Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don’t have good Calculus I skills, and you are constantly getting stuck on the Calculus I portion of the problem, you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone; it is simply an acknowledgment that you can’t just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general. You will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you’ve got it memorized doesn’t mean that you can use it. You’ll need to be able to look at an integral and realize that integration by parts can be used (which isn’t always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you’ll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

So, with all that out of the way let me also get a couple of warnings out of the way to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus II many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often
don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Approximating Definite Integrals

In this chapter we’ve spent quite a bit of time on computing the values of integrals. However, not all integrals can be computed. A perfect example is the following definite integral.

\[ \int_0^2 e^{x^2} \, dx \]

We now need to talk a little bit about estimating values of definite integrals. We will look at three different methods, although one should already be familiar to you from your Calculus I days. We will develop all three methods for estimating

\[ \int_a^b f(x) \, dx \]

by thinking of the integral as an area problem and using known shapes to estimate the area under the curve.

Let’s get first develop the methods and then we’ll try to estimate the integral shown above.

Midpoint Rule

This is the rule that should be somewhat familiar to you. We will divide the interval \([a, b]\) into \(n\) subintervals of equal width,

\[ \Delta x = \frac{b - a}{n} \]

We will denote each of the intervals as follows,

\[ [x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n] \]

where \(x_0 = a\) and \(x_n = b\)

Then for each interval let \(x_i^*\) be the midpoint of the interval. We then sketch in rectangles for each subinterval with a height of \(f(x_i^*)\). Here is a graph showing the set up using \(n = 6\).

We can easily find the area for each of these rectangles and so for a general \(n\) we get that,

\[ \int_a^b f(x) \, dx \approx \Delta x f(x_1^*) + \Delta x f(x_2^*) + \cdots + \Delta x f(x_n^*) \]

Or, upon factoring out a \(\Delta x\) we get the general Midpoint Rule.
\[ \int_a^b f(x) \, dx \approx \Delta x \left[ f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*) \right] \]

**Trapezoid Rule**

For this rule we will do the same set up as for the Midpoint Rule. We will break up the interval \([a,b]\) into \(n\) subintervals of width,

\[
\Delta x = \frac{b-a}{n}
\]

Then on each subinterval we will approximate the function with a straight line that is equal to the function values at either endpoint of the interval. Here is a sketch of this case for \(n = 6\).

Each of these objects is a trapezoid (hence the rule’s name…) and as we can see some of them do a very good job of approximating the actual area under the curve and others don’t do such a good job.

The area of the trapezoid in the interval \([x_{i-1}, x_i]\) is given by,

\[
A_i = \frac{\Delta x}{2} \left( f(x_{i-1}) + f(x_i) \right)
\]

So, if we use \(n\) subintervals the integral is approximately,

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left( f(x_0) + f(x_1) \right) + \frac{\Delta x}{2} \left( f(x_1) + f(x_2) \right) + \cdots + \frac{\Delta x}{2} \left( f(x_{n-1}) + f(x_n) \right)
\]

Upon doing a little simplification we arrive at the general Trapezoid Rule.

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{2} \left[ f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n) \right]
\]

Note that all the function evaluations, with the exception of the first and last, are multiplied by 2.

Simpson’s Rule
This is the final method we’re going to take a look at and in this case we will again divide up the interval \([a, b]\) into \(n\) subintervals. However unlike the previous two methods we need to require that \(n\) be even. The reason for this will be evident in a bit. The width of each subinterval is, \[
\Delta x = \frac{b - a}{n}
\]

In the Trapezoid Rule we approximated the curve with a straight line. For Simpson’s Rule we are going to approximate the function with a quadratic and we’re going to require that the quadratic agree with three of the points from our subintervals. Below is a sketch of this using \(n = 6\). Each of the approximations is colored differently so we can see how they actually work.

Notice that each approximation actually covers two of the subintervals. This is the reason for requiring \(n\) to be even. Some of the approximations look more like a line than a quadratic, but they really are quadratics. Also note that some of the approximations do a better job than others. It can be shown that the area under the approximation on the intervals \([x_{i-1}, x_i]\) and \([x_i, x_{i+1}]\) is,

\[
A_i = \frac{\Delta x}{3} \left( f(x_{i-1}) + 4f(x_i) + f(x_{i+1}) \right)
\]

If we use \(n\) subintervals the integral is then approximately,

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} \left( f(x_0) + 4f(x_1) + f(x_2) + \frac{\Delta x}{3} \left( f(x_2) + 4f(x_3) + f(x_4) \right) + \cdots + \frac{\Delta x}{3} \left( f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right) \right)
\]

Upon simplifying we arrive at the general Simpson’s Rule.

\[
\int_a^b f(x) \, dx \approx \frac{\Delta x}{3} \left[ f(x_0) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n) \right]
\]
In this case notice that all the function evaluations at points with odd subscripts are multiplied by 4 and all the function evaluations at points with even subscripts (except for the first and last) are multiplied by 2. If you can remember this, this is a fairly easy rule to remember.

