \sin^2 \varphi \cos \varphi \sin \theta - 162 \sin^2 \varphi \cos \varphi \cos \theta \right)$$

$$= \frac{1}{\|\vec{r}_\theta \times \vec{r}_\varphi\|} \left( -\frac{9}{2} \left( 1 - \cos(2\varphi) \right) \cos \theta - \sin^2 \varphi \cos \varphi \left( 27 \sin \theta + 162 \cos \theta \right) \right)$$

Note that we did a little simplification for the integration process in the last step above.

The integral is then,

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \frac{1}{\|\vec{r}_\theta \times \vec{r}_\varphi\|} \left( -\frac{9}{2} \left( 1 - \cos(2\varphi) \right) \cos \theta - \sin^2 \varphi \cos \varphi \left( 27 \sin \theta + 162 \cos \theta \right) \right) dS$$

$$= \iint_D \frac{1}{\|\vec{r}_\theta \times \vec{r}_\varphi\|} \left( -\frac{9}{2} \left( 1 - \cos(2\varphi) \right) \cos \theta - \sin^2 \varphi \cos \varphi \left( 27 \sin \theta + 162 \cos \theta \right) \right) \left\| \vec{r}_\theta \times \vec{r}_\varphi \right\| dA$$

$$= \iint_D -\frac{9}{2} \left( 1 - \cos(2\varphi) \right) \cos \theta - \sin^2 \varphi \cos \varphi \left( 27 \sin \theta + 162 \cos \theta \right) dA$$

As noted above we didn’t need to compute the magnitude of the cross product since it would just cancel out when we converted the surface integral into a “normal” double integral.

Also, recall that $D$ is given by the limits on $\theta$ and $\varphi$ we found at the start of Step 2.

Step 4
Now all that we need to do is evaluate the double integral and that shouldn’t be too difficult at this point.

The integral is then,
\[
\int_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(1 - \cos(2\varphi)\right) \cos \theta - \sin^2 \varphi \cos \varphi (27 \sin \theta + 162 \cos \theta) \, dA \\
= \int_0^\pi \int_0^\frac{\pi}{2} \left(1 - \cos(2\varphi)\right) \cos \theta - \sin^2 \varphi \cos \varphi (27 \sin \theta + 162 \cos \theta) \, d\varphi \, d\theta \\
= \int_0^\pi -\frac{9}{2} \left(\varphi - \frac{1}{2} \sin(2\varphi)\right) \cos \theta - \frac{1}{3} \sin^3 \varphi (27 \sin \theta + 162 \cos \theta) \bigg|_0^{\frac{\pi}{2}} \, d\theta \\
= \int_0^\pi -\frac{9}{4} \pi \cos \theta - 9 \sin \theta - 54 \cos \theta \, d\theta \\
= \left(-\frac{9}{4} \pi \sin \theta + 9 \cos \theta - 54 \sin \theta\right) \bigg|_0^\pi = \frac{25}{4} \pi + 45
\]

5. Evaluate \[\int_S \mathbf{F} \cdot d\mathbf{S}\] where \(\mathbf{F} = y \mathbf{i} + 2x \mathbf{j} + (z - 8) \mathbf{k}\) and \(S\) is the surface of the solid bounded by \(4x + 2y + z = 8\), \(z = 0\), \(y = 0\) and \(x = 0\) with the positive orientation. Note that all four surfaces of this solid are included in \(S\).

**Step 1**
Let’s start off with a quick sketch of the surface we are working with in this problem.
Okay, as noted in the problem statement all four surfaces in the sketch (two not shown) are part of $S$ so let’s define each of them as follows.

$S_1$ : Plane given by $4x + 2y + z = 8$ (i.e the top of the solid)

$S_2$ : Plane given by $y = 0$ (i.e the triangle on right side of the solid)

$S_3$ : Plane given by $x = 0$ (i.e the triangle at back of the solid - not shown in sketch)

$S_4$ : Plane given by $z = 0$ (i.e the triangle on bottom of the solid - not shown in sketch)

As noted in the definitions above the first two surfaces are shown in the sketch but the last two are not actually shown due to the orientation of the solid. Below are sketches of each of the three surfaces that correspond to the coordinates planes.
With each of the sketches we gave limits on the variables for each of them since we’ll eventually need that when we start doing the surface integral along each surface.

Now we need to go through and do the integral for each of these surfaces and we’re going to go through these a little quicker than we did for the first few problems in this section.

Step 2
Let’s start with $S_1$. In this case we can write the equation of the plane as follows,

$$f(x, y, z) = 4x + 2y + z - 8 = 0$$

A unit normal vector for the surface is then,

$$\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle 4, 2, 1 \rangle}{\|\nabla f\|}$$

We didn’t compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.
The surface has the positive orientation and so must point outwards from the region enclosed by the surface. This means that normal vectors on this plane will need to be pointing generally upwards (i.e. have a positive $z$ component) which this normal vector does.

Now we’ll need the following dot product and don’t forget to plug in the equation of the plane (solved for $z$ of course) into $z$ in the vector field.

$$\vec{F}(x, y, 8 - 4x - 2y) \cdot \vec{n} = \langle y, 2x, -4x - 2y \rangle \cdot \frac{\langle 4, 2, 1 \rangle}{\| \nabla f \|}$$

$$= \frac{1}{\sqrt{41}} (2y)$$

The integral is then,

$$\iint \vec{F} \cdot d\vec{S} = \iint_S \frac{1}{\sqrt{41}} (2y) dS$$

$$= \iint_D \frac{1}{\sqrt{41}} (2y) \| \nabla f \| dA$$

$$= \iint_D 2y dA$$

In this case $D$ is just $S_4$ and so we can now finish out the integral.

$$\iint \vec{F} \cdot d\vec{S} = \iint_D 2y dA$$

$$= \int_0^2 \int_0^{4-2x} 2y dy \, dx$$

$$= \int_0^2 y^2 \bigg|_0^{4-2x} \, dx$$

$$= \int_0^2 (4-2x)^2 \, dx$$

$$= -\frac{1}{6} (4-2x)^3 \bigg|_0^3 = \frac{23}{3}$$

Step 3
Next we’ll take care of $S_2$. In this case the equation for the surface is simply $y = 0$ and $D$ is given in the sketch of $S_2$ in Step 1.

In this case $S_2$ is simply a portion of the $xz$-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply $\vec{n} = -\vec{j} = \langle 0, -1, 0 \rangle$.

The dot product for this surface is,
Calculus II

\[ \vec{F}(x,0,z) \cdot \vec{n} = \langle 0, 2x, z - 8 \rangle \cdot \frac{\langle 0, -1, 0 \rangle}{1} = -2x \]

Don’t forget to plug \( y = 0 \) into the vector field and note that the magnitude of the gradient is,

\[ \| \nabla f \| = \sqrt{(0)^2 + (1)^2 + (0)^2} = 1 \]

The integral is then,

\[ \iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} -2x \, dS \]
\[ = \iint_{D} -2x(1) \, dA \]
\[ = \int_{0}^{2} \int_{0}^{8-4x} -2x \, dz \, dx \]
\[ = \int_{0}^{2} -2xz\bigg|_{0}^{8-4x} \, dx \]
\[ = \int_{0}^{2} 8x^2 - 16x \, dx \]
\[ = \left( \frac{8}{3} x^3 - 8x^2 \right)\bigg|_{0}^{1} = \frac{-32}{3} \]

Step 4

Now we can deal with \( S_3 \). In this case the equation for the surface is simply \( x = 0 \) and \( D \) is given in the sketch of \( S_3 \) in Step 1.

