Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Implicit Differentiation

1. For \( \frac{x}{y^3} = 1 \) do each of the following.
   (a) Find \( y' \) by solving the equation for \( y \) and differentiating directly.
   (b) Find \( y' \) by implicit differentiation.
   (c) Check that the derivatives in (a) and (b) are the same.

(a) Find \( y' \) by solving the equation for \( y \) and differentiating directly.
Step 1
First we just need to solve the equation for \( y \).
\[ y^3 = x \quad \Rightarrow \quad y = x^{\frac{1}{3}} \]
Step 2
Now differentiate with respect to \( x \).
\[ y' = \frac{1}{3} x^{-\frac{2}{3}} \]

(b) Find \( y' \) by implicit differentiation.
Hint: Don’t forget that $y$ is really $y(x)$ and so we’ll need to use the Chain Rule when taking the derivative of terms involving $y$! Also, don’t forget that because $y$ is really $y(x)$ we may well have a Product and/or a Quotient Rule buried in the problem.

Step 1
First, we just need to take the derivative of everything with respect to $x$ and we’ll need to recall that $y$ is really $y(x)$ and so we’ll need to use the Chain Rule when taking the derivative of terms involving $y$.

Also, prior to taking the derivative a little rewrite might make this a little easier.

$$x y^{-3} = 1$$

Now take the derivative and don’t forget that we actually have a product of functions of $x$ here and so we’ll need to use the Product Rule when differentiating the left side.

$$y^{-3} - 3x y^{-4} y' = 0$$

Step 2
Finally all we need to do is solve this for $y'$.

$$y' = \frac{y^{-3}}{3x y^{-4}} = \frac{y}{3x}$$

(c) Check that the derivatives in (a) and (b) are the same.

Hint: To show they are the same all we need is to plug the formula for $y$ (which we already have....) into the derivative we found in (b) and, potentially with a little work, show that we get the same derivative as we got in (a).

From (a) we have a formula for $y$ written explicitly as a function of $x$ so plug that into the derivative we found in (b) and, with a little simplification/work, show that we get the same derivative as we got in (a).

$$y' = \frac{y}{3x} = \frac{\frac{1}{3} x^{3} }{3x} = \frac{1}{3} x^{-\frac{2}{3}}$$

So, we got the same derivative as we should.

2. For $x^2 + y^3 = 4$ do each of the following.

(d) Find $y'$ by solving the equation for $y$ and differentiating directly.

(e) Find $y'$ by implicit differentiation.

(f) Check that the derivatives in (a) and (b) are the same.
(a) Find $y'$ by solving the equation for $y$ and differentiating directly.

Step 1
First we just need to solve the equation for $y$.

\[ y^3 = 4 - x^2 \quad \Rightarrow \quad y = \left(4 - x^2\right)^{\frac{1}{3}} \]

Step 2
Now differentiate with respect to $x$.

\[ y' = -\frac{2}{3} x \left(4 - x^2\right)^{-\frac{2}{3}} \]

(b) Find $y'$ by implicit differentiation.

Hint: Don’t forget that $y$ is really $y(x)$ and so we’ll need to use the Chain Rule when taking the derivative of terms involving $y$!

Step 1
First, we just need to take the derivative of everything with respect to $x$ and we’ll need to recall that $y$ is really $y(x)$ and so we’ll need to use the Chain Rule when taking the derivative of terms involving $y$.

Taking the derivative gives,

\[ 2x + 3y^2 y' = 0 \]

Step 2
Finally all we need to do is solve this for $y'$.

\[ y' = \frac{-2x}{3y^2} \]

(c) Check that the derivatives in (a) and (b) are the same.

Hint: To show they are the same all we need is to plug the formula for $y$ (which we already have….) into the derivative we found in (b) and, potentially with a little work, show that we get the same derivative as we got in (a).

From (a) we have a formula for $y$ written explicitly as a function of $x$ so plug that into the derivative we found in (b) and, with a little simplification/work, show that we get the same derivative as we got in (a).

\[ y' = -\frac{2x}{3y^2} = -\frac{2x}{3 \left(4 - x^2\right)^{\frac{2}{3}}} = -\frac{2}{3} x \left(4 - x^2\right)^{-\frac{2}{3}} \]

So, we got the same derivative as we should.
3. For \( x^2 + y^2 = 2 \) do each of the following.

