Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.

**Boundary Value Problems**

Before we start off this section we need to make it very clear that we are only going to scratch the surface of the topic of boundary value problems. There is enough material in the topic of boundary value problems that we could devote a whole class to it. The intent of this section is to give a brief (and we mean very brief) look at the idea of boundary value problems and to give enough information to allow us to do some basic partial differential equations in the next chapter.

Now, with that out of the way, the first thing that we need to do is to define just what we mean by a boundary value problem (BVP for short). With initial value problems we had a differential equation and we specified the value of the solution and an appropriate number of derivatives at the same point (collectively called initial conditions). For instance for a second order differential equation the initial conditions are,

\[ y(t_0) = y_0 \quad \text{and} \quad y'(t_0) = y'_0. \]

With boundary value problems we will have a differential equation and we will specify the function and/or derivatives at different points, which we’ll call boundary values. For second order differential equations, which will be looking at pretty much exclusively here, any of the following can, and will, be used for boundary conditions.

\[ y(x_0) = y_0, \quad y(x_1) = y_1 \quad \text{(1)} \]
\[ y'(x_0) = y_0, \quad y'(x_1) = y_1 \quad \text{(2)} \]
\[ y'(x_0) = y_0, \quad y(x_1) = y_1 \quad \text{(3)} \]
\[ y(x_0) = y_0, \quad y'(x_1) = y_1 \quad \text{(4)} \]

As mentioned above we’ll be looking pretty much exclusively at second order differential equations. We will also be restricting ourselves down to linear differential equations. So, for the purposes of our discussion here we’ll be looking almost exclusively at differential equations in the form,

\[ y'' + p(x)y' + q(x)y = g(x) \quad \text{(5)} \]

along with one of the sets of boundary conditions given in (1) – (4). We will, on occasion, look at some different boundary conditions but the differential equation will always be on that can be written in this form.

As we’ll soon see much of what we know about initial value problems will not hold here. We can, of course, solve (5) provided the coefficients are constant and for a few cases in which they aren’t. None of that will change. The changes (and perhaps the problems) arise when we move from initial conditions to boundary conditions.

One of the first changes is a definition that we saw all the time in the earlier chapters. In the earlier chapters we said that a differential equation was homogeneous if \( g(x) = 0 \) for all \( x \). Here we will say that a boundary value problem is **homogeneous** if in addition to \( g(x) = 0 \) we also have \( y_0 = 0 \) and \( y_1 = 0 \) (regardless of the boundary conditions we use). If any of these are not zero we will call the BVP **nonhomogeneous**.
It is important to now remember that when we say homogeneous (or nonhomogeneous) we are saying something not only about the differential equation itself but also about the boundary conditions as well.

The biggest change that we’re going to see here comes when we go to solve the boundary value problem. When solving linear initial value problems a unique solution will be guaranteed under very mild conditions. We only looked at this idea for first order IVP’s but the idea does extend to higher order IVP’s. In that section we saw that all we needed to guarantee a unique solution was some basic continuity conditions. With boundary value problems we will often have no solution or infinitely many solutions even for very nice differential equations that would yield a unique solution if we had initial conditions instead of boundary conditions.

Before we get into solving some of these let’s next address the question of why we’re even talking about these in the first place. As we’ll see in the next chapter in the process of solving some partial differential equations we will run into boundary value problems that will need to be solved as well. In fact, a large part of the solution process there will be in dealing with the solution to the BVP. In these cases the boundary conditions will represent things like the temperature at either end of a bar, or the heat flow into/out of either end of a bar. Or maybe they will represent the location of ends of a vibrating string. So, the boundary conditions there will really be conditions on the boundary of some process.

So, with some of basic stuff out of the way let’s find some solutions to a few boundary value problems. Note as well that there really isn’t anything new here yet. We know how to solve the differential equation and we know how to find the constants by applying the conditions. The only difference is that here we’ll be applying boundary conditions instead of initial conditions.

**Example 1** Solve the following BVP.

\[ y'' + 4y = 0 \quad y(0) = -2 \quad y\left(\frac{\pi}{4}\right) = 10 \]

**Solution**

Okay, this is a simple differential equation to solve and so we’ll leave it to you to verify that the general solution to this is,

\[ y(x) = c_1 \cos(2x) + c_2 \sin(2x) \]

Now all that we need to do is apply the boundary conditions.

\[-2 = y(0) = c_1 \]
\[10 = y\left(\frac{\pi}{4}\right) = c_2 \]

The solution is then,

\[ y(x) = -2 \cos(2x) + 10 \sin(2x) \]

We mentioned above that some boundary value problems can have no solutions or infinite solutions we had better do a couple of examples of those as well here. This next set of examples will also show just how small of a change to the BVP it takes to move into these other possibilities.

