Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Systems of Differential Equations

In the introduction to this section we briefly discussed how a system of differential equations can arise from a population problem in which we keep track of the population of both the prey and the predator. It makes sense that the number of prey present will affect the number of the predator present. Likewise, the number of predator present will affect the number of prey present. Therefore the differential equation that governs the population of either the prey or the predator should in some way depend on the population of the other. This will lead to two differential equations that must be solved simultaneously in order to determine the population of the prey and the predator.

The whole point of this is to notice that systems of differential equations can arise quite easily from naturally occurring situations. Developing an effective predator-prey system of differential equations is not the subject of this chapter. However, systems can arise from \( n^{th} \) order linear differential equations as well. Before we get into this however, let’s write down a system and get some terminology out of the way.

We are going to be looking at first order, linear systems of differential equations. These terms mean the same thing that they have meant up to this point. The largest derivative anywhere in the system will be a first derivative and all unknown functions and their derivatives will only occur to the first power and will not be multiplied by other unknown functions. Here is an example of a system of first order, linear differential equations.

\[
\begin{align*}
\frac{dx_1}{dt} &= x_1 + 2x_2 \\
\frac{dx_2}{dt} &= 3x_1 + 2x_2
\end{align*}
\]

We call this kind of system a coupled system since knowledge of \( x_2 \) is required in order to find \( x_1 \) and likewise knowledge of \( x_1 \) is required to find \( x_2 \). We will worry about how to go about solving these later. At this point we are only interested in becoming familiar with some of the basics of systems.

Now, as mentioned earlier, we can write an \( n^{th} \) order linear differential equation as a system. Let’s see how that can be done.

**Example 1** Write the following 2\(^{nd}\) order differential equation as a system of first order, linear differential equations.

\[
2y'' - 5y' + y = 0 \quad y(3) = 6 \quad y'(3) = -1
\]

**Solution**

We can write higher order differential equations as a system with a very simple change of variable. We’ll start by defining the following two new functions.

\[
\begin{align*}
x_1(t) &= y'(t) \\
x_2(t) &= y(t)
\end{align*}
\]

Now notice that if we differentiate both sides of these we get,

\[
\begin{align*}
x_1' &= y'' = x_2 \\
x_2' &= y' = -\frac{1}{2}y + \frac{5}{2}y' = -\frac{1}{2}x_1 + \frac{5}{2}x_2
\end{align*}
\]
Note the use of the differential equation in the second equation. We can also convert the initial conditions over to the new functions.

\[ x_1(3) = y(3) = 6 \]
\[ x_2(3) = y'(3) = -1 \]

Putting all of this together gives the following system of differential equations.

\[
\begin{align*}
& x_1' = x_2 & & x_1(3) = 6 \\
& x_2' = -\frac{1}{2}x_1 + \frac{5}{2}x_2 & & x_2(3) = -1
\end{align*}
\]

We will call the system in the above example an **Initial Value Problem** just as we did for differential equations with initial conditions.

Let’s take a look at another example.

**Example 2** Write the following 4th order differential equation as a system of first order, linear differential equations.

\[ y^{(4)} + 3y'' - \sin(t)y' + 8y = t^2 \quad y(0) = 1 \quad y'(0) = 2 \quad y''(0) = 3 \quad y'''(0) = 4 \]

**Solution**
Just as we did in the last example we’ll need to define some new functions. This time we’ll need 4 new functions.

\[
\begin{align*}
x_1 &= y & \Rightarrow & & x_1' &= y' = x_2 \\
x_2 &= y' & \Rightarrow & & x_2' &= y'' = x_3 \\
x_3 &= y'' & \Rightarrow & & x_3' &= y''' = x_4 \\
x_4 &= y''' & \Rightarrow & & x_4' &= y^{(4)} = -8y + \sin(t)y' - 3y'' + t^2 = -8x_1 + \sin(t)x_2 - 3x_3 + t^2
\end{align*}
\]

The system along with the initial conditions is then,

\[
\begin{align*}
x_1' &= x_2 & & x_1(0) = 1 \\
x_2' &= x_3 & & x_2(0) = 2 \\
x_3' &= x_4 & & x_3(0) = 3 \\
x_4' &= -8x_1 + \sin(t)x_2 - 3x_3 + t^2 & & x_4(0) = 4
\end{align*}
\]