In this case \( S_3 \) is simply a portion of the \( yz \)-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply \( \vec{n} = -\vec{i} = \langle -1, 0, 0 \rangle \).

The dot product for this surface is,

\[ \vec{F}(0,y,z) \cdot \vec{n} = \langle y, 0, z - 8 \rangle \cdot \frac{\langle -1, 0, 0 \rangle}{1} = -y \]

Don’t forget to plug \( x = 0 \) into the vector field and note that the magnitude of the gradient is,

\[ \| \nabla f \| = \sqrt{(1)^2 + (0)^2 + (0)^2} = 1 \]

The integral is then,
\[
\iint_{S_4} \vec{F} \cdot d\vec{S} = \iint_{S_4} -y \, dS \\
= \iint_D -y(1) \, dA \\
= \int_0^4 \int_0^{8-2y} -y \, dz \, dy \\
= \int_0^4 -yz \bigg|_0^{8-2y} \, dy \\
= \int_0^4 2y^2 - 8y \, dy \\
= \left( \frac{y^3}{3} - 4y^2 \right) \bigg|_0^4 = -\frac{64}{3}
\]

Step 5

Finally let’s take care of \( S_4 \). In this case the equation for the surface is simply \( z = 0 \) and \( D \) is given in the sketch of \( S_4 \) in Step 1.

In this case \( S_4 \) is simply a portion of the xy-plane and we have the positive orientation and so the normal vector must point away from the region enclosed by the surface. That means that, in this case, the normal vector is simply \( \vec{n} = -\vec{k} = \langle 0, 0, -1 \rangle \).

The dot product for this surface is,

\[
\vec{F}(x, y, 0) \cdot \vec{n} = \langle y, 2x, -8 \rangle \cdot \langle 0, 0, -1 \rangle = 8
\]

Don’t forget to plug \( z = 0 \) into the vector field and note that the magnitude of the gradient is,

\[
\left\| \nabla f \right\| = \sqrt{(0)^2 + (0)^2 + (1)^2} = 1
\]

The integral is then,

\[
\iint_{S_4} \vec{F} \cdot d\vec{S} = \iint_{S_4} 8 \, dS \\
= \iint_D 8 \, dA \\
= 8 \iint_D dA \\
= 8 \text{(Area of } D \text{)} \\
= 8 \left( \frac{1}{2} \right)(2)(4) = 32
\]

In this case notice that we didn’t have to actually compute the double integral since \( D \) was just a right triangle and we can easily compute its area.
Step 6
Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the four surfaces above. Doing this gives,

\[ \iint_S \vec{F} \cdot d\vec{S} = \left( \frac{32}{3} \right) + \left( -\frac{32}{3} \right) + \left( -\frac{64}{3} \right) + (32) = \frac{32}{3} \]

We put parenthesis around each of the individual integral values just to indicate where each came from. In general these aren’t needed of course.

6. Evaluate \( \iint_S \vec{F} \cdot d\vec{S} \) where \( \vec{F} = yz \hat{i} + x \hat{j} + 3y^2 \hat{k} \) and \( S \) is the surface of the solid bounded by 
\( x^2 + y^2 = 4 \), \( z = x - 3 \), and \( z = x + 2 \) with the negative orientation. Note that all three surfaces of this solid are included in \( S \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
As noted in the problem statement there are three surfaces here. The “top” of the cylinder is a little hard to see. We made the walls of the cylinder slightly transparent and the top of the cylinder can be seen as a darker ellipse along the top of the surface.

To help visualize the relationship between the top and bottom of the cylinder here is a different view of the surface.
From this view we can see that the top and bottom planes that “cap” the cylinder are parallel.

Let’s define the three surfaces in the sketch as follows.

\( S_1 \): Cylinder given by \( x^2 + y^2 = 4 \) (\( i.e \) the walls of the solid)
\( S_2 \): Plane given by \( z = x + 2 \) (\( i.e \) the top "cap" of the cylinder)
\( S_3 \): Plane given by \( z = x - 3 \) (\( i.e \) the bottom "cap" of the cylinder)

Now we need to go through and do the integral for each of these surfaces and we’re going to go through these a little quicker than we did for the first few problems in this section.

Step 2
Let’s start with \( S_1 \). The surface in this case is a cylinder and so we’ll need to parameterize it. The parameterization of the surface is,

\[
\vec{r}(z, \theta) = \langle 2 \cos \theta, 2 \sin \theta, z \rangle
\]

The limits on \( z \) and \( \theta \) are,
0 ≤ θ ≤ 2π, 2 cos θ - 3 = x - 3 ≤ z ≤ x + 2 = 2 cos θ + 2

With the z limits we’ll need to make sure that we convert the x’s into their parameterized form.

In order to evaluate the integral in this case we’ll need the cross product \( \vec{r}_z \times \vec{r}_\theta \) so here is that work.

\[
\vec{r}_z = \langle 0, 0, 1 \rangle \quad \vec{r}_\theta = \langle -2 \sin \theta, 2 \cos \theta, 0 \rangle
\]

\[
\vec{r}_z \times \vec{r}_\theta = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
0 & 0 & 1 \\
-2 \sin \theta & 2 \cos \theta & 0 \\
\end{vmatrix} = -2 \cos \theta \hat{i} - 2 \sin \theta \hat{j}
\]

A unit normal vector for the surface is then,

\[
\vec{n} = \frac{\vec{r}_z \times \vec{r}_\theta}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|} = \frac{\langle -2 \cos \theta, -2 \sin \theta, 0 \rangle}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|}
\]

We didn’t compute the magnitude of the cross product since we know that it will just cancel out when we start working with the integral.

The surface has the negative orientation and so must point in towards the region enclosed by the surface. This means that normal vectors on cylinder will need to point in towards the z-axis and this vector does point in that direction.

To see that this vector points in towards the z-axis consider the \( 0 ≤ θ ≤ \frac{π}{2} \). In this range both sine and cosine are positive and so the x and y component of the normal vector will be negative and so will point in towards the z-axis.