(a) Find \( y' \) by solving the equation for \( y \) and differentiating directly.

(b) Find \( y' \) by implicit differentiation.

(i) Check that the derivatives in (a) and (b) are the same.

(a) Find \( y' \) by solving the equation for \( y \) and differentiating directly.

Step 1
First we just need to solve the equation for \( y \).

\[
y^2 = 2 - x^2 \quad \Rightarrow \quad y = \pm \left(2 - x^2\right)^{\frac{1}{2}}
\]

Note that because we have no restriction on \( y \) (i.e. we don’t know if \( y \) is positive or negative) we really do need to have the “\( \pm \)” there and that does lead to issues when taking the derivative.

Hint : Two formulas for \( y \) and so two derivatives.

Step 2
Now, because there are two formulas for \( y \) we will also have two formulas for the derivative, one for each formula for \( y \).

The derivatives are then,

\[
y = \left(2 - x^2\right)^{\frac{1}{2}} \quad \Rightarrow \quad y' = -x \left(2 - x^2\right)^{\frac{1}{2}} \quad (y > 0)
\]

\[
y = -\left(2 - x^2\right)^{\frac{1}{2}} \quad \Rightarrow \quad y' = x \left(2 - x^2\right)^{-\frac{1}{2}} \quad (y < 0)
\]

As noted above the first derivative will hold for \( y > 0 \) while the second will hold for \( y < 0 \) and we can use either for \( y = 0 \) as the plus/minus won’t affect that case.

(b) Find \( y' \) by implicit differentiation.

Hint : Don’t forget that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \).

Step 1
First, we just need to take the derivative of everything with respect to \( x \) and we’ll need to recall that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \).

Taking the derivative gives,

\[
2x + 2y y' = 0
\]

Step 2
Finally all we need to do is solve this for \( y' \).
(c) Check that the derivatives in (a) and (b) are the same. 
Hint: To show they are the same all we need is to plug the formula for \( y \) (which we already have…) into the derivative we found in (b) and, potentially with a little work, show that we get the same derivative as we got in (a). Again, two formulas for \( y \) so two derivatives…

From (a) we have a formula for \( y \) written explicitly as a function of \( x \) so plug that into the derivative we found in (b) and, with a little simplification/work, show that we get the same derivative as we got in (a).

Also, because we have two formulas for \( y \) we will have two formulas for the derivative.

First, if \( y > 0 \) we will have,
\[
y = (2 - x^2)^{\frac{1}{2}} \quad \Rightarrow \quad y' = -\frac{x}{y} = -\frac{x}{(2 - x^2)^{\frac{1}{2}}} = -x(2 - x^2)^{\frac{-1}{2}}
\]

Next, if \( y < 0 \) we will have,
\[
y = -(2 - x^2)^{\frac{1}{2}} \quad \Rightarrow \quad y' = -\frac{x}{y} = -\frac{x}{-(2 - x^2)^{\frac{1}{2}}} = x(2 - x^2)^{\frac{-1}{2}}
\]

So, in both cases, we got the same derivative as we should.

4. Find \( y' \) by implicit differentiation for \( 2y^3 + 4x^2 - y = x^6 \).

Hint: Don’t forget that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y! \)

Step 1
First, we just need to take the derivative of everything with respect to \( x \) and we’ll need to recall that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \).

Differentiating with respect to \( x \) gives,
\[
6y^2 \ y' + 8x - y' = 6x^5
\]

Step 2
Finally all we need to do is solve this for \( y' \).
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\[
(6y^2 - 1)y' = 6x^5 - 8x \quad \Rightarrow \quad y' = \frac{6x^5 - 8x}{6y^2 - 1}
\]

5. Find \( y' \) by implicit differentiation for \( 7y^2 + \sin(3x) = 12 - y^4 \).

Hint: Don’t forget that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \)!

Step 1
First, we just need to take the derivative of everything with respect to \( x \) and we’ll need to recall that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \).