Okay, it’s time to work an example and see how these rules work.

**Example 1** Using $n = 4$ and all three rules to approximate the value of the following integral.

$$\int_{0}^{2} e^{x^2} \, dx$$

**Solution**

First, for reference purposes, Maple gives the following value for this integral.

$$\int_{0}^{2} e^{x^2} \, dx = 16.45262776$$

In each case the width of the subintervals will be,

$$\Delta x = \frac{2 - 0}{4} = \frac{1}{2}$$

and so the subintervals will be,

$$[0, 0.5], [0.5, 1], [1, 1.5], [1.5, 2]$$

Let’s go through each of the methods.

**Midpoint Rule**

$$\int_{0}^{2} e^{x^2} \, dx \approx \frac{1}{2} \left( e^{(0.25)^2} + e^{(0.75)^2} + e^{(1.25)^2} + e^{(1.75)^2} \right) = 14.48561253$$

Remember that we evaluate at the midpoints of each of the subintervals here! The Midpoint Rule has an error of 1.96701523.

**Trapezoid Rule**

$$\int_{0}^{2} e^{x^2} \, dx \approx \frac{1}{2} \left( e^{(0)^2} + 2e^{(0.5)^2} + 2e^{(1)^2} + 2e^{(1.5)^2} + e^{(2)^2} \right) = 20.64455905$$

The Trapezoid Rule has an error of 4.19193129

**Simpson’s Rule**

$$\int_{0}^{2} e^{x^2} \, dx \approx \frac{1}{3} \left( e^{(0)^2} + 4e^{(0.5)^2} + 2e^{(1)^2} + 4e^{(1.5)^2} + e^{(2)^2} \right) = 17.35362645$$

The Simpson’s Rule has an error of 0.90099869.

None of the estimations in the previous example are all that good. The best approximation in this case is from the Simpson’s Rule and yet it still had an error of almost 1. To get a better estimation we would need to use a larger $n$. So, for completeness sake here are the estimates for some larger value of $n$. 

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In this case we were able to determine the error for each estimate because we could get our hands on the exact value. Often this won’t be the case and so we’d next like to look at error bounds for each estimate.

These bounds will give the largest possible error in the estimate, but it should also be pointed out that the actual error may be significantly smaller than the bound. The bound is only there so we can say that we know the actual error will be less than the bound.

So, suppose that \( |f''(x)| \leq K \) and \( |f^{(4)}(x)| \leq M \) for \( a \leq x \leq b \) then if \( E_M, E_T, \) and \( E_S \) are the actual errors for the Midpoint, Trapezoid and Simpson’s Rule we have the following bounds,

\[
|E_M| \leq \frac{K(b-a)^3}{24n^2} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2} \quad |E_S| \leq \frac{M(b-a)^5}{180n^4}
\]

**Example 2** Determine the error bounds for the estimations in the last example.

**Solution**

We already know that \( n = 4, \ a = 0, \) and \( b = 2 \) so we just need to compute \( K \) (the largest value of the second derivative) and \( M \) (the largest value of the fourth derivative). This means that we’ll need the second and fourth derivative of \( f(x) \).

\[
f''(x) = 2e^{x^2} \left( 1 + 2x^2 \right) \]

\[
f^{(4)}(x) = 4e^{x^2} \left( 3 + 12x^2 + 4x^4 \right)
\]

Here is a graph of the second derivative.

Here is a graph of the fourth derivative.
So, from these graphs it’s clear that the largest value of both of these are at \( x = 2 \). So,

\[
f''(2) = 982.76667 \quad \Rightarrow \quad K = 983
\]

\[
f^{(4)}(2) = 25115.14901 \quad \Rightarrow \quad M = 25116
\]

We rounded to make the computations simpler.

Here are the bounds for each rule.

\[
|E_M| \leq \frac{983(2-0)^3}{24(4)^2} = 20.4791666667
\]

\[
|E_T| \leq \frac{983(2-0)^3}{12(4)^2} = 40.9583333333
\]

\[
|E_S| \leq \frac{25116(2-0)^5}{180(4)^4} = 17.4416666667
\]

In each case we can see that the errors are significantly smaller than the actual bounds.