Now we’ll need the following dot product and don’t forget to plug in the parameterization of the surface in the vector field.

\[
\vec{F} \left( \vec{r}(z, \theta) \right) \cdot \vec{n} = \left( 2z \sin \theta, 2 \cos \theta, 12 \sin^2 \theta \right) \cdot \frac{\langle -2 \cos \theta, -2 \sin \theta, 0 \rangle}{\left\| \vec{r}_z \times \vec{r}_\theta \right\|}
\]

\[
= \frac{1}{V_{z \times \theta}} \left( -4z \sin \theta \cos \theta - 4 \sin \theta \cos \theta \right)
\]

\[
= \frac{1}{V_{z \times \theta}} \left( -4 \sin \theta \cos (z + 1) \right)
\]

The integral is then,
\[
\iiint_{S_1} \vec{F} \cdot d\vec{S} = \iiint_{S_1} \frac{1}{|\vec{r}_z \times \vec{r}_\theta|} \left[-2 \sin(2\theta)(z+1)\right] dS \\
= \oiint_{D} \frac{1}{|\vec{r}_z \times \vec{r}_\theta|} \left[-2 \sin(2\theta)(z+1)\right] \|\vec{r}_z \times \vec{r}_\theta\| dA \\
= \oiint_{D} -2 \sin(2\theta)(z+1) dA
\]

In this case \(D\) is is nothing more than the limits on \(z\) and \(\theta\) we gave above and so we can now finish out the integral.

\[
\iiint_{S_1} \vec{F} \cdot d\vec{S} = \oiint_{D} -4 \sin \theta \cos \theta (z+1) dA \\
= \int_0^{2\pi} \int_0^{2\cos \theta + 2} -4 \sin \theta \cos \theta (z+1) dz \ d\theta \\
= \int_0^{2\pi} -4 \sin \theta \cos \theta \left(\frac{1}{2}z^2 + z\right)_{0}^{2\cos \theta + 2} d\theta \\
= \int_0^{2\pi} -16 \sin \theta \cos \theta - 24 \sin \theta \cos^2 \theta - 8 \sin \theta \cos^3 \theta d\theta \\
= \int_0^{2\pi} -8 \sin(2\theta) - 24 \sin \theta \cos^2 \theta - 8 \sin \theta \cos^3 \theta d\theta \\
= \left(4 \cos(2\theta) + 8 \cos^3 \theta + 2 \cos^4 \theta\right)\bigg|_0^{2\pi} = 0
\]

Step 3
Next we'll take care of \(S_2\). In this case the equation for the surface is can be written as \(z-x-2=0\) and \(D\) is the disk \(x^2 + y^2 \leq 4\).

A unit normal vector for \(S_2\) is then,

\[
\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -1, 0, 1 \rangle}{\|\nabla f\|}
\]

The region has the negative orientation and so must point into the enclosed region and so must point downwards (since this is the top “cap” of the cylinder). The normal vector above points upwards (it has a positive \(z\) component) and so we’ll need to multiply this by minus one to get the normal vector we need for this surface.

The correct normal vector is then,

\[
\vec{n} = -\frac{\nabla f}{\|\nabla f\|} = -\frac{\langle 1, 0, -1 \rangle}{\|\nabla f\|}
\]

We didn’t compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.
Calculus II

The dot product we’ll need for this surface is,

\[
\vec{F}(x, y, x + 2) \cdot \vec{n} = \left(y(x + 2), x, 3y^2\right) \cdot \frac{(1, 0, -1)}{\|\nabla f\|} = \frac{1}{\|\nabla f\|}(xy + 2y - 3y^2)
\]

Don’t forget to plug the equation of the surface into \(z\) in the vector field.

The integral is then,

\[
\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{S_2} \frac{1}{\|\nabla f\|}(xy + 2y - 3y^2) dS
= \iint_{D} \frac{1}{\|\nabla f\|} (xy + 2y - 3y^2) \|\nabla f\| dA
= \iint_{D} xy + 2y - 3y^2 dA
\]

Note that we’ll need to finish this integral with polar coordinates and the polar limits will be,

\[
0 \leq \theta \leq 2\pi \\
0 \leq r \leq 2
\]

The integral is then,

\[
\iint_{S_2} \vec{F} \cdot d\vec{S} = \iint_{D} xy + 2y - 3y^2 dA
= \int_{0}^{2\pi} \int_{0}^{2} \left(r^2 \sin \theta \cos \theta + 2r \sin \theta - 3r^2 \sin^2 \theta\right)(r) \, dr \, d\theta
= \int_{0}^{2\pi} \int_{0}^{2} \frac{1}{2} r^3 \sin(2\theta) + 2r^2 \sin \theta - \frac{3}{8} r^3 \left(1 - \cos(2\theta)\right) \, dr \, d\theta
= \int_{0}^{2\pi} \frac{1}{8} r^4 \sin(2\theta) + \frac{3}{2} r^3 \sin \theta - \frac{3}{8} r^4 \left(1 - \cos(2\theta)\right) \, d\theta
= \int_{0}^{2\pi} 2 \sin(2\theta) + \frac{16}{3} \sin \theta - 6(1 - \cos(2\theta)) \, d\theta
= \left(-\cos(2\theta) - \frac{16}{3} \sin \theta - 6(\theta - \frac{1}{2} \sin(2\theta))\right)_{\theta=0}^{2\pi} = -12\pi
\]

Step 4

Finally, let’s integrate over \(S_3\). In this case the equation for the surface is can be written as \(z = x + 3 = 0\) and \(D\) is the disk \(x^2 + y^2 \leq 4\).

A unit normal vector for \(S_2\) is then,
\[ \vec{n} = \frac{\nabla f}{\| \nabla f \|} = \frac{\langle -1, 0, 1 \rangle}{\| \nabla f \|} \]

The region has the negative orientation and so must point into the enclosed region and so must point upwards (since this is the bottom “cap” of the cylinder). The normal vector above does point upwards (it has a positive \( z \) component) and so is the normal vector we’ll need.

We didn’t compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

The dot product we’ll need for this surface is,

\[ \vec{F}(x, y, x-3) \cdot \vec{n} = \langle y(x-3), x, 3y^2 \rangle \cdot \langle -1, 0, 1 \rangle \]

\[ = \frac{1}{\| \nabla f \|}(-xy + 3y + 3y^2) \]

Don’t forget to plug the equation of the surface into \( z \) in the vector field.

The integral is then,

\[ \iint_{S_3} \vec{F} \cdot d\vec{S} = \iint_{S_5} \frac{1}{\| \nabla f \|}(-xy + 3y + 3y^2) \, dS \]

\[ = \iint_{D} \frac{1}{\| \nabla f \|}(-xy + 3y + 3y^2) \| \nabla f \| \, dA \]

\[ = \iint_{D} -xy + 3y + 3y^2 \, dA \]

Note that we’ll need to finish this integral with polar coordinates and the polar limits will be,

\[ 0 \leq \theta \leq 2\pi \]

\[ 0 \leq r \leq 2 \]

The integral is then,
\[
\iint_S \vec{F} \cdot d\vec{S} = \iint_D -xy + 3y + 3y^2 \, dA
\]

\[
= \int_0^{2\pi} \int_0^2 \left(-r^2 \sin \theta \cos \theta + 3r \sin \theta + 3r^2 \sin^2 \theta\right)(r) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^2 \left(-\frac{1}{2} r^3 \sin (2\theta) + 3r^2 \sin \theta + \frac{3}{2} r^3 \left(1 - \cos (2\theta)\right)\right) \, dr \, d\theta
\]

\[
= \int_0^{2\pi} \left(-\frac{1}{2} r^4 \sin (2\theta) + r^3 \sin \theta + 3r^4 \left(1 - \cos (2\theta)\right)\right) \bigg|_0^2 \, d\theta
\]

\[
= \int_0^{2\pi} -2 \sin (2\theta) + 6 \sin \theta + 16 \left(1 - \cos (2\theta)\right) \, d\theta
\]

\[
= \left[\cos (2\theta) - 16 \cos \theta + 6 \left(\theta - \frac{1}{2} \sin (2\theta)\right)\right]_0^{2\pi} = 12\pi
\]

Step 5
Now, to get the value of the integral over the full surface all we need to do is sum up the values of each of the integrals over the three surfaces above. Doing this gives,

\[
\iint_S \vec{F} \cdot d\vec{S} = (0) + (-12\pi) + (12\pi) = 0
\]

We put parenthesis around each of the individual integral values just to indicate where each came from. In general these aren’t needed of course.