Differentiating with respect to \( x \) gives,

\[
14y \cdot y' + 3\cos(3x) = -4y^3y'
\]

Step 2
Finally all we need to do is solve this for \( y' \).

\[
(14y + 4y^3)y' = -3\cos(3x) \quad \Rightarrow \quad y' = \frac{-3\cos(3x)}{14y + 4y^3}
\]

6. Find \( y' \) by implicit differentiation for \( e^y - \sin(y) = x \).

Hint: Don’t forget that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \)!

Step 1
First, we just need to take the derivative of everything with respect to \( x \) and we’ll need to recall that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \).

Differentiating with respect to \( x \) gives,

\[
e^y - \cos(y)y' = 1
\]

Don’t forget the \( y' \) on the cosine after differentiating. Again, \( y \) is really \( y(x) \) and so when differentiating \( \sin(y) \) we really differentiating \( \sin[y(x)] \) and so we are differentiating using the Chain Rule!
Step 2
Finally all we need to do is solve this for \( y' \).

\[
y' = \frac{1-e^x}{-\cos(y)} = (e^x - 1) \sec(y)
\]

7. Find \( y' \) by implicit differentiation for \( 4x^2y^7 - 2x = x^5 + 4y^3 \).

Hint: Don't forget that \( y \) is really \( y(x) \) and so we'll need to use the Chain Rule when taking the derivative of terms involving \( y \)! Also, don't forget that because \( y \) is really \( y(x) \) we may well have a Product and/or a Quotient Rule buried in the problem.

Step 1
First, we just need to take the derivative of everything with respect to \( x \) and we'll need to recall that \( y \) is really \( y(x) \) and so we'll need to use the Chain Rule when taking the derivative of terms involving \( y \). This also means that the first term on the left side is really a product of functions of \( x \) and hence we will need to use the Product Rule when differentiating that term.

Differentiating with respect to \( x \) gives,

\[
8xy^7 + 28x^2y^6y' - 2 = 5x^4 + 12y^2y'
\]

Step 2
Finally all we need to do is solve this for \( y' \).

\[
8xy^7 - 5x^4 - 2 = (12y^2 - 28x^2y^6)y' \quad \Rightarrow \quad y' = \frac{8xy^7 - 5x^4 - 2}{12y^2 - 28x^2y^6}
\]

8. Find \( y' \) by implicit differentiation for \( \cos(x^2 + 2y) + x e^{y^2} = 1 \).

Hint: Don't forget that \( y \) is really \( y(x) \) and so we'll need to use the Chain Rule when taking the derivative of terms involving \( y \)! Also, don't forget that because \( y \) is really \( y(x) \) we may well have a Product and/or a Quotient Rule buried in the problem.

Step 1
First, we just need to take the derivative of everything with respect to \( x \) and we'll need to recall that \( y \) is really \( y(x) \) and so we'll need to use the Chain Rule when taking the derivative of terms involving \( y \). This also means that the second term on the left side is really a product of functions of \( x \) and hence we will need to use the Product Rule when differentiating that term.

Differentiating with respect to \( x \) gives,
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\[-(2x + 2y) \sin(x^2 + 2y) + e^{y^2} + 2ye^{y^2} = 0\]

Step 2
Finally all we need to do is solve this for \( y' \) (with some potentially messy algebra).

\[
-2x \sin(x^2 + 2y) - 2y' \sin(x^2 + 2y) + e^{y^2} + 2yy' e^{y^2} = 0
\]

\[
(2yy' e^{y^2} - 2 \sin(x^2 + 2y)) y' = 0 + 2x \sin(x^2 + 2y) - e^{y^2}
\]

\[
y' = \frac{2x \sin(x^2 + 2y) - e^{y^2}}{2yy' e^{y^2} - 2 \sin(x^2 + 2y)}
\]

9. Find \( y' \) by implicit differentiation for \( \tan(x^2 y^4) = 3x + y^2 \).

Hint: Don’t forget that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \)! Also, don’t forget that because \( y \) is really \( y(x) \) we may well have a Product and/or a Quotient Rule buried in the problem.

Step 1
First, we just need to take the derivative of everything with respect to \( x \) and we’ll need to recall that \( y \) is really \( y(x) \) and so we’ll need to use the Chain Rule when taking the derivative of terms involving \( y \). This also means that the when doing Chain Rule on the first tangent on the left side we will need to do Product Rule when differentiating the “inside term”.