**Example 2** Solve the following BVP.

\[ y'' + 4y = 0 \quad y(0) = -2 \quad y(2\pi) = -2 \]

**Solution**
We’re working with the same differential equation as the first example so we still have,

\[ y(x) = c_1 \cos(2x) + c_2 \sin(2x) \]

Upon applying the boundary conditions we get,

\[ \begin{align*}
-2 &= y(0) = c_1 \\
-2 &= y(2\pi) = c_1 
\end{align*} \]

So in this case, unlike previous example, both boundary conditions tell us that we have to have \( c_1 = -2 \) and neither one of them tell us anything about \( c_2 \). Remember however that all we’re asking for is a solution to the differential equation that satisfies the two given boundary conditions and the following function will do that,

\[ y(x) = -2 \cos(2x) + c_2 \sin(2x) \]

In other words, regardless of the value of \( c_2 \) we get a solution and so, in this case we get infinitely many solutions to the boundary value problem.

Example 3  Solve the following BVP.

\[ y^{\prime\prime} + 4y = 0 \quad y(0) = -2 \quad y(2\pi) = 3 \]

Solution

Again, we have the following general solution,

\[ y(x) = c_1 \cos(2x) + c_2 \sin(2x) \]

This time the boundary conditions give us,

\[ \begin{align*}
-2 &= y(0) = c_1 \\
3 &= y(2\pi) = c_1 
\end{align*} \]

In this case we have a set of boundary conditions each of which requires a different value of \( c_1 \) in order to be satisfied. This, however, is not possible and so in this case we have no solution.

So, with Examples 2 and 3 we can see that only a small change to the boundary conditions, in relation to each other and to Example 1, can completely change the nature of the solution. All three of these examples used the same differential equation and yet a different set of initial conditions yielded, no solutions, one solution, or infinitely many solutions.

Note that this kind of behavior is not always unpredictable however. If we use the conditions \( y(0) \) and \( y(2\pi) \) the only way we’ll ever get a solution to the boundary value problem is if we have,

\[ \begin{align*}
y(0) &= a \\
y(2\pi) &= a 
\end{align*} \]

for any value of \( a \). Also, note that if we do have these boundary conditions we’ll in fact get infinitely many solutions.
All the examples we’ve worked to this point involved the same differential equation and the same type of boundary conditions so let’s work a couple more just to make sure that we’ve got some more examples here. Also, note that with each of these we could tweak the boundary conditions a little to get any of the possible solution behaviors to show up (i.e. zero, one or infinitely many solutions).

**Example 4** Solve the following BVP.

\[ y^{\prime\prime} + 3y = 0 \quad y(0) = 7 \quad y(2\pi) = 0 \]

**Solution**

The general solution for this differential equation is,

\[ y(x) = c_1 \cos(\sqrt{3} x) + c_2 \sin(\sqrt{3} x) \]

Applying the boundary conditions gives,

\[ 7 = y(0) = c_1 \]

\[ 0 = y(2\pi) = c_1 \cos(2\sqrt{3} \pi) + c_2 \sin(2\sqrt{3} \pi) \quad \Rightarrow \quad c_2 = -7 \cot(2\sqrt{3} \pi) \]

In this case we get a single solution,

\[ y(x) = 7 \cos(\sqrt{3} x) - 7 \cot(2\sqrt{3} \pi) \sin(\sqrt{3} x) \]

**Example 5** Solve the following BVP.

\[ y^{\prime\prime} + 25y = 0 \quad y'(0) = 6 \quad y'(\pi) = -9 \]

**Solution**

Here the general solution is,

\[ y(x) = c_1 \cos(5x) + c_2 \sin(5x) \]

and we’ll need the derivative to apply the boundary conditions,

\[ y'(x) = -5c_1 \sin(5x) + 5c_2 \cos(5x) \]

Applying the boundary conditions gives,

\[ 6 = y'(0) = 5c_2 \quad \Rightarrow \quad c_2 = \frac{6}{5} \]

\[ -9 = y'(\pi) = -5c_2 \quad \Rightarrow \quad c_2 = \frac{9}{5} \]

This is not possible and so in this case have **no solution**.