---

**Stokes’ Theorem**

1. Use Stokes’ Theorem to evaluate \( \iint_S \text{curl} \vec{F} \cdot d\vec{S} \) where \( \vec{F} = y\hat{i} - x\hat{j} + yx^3\hat{k} \) and \( S \) is the portion of the sphere of radius 4 with \( z \geq 0 \) and the upwards orientation.

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface.

Because the orientation of the surface is upwards then all the normal vectors will be pointing outwards. So, if we walk along the edge of the surface, *i.e.* the curve $C$, in the direction indicated with our head pointed away from the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore the direction indicated in the sketch is the positive orientation of $C$.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it upside down on a piece of paper on a table. Sketch a set of axes on the piece of paper that will represent the plane the cup/bowl is sitting on to really help with the visualization. Then cut out a little stick figure and put a face on the “front” side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with its head pointing in the direction of the normal vectors (out away from the sphere/cup/bowl in our case) with its left hand over the surface. The direction that the “face” on the stick figure is facing is the direction you’d need to walk along the surface to get the positive orientation for $C$.

Step 2
We are going to use Stokes’ Theorem in the following direction.

\[ \int_S \text{curl} \vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} \]

We’ve been given the vector field in the problem statement so we don’t need to worry about that. We will need to deal with \( C \).

Because \( C \) is just the curve along the bottom of the upper half of the sphere we can see that \( C \) in fact will be the intersection of the sphere and the \( xy \)-plane (\( i.e. \ z = 0 \)). Therefore, \( C \) is just the circle of radius 4.

If we look at the sphere from above we get the following sketch of \( C \).

The parameterization of \( C \) is given by,

\[ \vec{r} (t) = \langle 4 \cos t, 4 \sin t, 0 \rangle \quad 0 \leq t \leq 2\pi \]

The \( z \) component of the parameterization is zero because \( C \) lies in the \( xy \)-plane.

Step 3
Since we know we’ll need to eventually do the line integral we know we’ll need the following dot product.

\[ \vec{F} \left( \vec{r} (t) \right) \cdot \vec{r}' (t) = \langle 4 \sin t, -4 \cos t, 256 \sin t \cos^3 t \rangle \cdot \langle -4 \sin t, 4 \cos t, 0 \rangle \]
\[ = -16 \sin^2 t - 16 \cos^2 t \]
\[ = -16 \]

Don’t forget to plug the parameterization of \( C \) into the vector field!

Step 4
Okay, let’s go ahead and evaluate the integral using Stokes’ Theorem.
2. Use Stokes’ Theorem to evaluate \[ \iint_S \text{curl} \vec{F} \cdot d\vec{S} \] where \( \vec{F} = (z^2 - 1)i + (z + xy^3)j + 6k \) and \( S \) is the portion of \( x = 6 - 4y^2 - 4z^2 \) in front of \( x = -2 \) with orientation in the negative \( x \)-axis direction.

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface.

Because the orientation of the surface is towards the negative $x$-axis all the normal vectors will be pointing into the region enclosed by the surface. So, if we walk along the edge of the surface, *i.e.* the curve $C$, in the direction indicated with our head pointed into the region enclosed by the surface (*i.e.* in the same direction as the normal vectors) then our left hand will be over the region. Therefore the direction indicated in the sketch is the positive orientation of $C$.

If you have trouble visualizing the direction of the curve simply get a cup or bowl and put it on its side with a piece of paper behind it. Sketch a set of axis on the piece of paper that will represent the plane the cup/bowl is sitting in front of to really help with the visualization. Then cut out a little stick figure and put a face on the “front” side of it and color the left hand a bright color so you can quickly see it. Now, on the edge of the cup/bowl/whatever you place the stick figure with it’s head pointing in the direction of the normal vectors (into the paraboloid/cup/bowl in our case) with its left hand over the surface. The direction that the “face” on the stick figure is facing is the direction you’d need to walk along the surface to get the positive orientation for $C$.

**Step 2**
We are going to use Stokes’ Theorem in the following direction.

$$\int\int_{S} \text{curl} \vec{F} \cdot d\vec{S} = \int_{C} \vec{F} \cdot d\vec{r}$$

We’ve been given the vector field in the problem statement so we don’t need to worry about that. We will need to deal with $C$. 

© 2007 Paul Dawkins
In this case \( C \) is the curve we get by setting the two equations in the problem statement equal. Doing this gives,

\[
-2 = 6 - 4y^2 - 4z^2 \quad \Rightarrow \quad 4y^2 + 4z^2 = 8 \quad \Rightarrow \quad y^2 + z^2 = 2
\]

We will see following sketch of \( C \) if we are in front of the paraboloid and look directly along the \( x \)-axis.

One possible parameterization of \( C \) is given by,

\[
\begin{align*}
\vec{r}(t) & = \left\langle -2, \sqrt{2}\sin t, \sqrt{2}\cos t \right\rangle \\
& \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 0 \leq t \leq 2\pi
\end{align*}
\]

The \( x \) component of the parameterization is \(-2\) because \( C \) lies at \( x = -2 \).

Step 3
Since we know we’ll need to eventually do the line integral we know we’ll need the following dot product.

\[
\begin{align*}
\vec{F} \left( \vec{r}\left( t \right) \right) \cdot \vec{r}'\left( t \right) & = \left\langle 2\cos^2 t - 1, \sqrt{2}\cos t + 4\sqrt{2}\sin^3 t, 6 \right\rangle \cdot \left\langle 0, \sqrt{2}\cos t, -\sqrt{2}\sin t \right\rangle \\
& = \sqrt{2}\cos t \left( \sqrt{2}\cos t + 4\sqrt{2}\sin^3 t \right) - 6\sqrt{2}\sin t \\
& = 2\cos^2 t + 8\cos t\sin^3 t - 6\sqrt{2}\sin t \\
& = \left( 1 - \sin^2 (2t) \right) + 8\cos t\sin^3 t - 6\sqrt{2}\sin t
\end{align*}
\]

Don’t forget to plug the parameterization of \( C \) into the vector field!

We also did a little simplification on the first term with an eye towards the integration.

