Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Review : Taylor Series

We are not going to be doing a whole lot with Taylor series once we get out of the review, but they are a nice way to get us back into the swing of dealing with power series. By time most students reach this stage in their mathematical career they’ve not had to deal with power series for at least a semester or two. Remembering how Taylor series work will be a very convenient way to get comfortable with power series before we start looking at differential equations.

Taylor Series

If \( f(x) \) is an infinitely differentiable function then the Taylor Series of \( f(x) \) about \( x=x_0 \) is,

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n
\]

Recall that

\[
f^{(0)}(x) = f(x) \quad \quad f^{(n)}(x) = \text{n}^{\text{th}} \text{ derivative of } f(x)
\]

Let’s take a look at an example.

Example 1  Determine the Taylor series for \( f(x) = e^x \) about \( x=0 \).

Solution

This is probably one of the easiest functions to find the Taylor series for. We just need to recall that,

\[
f^{(n)}(x) = e^x \quad n = 0,1,2,\ldots
\]

and so we get,

\[
f^{(n)}(0) = 1 \quad n = 0,1,2,\ldots
\]

The Taylor series for this example is then,

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}
\]

Of course, it’s often easier to find the Taylor series about \( x=0 \) but we don’t always do that.

Example 2  Determine the Taylor series for \( f(x) = e^x \) about \( x=-4 \).

Solution

This problem is virtually identical to the previous problem. In this case we just need to notice that,

\[
f^{(n)}(-4) = e^{-4} \quad n = 0,1,2,\ldots
\]

The Taylor series for this example is then,

\[
e^x = \sum_{n=0}^{\infty} \frac{e^{-4}}{n!}(x+4)^n
\]
Let's now do a Taylor series that requires a little more work.

**Example 3** Determine the Taylor series for \( f(x) = \cos(x) \) about \( x=0 \).

**Solution**
This time there is no formula that will give us the derivative for each \( n \) so let's start taking derivatives and plugging in \( x=0 \).

\[
\begin{align*}
  f^{(0)}(x) &= \cos(x) & f^{(0)}(0) &= 1 \\
  f^{(1)}(x) &= -\sin(x) & f^{(1)}(0) &= 0 \\
  f^{(2)}(x) &= -\cos(x) & f^{(2)}(0) &= -1 \\
  f^{(3)}(x) &= \sin(x) & f^{(3)}(0) &= 0 \\
  f^{(4)}(x) &= \cos(x) & f^{(4)}(0) &= 1 \\
  &\vdots & & \vdots
\end{align*}
\]

Once we reach this point it’s fairly clear that there is a pattern emerging here. Just what this pattern is has yet to be determined, but it does seem fairly clear that a pattern does exist.

Let's plug what we’ve got into the formula for the Taylor series and see what we get.

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{f^{(0)}(0)}{0!} + \frac{f^{(1)}(0)}{1!} x + \frac{f^{(2)}(0)}{2!} x^2 + \frac{f^{(3)}(0)}{3!} x^3 + \ldots
\]

\[
= \frac{1}{0!} + 0 - \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + 0 - \frac{x^6}{6!} + 0 + \frac{x^8}{8!} + \ldots
\]

So, every other term is zero.

We would like to write this in terms of a series, however finding a formula that is zero every other term and gives the correct answer for those that aren’t zero would be unnecessarily complicated. So, let’s rewrite what we’ve got above and while were at it renumber the terms as follows,

\[
\cos(x) = \frac{1}{0!} + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!} + \ldots
\]

With this “renumbering” we can fairly easily get a formula for the Taylor series of the cosine function about \( x=0 \).

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}
\]

For practice you might want to see if you can verify that the Taylor series for the sine function about \( x=0 \) is,
\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

We need to look at one more example of a Taylor series. This example is both tricky and very easy.

**Example 4** Determine the Taylor series for \( f(x) = 3x^2 - 8x + 2 \) about \( x=2 \).

**Solution**

There’s not much to do here except to take some derivatives and evaluate at the point.

\[
\begin{align*}
f(x) &= 3x^2 - 8x + 2 \\
f'(x) &= 6x - 8 \\
f''(x) &= 6 \\
f^{(n)}(x) &= 0, \ n \geq 3
\end{align*}
\]

So, in this case the derivatives will all be zero after a certain order. That happens occasionally and will make our work easier. Setting up the Taylor series then gives,

\[
3x^2 - 8x + 2 = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n
\]

\[
= \frac{f^{(0)}(2)}{0!} + \frac{f^{(1)}(2)}{1!} (x-2) + \frac{f^{(2)}(2)}{2!} (x-2)^2 + \frac{f^{(3)}(2)}{3!} (x-2)^3 + \cdots
\]

\[
= -2 + 4(x-2) + \frac{6}{2} (x-2)^2 + 0
\]

\[
= -2 + 4(x-2) + 3(x-2)^2
\]

In this case the Taylor series terminates and only had three terms. Note that since we are after the Taylor series we do not multiply the 4 through on the second term or square out the third term. All the terms with the exception of the constant should contain an \( x^2 \).

Note in this last example that if we were to multiply the Taylor series we would get our original polynomial. This should not be too surprising as both are polynomials and they should be equal.

We now need a quick definition that will make more sense to give here rather than in the next section where we actually need it since it deals with Taylor series.

**Definition**

A function, \( f(x) \), is called **analytic** at \( x=a \) if the Taylor series for \( f(x) \) about \( x=a \) has a positive radius of convergence and converges to \( f(x) \).

We need to give one final note before proceeding into the next section. We started this section out by saying that we weren’t going to be doing much with Taylor series after this section. While that is correct it is only correct because we are going to be keeping the problems fairly simple. For more complicated problems we would also be using quite a few Taylor series.