Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Vector Fields**

We need to start this chapter off with the definition of a vector field as they will be a major component of both this chapter and the next. Let’s start off with the formal definition of a vector field.

**Definition**

A vector field on two (or three) dimensional space is a function \( \vec{F} \) that assigns to each point \((x, y)\) (or \((x, y, z)\)) a two (or three dimensional) vector given by \( \vec{F}(x, y) \) (or \( \vec{F}(x, y, z) \)).

That may not make a lot of sense, but most people do know what a vector field is, or at least they’ve seen a sketch of a vector field. If you’ve seen a current sketch giving the direction and magnitude of a flow of a fluid or the direction and magnitude of the winds then you’ve seen a sketch of a vector field.

The standard notation for the function \( \vec{F} \) is,

\[
\vec{F}(x, y) = P(x, y)\hat{i} + Q(x, y)\hat{j}
\]

\[
\vec{F}(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}
\]

depending on whether or not we’re in two or three dimensions. The function \( P, Q, R \) (if it is present) are sometimes called scalar functions.

Let’s take a quick look at a couple of examples.

**Example 1** Sketch each of the following vector fields.

(a) \( \vec{F}(x, y) = -y\hat{i} + x\hat{j} \)  \[Solution\]

(b) \( \vec{F}(x, y, z) = 2x\hat{i} - 2y\hat{j} - 2z\hat{k} \)  \[Solution\]

**Solution**

(a) \( \vec{F}(x, y) = -y\hat{i} + x\hat{j} \)

Okay, to graph the vector field we need to get some “values” of the function. This means plugging in some points into the function. Here are a couple of evaluations.

\[
\vec{F}\left(\frac{1}{2}, \frac{1}{2}\right) = -\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}
\]

\[
\vec{F}\left(\frac{1}{2}, -\frac{1}{2}\right) = -\left(-\frac{1}{2}\right)\hat{i} + \frac{1}{2}\hat{j} = \frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}
\]

\[
\vec{F}\left(\frac{3}{2}, \frac{1}{4}\right) = -\frac{3}{4}\hat{i} + \frac{3}{2}\hat{j}
\]

So, just what do these evaluations tell us? Well the first one tells us that at the point \(\left(\frac{1}{2}, \frac{1}{2}\right)\) we will plot the vector \(-\frac{1}{2}\hat{i} + \frac{1}{2}\hat{j}\). Likewise, the third evaluation tells us that at the point \(\left(\frac{3}{2}, \frac{1}{4}\right)\) we will plot the vector \(-\frac{3}{4}\hat{i} + \frac{3}{2}\hat{j}\).
We can continue in this fashion plotting vectors for several points and we’ll get the following sketch of the vector field.

If we want significantly more points plotted then it is usually best to use a computer aided graphing system such as Maple or Mathematica. Here is a sketch with many more vectors included that was generated with Mathematica.

(b) \( \vec{F}(x, y, z) = 2x \hat{i} - 2y \hat{j} - 2x \hat{k} \)

In the case of three dimensional vector fields it is almost always better to use Maple, Mathematica, or some other such tool. Despite that let’s go ahead and do a couple of evaluations anyway.

\[
\vec{F}(1, -3, 2) = 2 \hat{i} + 6 \hat{j} - 2 \hat{k} \\
\vec{F}(0, 5, 3) = -10 \hat{j}
\]

Notice that \( z \) only affect the placement of the vector in this case and does not affect the direction.
or the magnitude of the vector. Sometimes this will happen so don’t get excited about it when it does.

Here is a couple of sketches generated by Mathematica. The sketch on the left is from the “front” and the sketch on the right is from “above”.

Now that we’ve seen a couple of vector fields let’s notice that we’ve already seen a vector field function. In the second chapter we looked at the gradient vector. Recall that given a function \( f(x, y, z) \) the gradient vector is defined by,

\[ \nabla f = \left\langle f_x, f_y, f_z \right\rangle \]

This is a vector field and is often called a gradient vector field.

In these cases the function \( f(x, y, z) \) is often called a scalar function to differentiate it from the vector field.

**Example 2** Find the gradient vector field of the following functions.

(a) \( f(x, y) = x^2 \sin(5y) \)

(b) \( f(x, y, z) = xe^{-xy} \)

**Solution**

(a) \( f(x, y) = x^2 \sin(5y) \)

Note that we only gave the gradient vector definition for a three dimensional function, but don’t forget that there is also a two dimension definition. All that we need to drop off the third component of the vector.

Here is the gradient vector field for this function.

\[ \nabla f = \left\langle 2x \sin(5y), 5x^2 \cos(5y) \right\rangle \]
(b) \( f(x, y, z) = ze^{-xy} \)

There isn’t much to do here other than take the gradient.

\[
\nabla f = \left\langle -yze^{-xy}, -xze^{-xy}, e^{-xy} \right\rangle
\]

Let’s do another example that will illustrate the relationship between the gradient vector field of a function and its contours.

**Example 3** Sketch the gradient vector field for \( f(x, y) = x^2 + y^2 \) as well as several contours for this function.

**Solution**

Recall that the contours for a function are nothing more than curves defined by,

\[
f(x, y) = k
\]

for various values of \( k \). So, for our function the contours are defined by the equation,

\[
x^2 + y^2 = k
\]

and so they are circles centered at the origin with radius \( \sqrt{k} \).

Here is the gradient vector field for this function.

\[
\nabla f(x, y) = 2x \mathbf{i} + 2y \mathbf{j}
\]

Here is a sketch of several of the contours as well as the gradient vector field.

Notice that the vectors of the vector field are all perpendicular (or orthogonal) to the contours. This will always be the case when we are dealing with the contours of a function as well as its gradient vector field.

The \( k \)'s we used for the graph above were 1.5, 3, 4.5, 6, 7.5, 9, 10.5, 12, and 13.5. Now notice that as we increased \( k \) by 1.5 the contour curves get closer together and that as the contour curves get closer together the larger the vectors become. In other words, the closer the contour curves
are (as \( k \) is increased by a fixed amount) the faster the function is changing at that point. Also recall that the direction of fastest change for a function is given by the gradient vector at that point. Therefore, it should make sense that the two ideas should match up as they do here.

The final topic of this section is that of conservative vector fields. A vector field \( \vec{F} \) is called a **conservative vector field** if there exists a function \( f \) such that \( \vec{F} = \nabla f \). If \( \vec{F} \) is a conservative vector field then the function, \( f \), is called a **potential function** for \( \vec{F} \).

All this definition is saying is that a vector field is conservative if it is also a gradient vector field for some function.

For instance the vector field \( \vec{F} = y\vec{i} + x\vec{j} \) is a conservative vector field with a potential function of \( f(x, y) = xy \) because \( \nabla f = (y, x) \).

On the other hand, \( \vec{F} = -y\vec{i} + x\vec{j} \) is not a conservative vector field since there is no function \( f \) such that \( \vec{F} = \nabla f \). If you’re not sure that you believe this at this point be patient, we will be able to prove this in a couple of sections. In that section we will also show how to find the potential function for a conservative vector field.