Step 4
Okay, let’s go ahead and evaluate the integral using Stokes’ Theorem.
3. Use Stokes’ Theorem to evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F} = -yz \mathbf{i} + (4y+1) \mathbf{j} + xy \mathbf{k} \) and \( C \) is the circle of radius 3 at \( y = 4 \) and perpendicular to the \( y \)-axis. \( C \) has a clockwise rotation if you are looking down the \( y \)-axis from the positive \( y \)-axis to the negative \( y \)-axis. See the figure below for a sketch of the curve.

\[
\begin{align*}
\iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S} &= \int_C \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} (1 - \sin(2t)) + 8 \cos t \sin^3 t - 6\sqrt{2} \sin t \, dt \\
&= \left( t + \frac{1}{2} \cos(2t) + 2 \sin^4 t + 6\sqrt{2} \sin t \right) \bigg|_0^{2\pi} = 2\pi
\end{align*}
\]

Step 1
Okay, we are going to use Stokes’ Theorem in the following direction.

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \text{curl} \mathbf{F} \cdot d\mathbf{S}
\]

So, let’s first compute \( \text{curl} \mathbf{F} \) since that is easy enough to compute and might be useful to have when we go to determine the surface \( S \) we’re going to integrate over.

The curl of the vector field is then,
Step 2
Now we need to find a surface $S$ with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation. This can seem to be a daunting task at times but it’s not as bad as it might appear to be. First we know that the boundary curve needs to be a circle. This means that we’re going to be looking for a surface whose cross section is a circle and we know of several surfaces that meet this requirement. We know that spheres, cones and elliptic paraboloids all have circles as cross sections.

The question becomes which of these surfaces would be best for us in this problem. To make this decision remember that we’ll eventually need to plug this surface into the vector field and then take the dot product of this with normal vector (which will also come from the surface of course).

In general it won’t be immediately clear from the curl of the vector field by itself which surface we should use and that is the case here. The curl of the vector field has all three components and none of them are that difficult to deal with but there isn’t anything that suggests one surface might be easier than the other.

So, let’s consider a sphere first. The issue with spheres is that its parameterization and normal vector are lengthy and many lead to messy integrands. So, because the curl of the vector field does have all three components to it which may well lead to long and/or messy integrands we’ll not work with a sphere for this problem.

Now let’s think about a cone. Equations of cones aren’t that bad but they will involve a square root and in this case would need to be in the form $y = \sqrt{ax^2 + bz^2}$ because the boundary curve is centered on the $y$-axis. The normal vector will also contain roots and this will often lead to messy integrands. So, let’s not work with a cone either in this problem.

That leaves elliptic paraboloids and we probably should have considered them first. The equations are simple and the normal vectors are even simpler so they seem like a good choice of surface for this problem.

Note that we’re not saying that spheres and cones are never good choices for the surface. For some vector fields the curl may end up being very simple with one of these surfaces and so they would be perfectly good choices.

Step 3
We have two possibilities for elliptic paraboloids that we could use here. Both will be centered on the $y$-axis but one will open in the negative $y$ direction while the other will open in the positive $y$ direction.

Here is a couple of sketches of possible elliptic paraboloids we could use here.
Let’s get an equation for each of these. Note that for each of these if we set the equation of the paraboloid and the plane \( y = 4 \) equal we need to get the circle \( x^2 + z^2 = 9 \) since this is the boundary curve that should occur at \( y = 4 \).

Let’s get the equation of the first paraboloid (the one that opens in the negative \( y \) direction. We know that the equation of this paraboloid should be \( y = a - x^2 - z^2 \) for some value of \( a \). As noted if we set this equal to \( y = 4 \) and do some simplification we know what equation we should get. So, let’s set the two equations equal.

\[
4 = a - x^2 - z^2 \quad \rightarrow \quad x^2 + z^2 = a - 4 = 9 \quad \rightarrow \quad a = 13
\]

As shown we know that the \( a - 4 \) should be 9 and so we must have \( a = 13 \). Therefore the equation of the paraboloid that open in the negative \( y \) direction is,
Next, let’s get the equation of the paraboloid that opens in the positive $y$ direction. The equation of this paraboloid will be in the form $y = x^2 + y^2 + a$ for some $a$. Setting this equal to $y = 4$ gives,

$$4 = x^2 + z^2 + a \quad \rightarrow \quad x^2 + z^2 = 4 - a = 9 \quad \rightarrow \quad a = -5$$

The equation of the paraboloid that opens in the positive $y$ direction is then,

$$y = x^2 + z^2 - 5$$

Either of these surfaces could be used to do this problem.

Step 4
We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, $C$, that matches the orientation that was given in the problem statement.

We’ll find the normal vectors for each surface despite the fact that we really only need to do it for one of them since we only need one of the surfaces to do the problem as noted in the previous step. Determining the orientation of the surface can be a little tricky for some folks so doing an extra one might help see what’s going on here.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the surface itself. We now need to determine if we are walking along the outside of the surface or the inside of the surface.

If we are walking along the outside of the surface then our heads, and hence the normal vectors, will be pointing away from the region enclosed by the surface. On the other hand if we are walking along the inside of the surface then our heads, and hence the normal vectors, will be pointing into the region enclosed by the surface.

To help visualize this for our two surfaces it might help to get a cup or bowl that we can use to represent the surface. The edge of the cup/bowl will then represent the boundary curve. Next cut out a stick figure and put a face on one side so we know which direction we’ll be walking and brightly color the left hand to make it really clear which side is the left side.

Now, put the cup/bowl on its side so it looks vaguely like the surface we’re working with and put the stick figure on the edge with the face pointing in the direction the curve is moving and the left hand over the cup/bowl. Do we need to put the stick figure on the inside or outside of the cup/bowl to do this?

Okay, let’s do this for the first surface, $y = 13 - x^2 - z^2$. In this case our stick figure would need to be standing on the inside of the cup/bowl/surface. Therefore the normal vectors on the surface would all need to be point in towards the region enclosed by the surface. This also will mean that all the normal vectors will need to have a negative $y$ component. Again, to visualize this take the stick figure and move it into the region and toward the end of cup/bowl/surface and you’ll see it start to point more and more in the negative $y$ direction (and hence will have a negative $y$ component). Note that the $x$ and $z$ component can be either positive or negative depending on just where we are on the interior of the surface.
Now, let’s take a look at the first surface, \( y = x^2 + z^2 - 5 \). For this surface our stick figure would need to be standing on the outside of the cup/bowl/surface. So, in this case, the normal vectors would point out away from the region enclosed by the surface. These will also have a negative \( y \) component and you can use the method we discussed in the above paragraph to help visualize this.

Step 5
We now need to start thinking about actually computing the integral. We’ll write the equation of the surface as,

\[
f(x, y, z) = 13 - x^2 - z^2 - y = 0
\]

A unit normal vector for the surface is then,

\[
\vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\langle -2x, -1, -2z \rangle}{\|\nabla f\|}
\]

We didn’t compute the magnitude of the gradient since we know that it will just cancel out when we start working with the integral.

Note as well that this does have the correct orientation because the \( y \) component is negative.