All of the examples worked to this point have been nonhomogeneous because at least one of the boundary conditions have been non-zero. Let’s work one nonhomogeneous example where the differential equation is also nonhomogeneous before we work a couple of homogeneous examples.

**Example 6** Solve the following BVP.

\[ y^{\prime\prime} + 9y = \cos x \quad y'(0) = 5 \quad y\left(\frac{\pi}{2}\right) = -\frac{5}{3} \]

**Solution**

The complementary solution for this differential equation is,
\[ y'(x) = c_1 \cos(3x) + c_2 \sin(3x) \]

Using **Undetermined Coefficients** or **Variation of Parameters** it is easy to show (we’ll leave the details to you to verify) that a particular solution is,

\[ Y_p(x) = \frac{1}{5} \cos x \]

The general solution and its derivative (since we’ll need that for the boundary conditions) are,

\[ y(x) = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{5} \cos x \]

\[ y'(x) = -3c_1 \sin(3x) + 3c_2 \cos(3x) - \frac{1}{5} \sin x \]

Applying the boundary conditions gives,

\[ 0 = y'(0) = 3c_2 \quad \Rightarrow \quad c_2 = \frac{5}{3} \]

\[ -\frac{5}{3} = y\left(\frac{\pi}{3}\right) = -c_2 \quad \Rightarrow \quad c_2 = \frac{5}{3} \]

The boundary conditions then tell us that we must have \( c_2 = \frac{5}{3} \) and they don’t tell us anything about \( c_1 \), and so it is can be arbitrarily chosen. The solution is then,

\[ y(x) = c_1 \cos(3x) + \frac{5}{3} \sin(3x) + \frac{1}{5} \cos x \]

and there will be infinitely many solutions to the BVP.

Let’s now work a couple of homogeneous examples that will also be helpful to have worked once we get to the next section.

**Example 7** Solve the following BVP.

\[ y'' + 4y = 0 \quad y(0) = 0 \quad y(2\pi) = 0 \]

**Solution**

Here the general solution is,

\[ y(x) = c_1 \cos(2x) + c_2 \sin(2x) \]

Applying the boundary conditions gives,

\[ 0 = y(0) = c_1 \]

\[ 0 = y(2\pi) = c_1 \]

So \( c_2 \) is arbitrary and the solution is,

\[ y(x) = c_2 \sin(2x) \]

and in this case we’ll get infinitely many solutions.

**Example 8** Solve the following BVP.

\[ y'' + 3y = 0 \quad y(0) = 0 \quad y(2\pi) = 0 \]

**Solution**

The general solution in this case is,

\[ y(x) = c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x) \]
Applying the boundary conditions gives,

\[ 0 = y(0) = c_1 \]
\[ 0 = y(2\pi) = c_2 \sin(2\sqrt{3}\pi) \quad \Rightarrow \quad c_2 = 0 \]

In this case we found both constants to be zero and so the solution is,

\[ y(x) = 0 \]

In the previous example the solution was \( y(x) = 0 \). Notice however, that this will always be a solution to any homogenous system given by (5) and any of the (homogeneous) boundary conditions given by (1) – (4). Because of this we usually call this solution the **trivial solution**. Sometimes, as in the case of the last example the trivial solution is the only solution however we generally prefer solutions to be non-trivial. This will be a major idea in the next section.

Before we leave this section an important point needs to be made. In each of the examples, with one exception, the differential equation that we solved was in the form,

\[ y'' + \lambda y = 0 \]

The one exception to this still solved this differential equation except it was not a homogeneous differential equation and so we were still solving this basic differential equation in some manner.

So, there are probably several natural questions that can arise at this point. Do all BVP’s involve this differential equation and if not why did we spend so much time solving this one to the exclusion of all the other possible differential equations?

The answers to these questions are fairly simple. First, this differential equation is most definitely not the only one used in boundary value problems. It does however exhibit all of the behavior that we wanted to talk about here and has the added bonus of being very easy to solve. So, by using this differential equation almost exclusively we can see and discuss the important behavior that we need to discuss and frees us up from lots of potentially messy solution details and or messy solutions. We will, on occasion, look at other differential equations in the rest of this chapter, but we will still be working almost exclusively with this one.

There is another important reason for looking at this differential equation. When we get to the next chapter and take a brief look at solving partial differential equations we will see that almost every one of the examples that we’ll work there come down to exactly this differential equation. Also, in those problems we will be working some “real” problems that are actually solved in places and so are not just “made up” problems for the purposes of examples. Admittedly they will have some simplifications in them, but they do come close to realistic problem in some cases.