Now, when we finally get around to solving these we will see that we generally don’t solve systems in the form that we’ve given them in this section. Systems of differential equations can be converted to **matrix form** and this is the form that we usually use in solving systems.
Example 3  Convert the following system to matrix form.

\[
\begin{align*}
    x_1' & = 4x_1 + 7x_2 \\
    x_2' & = -2x_1 - 5x_2
\end{align*}
\]

Solution
First write the system so that each side is a vector.

\[
\begin{pmatrix}
    x_1' \\
    x_2'
\end{pmatrix} = \begin{pmatrix}
    4x_1 + 7x_2 \\
    -2x_1 - 5x_2
\end{pmatrix}
\]

Now the right side can be written as a matrix multiplication,

\[
\begin{pmatrix}
    x_1' \\
    x_2'
\end{pmatrix} = \begin{pmatrix}
    4 & 7 \\
    -2 & -5
\end{pmatrix} \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

Now, if we define,

\[
\vec{x} = \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

then,

\[
\vec{x}' = \begin{pmatrix}
    x_1' \\
    x_2'
\end{pmatrix}
\]

The system can then be wrote in the matrix form,

\[
\vec{x}' = \begin{pmatrix}
    4 & 7 \\
    -2 & -5
\end{pmatrix} \vec{x}
\]

Example 4  Convert the systems from Examples 1 and 2 into matrix form.

Solution
We’ll start with the system from Example 1.

\[
\begin{align*}
    x_1' & = x_2 \\
    x_2' & = -\frac{1}{2}x_1 + \frac{5}{2}x_2
\end{align*}
\]

First define,

\[
\vec{x} = \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\]

The system is then,

\[
\vec{x}' = \begin{pmatrix}
    0 & 1 \\
    -\frac{1}{2} & \frac{5}{2}
\end{pmatrix} \vec{x}
\]

\[
\vec{x}(3) = \begin{pmatrix}
    x_1(3) \\
    x_2(3)
\end{pmatrix} = \begin{pmatrix}
    6 \\
    -1
\end{pmatrix}
\]

Now, let’s do the system from Example 2.
\[\begin{align*}
x'_1 &= x_2 & x_1(0) &= 1 \\
x'_2 &= x_3 & x_2(0) &= 2 \\
x'_3 &= x_4 & x_3(0) &= 3 \\
x'_4 &= -8x_1 + \sin(t)x_2 - 3x_3 + t^2 & x_4(0) &= 4
\end{align*}\]

In this case we need to be careful with the \( t^2 \) in the last equation. We'll start by writing the system as a vector again and then break it up into two vectors, one vector that contains the unknown functions and the other that contains any known functions.

\[
\begin{pmatrix}
x'_1 \\
x'_2 \\
x'_3 \\
x'_4
\end{pmatrix} =
\begin{pmatrix}
x_2 \\
x_3 \\
x_4 \\
-8x_1 + \sin(t)x_2 - 3x_3 + t^2
\end{pmatrix} =
\begin{pmatrix}
x_2 \\
x_3 \\
x_4 \\
-8x_1 + \sin(t)x_2 - 3x_3
\end{pmatrix} +
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Now, the first vector can now be written as a matrix multiplication and we'll leave the second vector alone.

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-8 & \sin(t) & -3 & 0
\end{pmatrix}
\begin{pmatrix}
x(t)_1 \\
x(t)_2 \\
x(t)_3 \\
x(t)_4
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
t^2
\end{pmatrix}
\]

where,

\[
\begin{pmatrix}
x(t)_1 \\
x(t)_2 \\
x(t)_3 \\
x(t)_4
\end{pmatrix}
= \begin{pmatrix}
x(0)_1 \\
x(0)_2 \\
x(0)_3 \\
x(0)_4
\end{pmatrix}
\]

Note that occasionally for “large” systems such as this we will go one step farther and write the system as,

\[
\ddot{x}' = A\ddot{x} + \ddot{g}(t)
\]

The last thing that we need to do in this section is get a bit of terminology out of the way.

Starting with

\[
\ddot{x}' = A\ddot{x} + \ddot{g}(t)
\]

we say that the system is **homogeneous** if \( \ddot{g}(t) = \ddot{0} \) and we say the system is **nonhomogeneous** if \( \ddot{g}(t) \neq \ddot{0} \).