Next, we’ll need to compute the following dot product.

\[
\text{curl } \vec{F} \cdot \vec{n} = \langle x, -2(13 - x^2 - z^2), z \rangle \cdot \frac{\langle -2x, -1, -2z \rangle}{\|\nabla f\|}
\]

\[
= \frac{1}{\|\nabla f\|} \left( -2x^2 + 2(13 - x^2 - z^2) - 2z^2 \right)
\]

\[
= \frac{1}{\|\nabla f\|} \left( 26 - 4x^2 - 4z^2 \right)
\]

Step 6
Now, applying Stokes’ Theorem to the integral and converting to a “normal” double integral gives,

\[
\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S}
\]

\[
= \iint_S \frac{1}{\|\nabla f\|} \left( 26 - 4x^2 - 4z^2 \right) dS
\]

\[
= \iint_D \left( 26 - 4x^2 - 4z^2 \right) \|\nabla f\| dA
\]

\[
= \iint_D 26 - 4x^2 - 4z^2 dA
\]

Step 7
To finish this integral out then we’ll need to convert to polar coordinates using the following polar coordinates.
Calculus II

\[ x = r \cos \theta \quad \quad z = r \sin \theta \quad \quad x^2 + z^2 = r^2 \]

In this case \( D \) is just the disk \( x^2 + z^2 \leq 9 \) and so the limits for the integral are,

\[
0 \leq \theta \leq 2\pi \\
0 \leq r \leq 3
\]

The integral is then,

\[
\int_{C} \int_{S} \text{curl} \vec{F} \cdot d\vec{S} = \int_{S} \int \text{curl} \vec{F} \cdot d\vec{S} \\
= \int_{0}^{2\pi} \int_{0}^{3} (26 - 4r^2) r \, dr \, d\theta \\
= \int_{0}^{2\pi} \int_{0}^{3} 26r - 4r^3 \, dr \, d\theta \\
= \left[ \int_{0}^{2\pi} \left( 13r^2 - r^4 \right) \right]_{0}^{3} d\theta \\
= \int_{0}^{2\pi} 36 \, d\theta \\
= 72\pi
\]

Don’t forget to pick up an extra \( r \) from converting the \( dA \) to polar coordinates.

4. Use Stokes’ Theorem to evaluate \( \int_{C} \vec{F} \cdot d\vec{r} \) where \( \vec{F} = \left( 3yx^2 + z^3 \right) \hat{i} + y^2 \hat{j} + 4yx^2 \hat{k} \) and \( C \) is is triangle with vertices \((0,0,3), (0,2,0)\) and \((4,0,0)\). \( C \) has a counter clockwise rotation if you are above the triangle and looking down towards the \( xy \)-plane. See the figure below for a sketch of the curve.
Step 1
Okay, we are going to use Stokes’ Theorem in the following direction.

\[ \int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl} \vec{F} \cdot d\vec{S} \]

So, let’s first compute \( \text{curl} \vec{F} \) since that is easy enough to compute and might be useful to have when we go to determine the surface \( S \) we’re going to integrate over.

The curl of the vector field is then,

\[
\text{curl} \vec{F} = \begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
3y^2 + z^3 & y^2 & 4yx^2
\end{vmatrix} = 4x^2\vec{i} + 3z^2\vec{j} - 3x^2\vec{k} - 8yx\vec{j} = 4x^2\vec{i} + (3z^2 - 8yx)\vec{j} - 3x^2\vec{k}
\]

Step 2
Now we need to find a surface \( S \) with an orientation that will have a boundary curve that is the curve shown in the problem statement, including the correct orientation.

In this case we can see that the triangle looks like the portion of a plane and so it makes sense that we use the equation of the plane containing the three vertices for the surface here.

The curl of the vector field looks a little messy so using a plane here might be the best bet from this perspective as well. It will (hopefully) not make the curl of the vector field any messier and the normal vector, which we’ll get from the equation of the plane, will be simple and so shouldn’t make the curl of the vector field any worse.
Calculus II

Step 3
Determining the equation of the plane is pretty simple. We have three points on the plane, the vertices, and so we can quickly determine the equation.

First, let’s “label” the points as follows,

\[ P = (4, 0, 0) \quad Q = (0, 2, 0) \quad R = (0, 0, 3) \]

Then two vectors that must lie in the plane are,

\[ \overrightarrow{QP} = \langle 4, -2, 0 \rangle \quad \overrightarrow{QR} = \langle 0, -2, 3 \rangle \]

To write the equation of a plane recall that we need a normal vector to the plane. Now, we know that the cross product of these two vectors will be orthogonal to both of the vectors. Also, since both of the vectors lie in the plane the cross product will also be orthogonal, or normal, to the plane. In other words, we can use the cross product of these two vectors as the normal vector to the plane.

The cross product is,

\[
\begin{vmatrix}
\vec{i} & \vec{j} & \vec{k} \\
4 & -2 & 0 \\
0 & -2 & 3 \\
\end{vmatrix}
= -6\vec{i} - 12\vec{j} - 8\vec{k}
\]

Now we can use any of the points in our equation. We’ll use \( Q \) for our point. The equation of the plane is then,

\[-6(x - 0) - 12(y - 2) - 8(z - 0) = 0 \quad \rightarrow \quad -6x - 12y - 8z = -24 \]

\[ 3x + 6y + 4z = 12 \]

Note that we divided the equation by –2 to make the equation a little “nicer” to work with.

Step 4
We now need to determine the orientation of the normal vectors that will induce a positive orientation of the boundary curve, \( C \), that matches the orientation that was given in the problem statement.

Remember that what we want to do here is think of ourselves as walking along the boundary curve of the surface in the direction indicated while our left hand is over the plane. We now need to determine if we are walking along the top or bottom of the plane.

If we are walking along the top of the plane then our heads, and hence the normal vectors, will be pointing in a generally upwards direction. On the other hand if we are walking along the bottom of the plane then our heads, and hence the normal vectors, will be pointing generally downwards.

To help visualize this for our plane it might help to cut out a triangular piece of paper that we can use to represent the plane. The edge of the piece of paper will then represent the boundary curve. Next cut out a
stick figure and put a face on one side so we know which direction we’ll be walking and brightly color the
to make it really clear which side is the left side.

Now, hold the piece of paper so that it looks vaguely like the surface we’re working with and put the stick
figure on the edge with the face pointing in the direction the curve is moving and the left hand over the
cup/bowl. Doing this we’ll quickly see that we must be walking along the top of the surface. Therefore
the normal vectors on the surface need to be pointing in a generally upwards direction (and hence will
have a positive z component).

Step 5
We now need to start thinking about actually computing the integral. We’ll write the equation of the
surface as,

\[ z = 3 - \frac{3}{4}x - \frac{3}{2}y \]

Recall that if we aren’t going to parameterize the surface we need it to be written as \( z = g(x, y) \) so that
the magnitude of the normal vector will eventually cancel.

Now, that we have the surface written in the “proper” form let’s define,

\[ f(x, y, z) = z - 3 + \frac{3}{4}x + \frac{3}{2}y = 0 \]

The unit normal vector for the surface is then,

\[ \vec{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{\left( \frac{3}{4}, \frac{3}{2}, 1 \right)}{\|\nabla f\|} \]

We didn’t compute the magnitude of the gradient since we know that it will just cancel out when we start
working with the integral.

Note as well that this does have the correct orientation because the \( z \) component is positive.