Differentiating with respect to \( x \) gives,

\[
(2x y^4 + 4x^2 y^3 y') \sec^2(x^2 y^4) = 3 + 2y y'
\]

Step 2
Finally all we need to do is solve this for \( y' \) (with some potentially messy algebra).

\[
2x y^4 \sec^2(x^2 y^4) + 4x^2 y^3 y' \sec^2(x^2 y^4) = 3 + 2y y'
\]

\[
(4x^2 y^3 \sec^2(x^2 y^4) - 2y) y' = 3 - 2x y^4 \sec^2(x^2 y^4)
\]

\[
y' = \frac{3 - 2x y^4 \sec^2(x^2 y^4)}{4x^2 y^3 \sec^2(x^2 y^4) - 2y}
\]

10. Find the equation of the tangent line to \( x^4 + y^2 = 3 \) at \( (1, -\sqrt{2}) \).

Hint: We know how to compute the slope of tangent lines and with implicit differentiation that shouldn’t be too hard at this point.
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Step 1
The first thing to do is use implicit differentiation to find \( y' \) for this function.

\[
4x^3 + 2y y' = 0 \quad \Rightarrow \quad y' = -\frac{2x^3}{y}
\]

Step 2
Evaluating the derivative at the point in question to get the slope of the tangent line gives,

\[
m = y'|_{x=1, y=-\sqrt{2}} = -\frac{2}{-\sqrt{2}} = \sqrt{2}
\]

Step 3
Now, we just need to write down the equation of the tangent line.

\[
y - (-\sqrt{2}) = \sqrt{2} (x - 1) \quad \Rightarrow \quad y = \sqrt{2} (x - 1) - \sqrt{2} = \sqrt{2} (x - 2)
\]

11. Find the equation of then tangent line to \( y^2 e^{2x} = 3y + x^2 \) at \((0,3)\).

Hint : We know how to compute the slope of tangent lines and with implicit differentiation that shouldn’t be too hard at this point.

Step 1
The first thing to do is use implicit differentiation to find \( y' \) for this function.

\[
2yy' e^{2x} + 2y^2 e^{2x} = 3y' + 2x \quad \Rightarrow \quad y' = \frac{2x - 2y^2 e^{2x}}{2y e^{2x} - 3}
\]

Step 2
Evaluating the derivative at the point in question to get the slope of the tangent line gives,

\[
m = y'|_{x=0, y=3} = \frac{-18}{3} = -6
\]

Step 3
Now, we just need to write down the equation of the tangent line.

\[
y - 3 = -6(x - 0) \quad \Rightarrow \quad y = -6x + 3
\]

12. Assume that \( x = x(t) \), \( y = y(t) \) and \( z = z(t) \) and differentiate \( x^2 - y^3 + z^4 = 1 \) with respect to \( t \).
Hint: This is just implicit differentiation like we’ve been doing to this point. The only difference is that now all the functions are functions of some fourth variable, \( t \). Outside of that there is nothing different between this and the previous problems.

Solution
Differentiating with respect to \( t \) gives,

\[
2x x' - 3y^2 y' + 4z^3 z' = 0
\]

Note that because we were not asked to give the formula for a specific derivative we don’t need to go any farther. We could however, if asked, solved this for any of the three derivatives that are present.

13. Assume that \( x = x(t) \), \( y = y(t) \) and \( z = z(t) \) and differentiate \( x^2 \cos(y) = \sin(y^3 + 4z) \) with respect to \( t \).

Hint: This is just implicit differentiation like we’ve been doing to this point. The only difference is that now all the functions are functions of some fourth variable, \( t \). Outside of that there is nothing different between this and the previous problems.

Solution
Differentiating with respect to \( t \) gives,

\[
2x x' \cos(y) - x^2 \sin(y) \ y' = \left(3y^2 y' + 4z' \right) \cos(y^3 + 4z)
\]

Note that because we were not asked to give the formula for a specific derivative we don’t need to go any farther. We could however, if asked, solved this for any of the three derivatives that are present.