Next, we’ll need to compute the following dot product.

\[ \text{curl} \vec{F} \cdot \vec{n} = \left( 4x^2, \left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 8yx, -3x^2 \right) \cdot \frac{\left( \frac{3}{4}, \frac{3}{2}, 1 \right)}{\|\nabla f\|} \]

\[ = \frac{1}{\|\nabla f\|} \left( 3x^2 + \frac{9}{2} \left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy - 3x^2 \right) \]

\[ = \frac{1}{\|\nabla f\|} \left[ \frac{9}{2} \left(3 - \frac{3}{4}x - \frac{3}{2}y\right)^2 - 12xy \right] \]

Don’t forget to plug the equation of the surface into the curl of the vector field.

Step 6
Now, applying Stokes’ Theorem to the integral and converting to a “normal” double integral gives,
Calculus II

\[ \int_{c} \vec{F} \cdot d\vec{r} = \iint_{S} \text{curl} \vec{F} \cdot d\vec{S} \]

\[ = \iint_{S} \frac{1}{\sqrt{1}} \left[ \frac{9}{2} \left( 3 - \frac{1}{4} x - \frac{1}{2} y \right)^2 - 12xy \right] dS \]

\[ = \iint_{D} \frac{9}{2} \left( 3 - \frac{1}{4} x - \frac{1}{2} y \right)^2 - 12xy \| \nabla f \| dA \]

\[ = \iint_{D} \frac{9}{2} \left( 3 - \frac{1}{4} x - \frac{1}{2} y \right)^2 - 12xy dA \]

Step 7
To finish this integral we just need to determine \( D \). In this case \( D \) just the triangle in the \( xy \)-plane that lies below the plane. Here is a quick sketch of \( D \).

![Diagram of the triangle in the xy-plane]

3x + 6y = 12

The integral doesn’t seem to suggest one integration order over the other so let’s use the following set of limits for our integral.

\[ 0 \leq x \leq 4 \]
\[ 0 \leq y \leq 2 - \frac{1}{2} x \]

The integral is then,
\[ \int_C \vec{F} \cdot d\vec{r} = \int_S \text{curl} \vec{F} \cdot d\vec{S} \]
\[ = \int_0^4 \int_{2-x}^{2-2x} \left(3 - \frac{3}{4} x - \frac{3}{2} y\right)^2 - 12xy \, dy \, dx \]
\[ = \int_0^4 \left( -\left(3 - \frac{3}{4} x - \frac{3}{2} y\right)^3 - 6xy^2 \right) \bigg|_{2-x}^{2-2x} \, dx \]
\[ = \int_0^4 \left( 3 - \frac{3}{4} x \right)^3 - 6x(2 - \frac{1}{2} x) \, dx \]
\[ = \int_0^4 \left( 3 - \frac{3}{4} x \right)^3 - 24x + 12x^2 - \frac{3}{2} x^3 \, dx \]
\[ = \left( -\frac{1}{3} \left(3 - \frac{3}{4} x\right)^4 - 12x^2 + 4x^3 - \frac{3}{8} x^4 \right) \bigg|_0^4 \]
\[ = -5 \]

**Divergence Theorem**

1. Use the Divergence Theorem to evaluate \( \int_S \vec{F} \cdot d\vec{S} \) where \( \vec{F} = yx^2 \, \vec{i} + \left(x y^2 - 3z^4\right) \, \vec{j} + \left(x^3 + y^2\right) \, \vec{k} \)

and \( S \) is the surface of the sphere of radius 4 with \( z \leq 0 \) and \( y \leq 0 \). Note that all three surfaces of this solid are included in \( S \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface.

Note as well here that because we are including all three surfaces shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) the portion of the sphere shown above.

Step 2
We are going to use Stokes’ Theorem in the following direction.
Calculus II

\[ \iiint_{S} \vec{F} \cdot d\vec{S} = \iiint_{E} \text{div} \vec{F} \, dV \]

where \( E \) is just the solid shown in the sketches from Step 1.

Because \( E \) is a portion of a sphere we’ll be wanting to use spherical coordinates for the integration. Here are the spherical limits we’ll need to use for this region.

\[
\begin{align*}
\pi & \leq \theta \leq 2\pi \\
\frac{\pi}{2} & \leq \varphi \leq \pi \\
0 & \leq \rho \leq 4
\end{align*}
\]

One of the restrictions on the region in the problem statement was \( y \leq 0 \). This means that if we look at this from above we’d see the portion of the circle of radius 4 that is below the \( x \) axis and so we need the given range of \( \theta \) above to cover this region.

We were also told in the problem statement that \( z \leq 0 \) and so we only want the portion of the sphere that is below the \( xy \)-plane. We therefore need the given range of \( \varphi \) to make sure we are only below the \( xy \)-plane.

We’ll also need the divergence of the vector field so here is that.

\[
\text{div} \vec{F} = \frac{\partial}{\partial x} \left( yx^2 \right) + \frac{\partial}{\partial y} \left( xy^2 - 3z^4 \right) + \frac{\partial}{\partial z} \left( x^3 + y^2 \right) = 4xy
\]

Step 3
Now let’s apply the Divergence Theorem to the integral and get it converted to spherical coordinates while we’re at it.

\[
\iiint_{S} \vec{F} \cdot d\vec{S} = \iiint_{E} \text{div} \vec{F} \, dV = \iiint_{E} 4xy \, dV
\]

\[
= \int_{0}^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{4} 4(\rho \sin \varphi \cos \theta)(\rho \sin \varphi \sin \theta)(\rho^2 \sin \varphi) \, d\rho \, d\varphi \, d\theta
\]

\[
= \int_{0}^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_{0}^{4} 4\rho^4 \sin^3 \varphi \cos \theta \sin \theta \, d\rho \, d\varphi \, d\theta
\]

Don’t forget to pick up the \( \rho^2 \sin \varphi \) when converting the \( dV \) to spherical coordinates.

Step 4
All we need to do then in evaluate the integral.
\[ \iint_S \vec{F} \cdot d\vec{S} = \int_{\pi}^{2\pi} \int_{0}^{\pi} 4\rho^4 \sin^3 \varphi \cos \theta \sin \theta \rho \, d\varphi \, d\theta \]
\[ = \int_{\pi}^{2\pi} \int_{0}^{\pi} \left( \frac{1}{3} \rho^5 \sin \varphi \cos \theta \sin \theta \right) \rho \, d\varphi \, d\theta \]
\[ = \int_{\pi}^{2\pi} \int_{0}^{\pi} \frac{4096}{5} \sin \varphi \left( 1 - \cos^2 \varphi \right) \cos \theta \sin \theta \, d\varphi \, d\theta \]
\[ = \int_{\pi}^{2\pi} \left( -\frac{4096}{5} \left( \cos \varphi - \frac{1}{3} \cos^3 \varphi \right) \cos \theta \sin \theta \right) \bigg|^{\pi}_{\frac{\pi}{4}} \, d\theta \]
\[ = \int_{\pi}^{2\pi} \frac{8192}{15} \cos \theta \sin \theta \, d\theta \]
\[ = \int_{\pi}^{2\pi} \frac{4096}{15} \sin (2\theta) \, d\theta \]
\[ = \frac{2048}{15} \cos (2\theta) \bigg|^{2\pi}_{\pi} = 0 \]

Make sure you can do use your trig formulas as we did here to deal with these kinds of integrals!

---

2. Use the Divergence Theorem to evaluate \( \iint_S \vec{F} \cdot d\vec{S} \) where \( \vec{F} = \sin (\pi x) \hat{i} + zy^3 \hat{j} + \left( z^2 + 4x \right) \hat{k} \) and \( S \) is the surface of the box with \(-1 \leq x \leq 2\), \(0 \leq y \leq 1\) and \(1 \leq z \leq 4\). Note that all six sides of the box are included in \( S \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface.

Note as well here that because we are including all six sides of the box shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) for the box.

Step 2
We are going to use Stokes’ Theorem in the following direction.

\[ \iint_s \mathbf{F} \cdot d\mathbf{S} = \iiint_E \text{div} \mathbf{F} \ dV \]

where \( E \) is just the solid shown in the sketches from Step 1.

\( E \) is just a box and the limits defining it where given in the problem statement. The limits for our integral will then be,
Calculus II

\[-1 \leq x \leq 2\]
\[0 \leq y \leq 1\]
\[1 \leq z \leq 4\]

We’ll also need the divergence of the vector field so here is that.

\[
\text{div} \vec{F} = \frac{\partial}{\partial x} (\sin(\pi x)) + \frac{\partial}{\partial y} (zy^3) + \frac{\partial}{\partial z} (z^2 + 4x) = \pi \cos(\pi x) + 3zy^2 + 2z
\]

Step 3
Now let’s apply the Divergence Theorem to the integral and get it converted to a triple integral.

\[
\iiint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div} \vec{F} \, dV
\]

\[
= \iiint_E \pi \cos(\pi x) + 3zy^2 + 2z \, dV
\]

\[
= \int_{-1}^{1} \int_{0}^{4} \int_{1}^{2} \pi \cos(\pi x) + 3zy^2 + 2z \, dz \, dy \, dx
\]

Step 4
All we need to do then in evaluate the integral.

\[
\iiint_S \vec{F} \cdot d\vec{S} = \int_{-1}^{1} \int_{0}^{4} \int_{1}^{2} \pi \cos(\pi x) + 3zy^2 + 2z \, dz \, dy \, dx
\]

\[
= \int_{-1}^{1} \int_{0}^{4} \left( \pi z \cos(\pi x) + \frac{3}{2} z^2 y^2 + z^2 \right) \, dy \, dx
\]

\[
= \int_{-1}^{1} \int_{0}^{4} 3\pi \cos(\pi x) + \frac{45}{2} y^2 + 15 \, dy \, dx
\]

\[
= \int_{-1}^{1} \left( 3y \pi \cos(\pi x) + \frac{15}{2} y^3 + 15y \right) \, dx
\]

\[
= \int_{-1}^{1} 3\pi \cos(\pi x) + \frac{45}{2} \, dx
\]

\[
= \left( 3 \sin(\pi x) + \frac{45}{2} x \right) \Big|_{-1}^{1} = \frac{13\pi}{2}
\]

3. Use the Divergence Theorem to evaluate \( \iint_S \vec{F} \cdot d\vec{S} \) where \( \vec{F} = 2xz\vec{i} + (1 - 4xy^2) \vec{j} + (2z - z^2) \vec{k} \) and \( S \) is the surface of the solid bounded by \( z = 6 - 2x^2 - 2y^2 \) and the plane \( z = 0 \). Note that both of the surfaces of this solid included in \( S \).

Step 1
Let’s start off with a quick sketch of the surface we are working with in this problem.
We included a sketch with traditional axes and a sketch with a set of “box” axes to help visualize the surface. The bottom “cap” of the elliptic paraboloid is also included in the surface but isn’t shown.

Note as well here that because we are including both of the surfaces shown above that the surface does enclose (or is the boundary curve if you want to use that terminology) the region.

Step 2
We are going to use Stokes’ Theorem in the following direction.

$$\iint_{S} \vec{F} \cdot d\vec{S} = \iiint_{E} \text{div} \vec{F} \ dV$$

where $E$ is just the solid shown in the sketches from Step 1.

The region $D$ for that we’ll need in converting the triple integral into iterated integrals is just the intersection of the two surfaces from the problem statement. This is,

$$0 = 6 - 2x^2 - 2y^2 \quad \rightarrow \quad x^2 + y^2 = 3$$

So, $D$ is a disk and so we’ll eventually be doing cylindrical coordinates for this integral. Here are the cylindrical limits for the region $E$.

$$0 \leq \theta \leq 2\pi$$

$$0 \leq r \leq \sqrt{3}$$

$$0 \leq z \leq 6 - 2x^2 - 2y^2 = 6 - 2r^2$$

Don’t forget to convert the $z$ limits into cylindrical coordinates as well!
We’ll also need the divergence of the vector field so here is that.

\[ \text{div } \vec{F} = \frac{\partial}{\partial x} (2xz) + \frac{\partial}{\partial y} (1 - 4xy^2) + \frac{\partial}{\partial z} (2z - z^2) = 2 - 8xy \]

Step 3
Now let’s apply the Divergence Theorem to the integral and get it converted to spherical coordinates while we’re at it.

\[
\iiint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div } \vec{F} \ dV
\]

\[
= \iiint_E (2 - 8xy) \ dV
\]

\[
= \int_0^{2\pi} \int_0^\pi \int_0^{\sqrt{6 - 2r^2}} (2 - 8r^2 \cos \theta \sin \theta) r 
\]

\[
= \int_0^{2\pi} \int_0^\pi \int_0^{\sqrt{6 - 2r^2}} 2r - 8r^3 \cos \theta \sin \theta \ dz \ dr \ d\theta
\]

Don’t forget to pick up the \( r \) when converting the \( dV \) to spherical coordinates.

Step 4
All we need to do then in evaluate the integral.

\[
\iiint_S \vec{F} \cdot d\vec{S} = \int_0^{2\pi} \int_0^\pi \int_0^{\sqrt{6 - 2r^2}} 2r - 8r^3 \cos \theta \sin \theta \ dz \ dr \ d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi \left( 2r - 8r^3 \cos \theta \sin \theta \right) \left|_{r=0}^{r=\sqrt{6 - 2r^2}} \right| \ dr \ d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi (2r - 8r^3 \cos \theta \sin \theta) (6 - 2r^2) \ dr \ d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi 12r - 4r^3 - (48r^3 - 16r^5) \cos \theta \sin \theta \ dr \ d\theta
\]

\[
= \int_0^{2\pi} \left( 6r^2 - r^4 - \left( 12r^4 - \frac{8}{5}r^6 \right) \cos \theta \sin \theta \right) \left|_0^{\sqrt{6 - 2r^2}} \right| \ d\theta
\]

\[
= \int_0^{2\pi} 9 - 36 \cos \theta \sin \theta \ d\theta
\]

\[
= \int_0^{2\pi} 9 - 18 \sin(2\theta) \ d\theta
\]

\[
= (9\theta - 9 \cos(2\theta)) \left|_0^{2\pi} \right. = 18\pi
\]