Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Inverse Laplace Transforms**

Finding the Laplace transform of a function is not terribly difficult if we’ve got a table of transforms in front of us to use as we saw in the last section. What we would like to do now is go the other way.

We are going to be given a transform, \( F(s) \), and ask what function (or functions) did we have originally. As you will see this can be a more complicated and lengthy process than taking transforms. In these cases we say that we are finding the **Inverse Laplace Transform** of \( F(s) \) and use the following notation.

\[
 f(t) = \mathcal{L}^{-1}\{F(s)\}
\]

As with Laplace transforms, we’ve got the following fact to help us take the inverse transform.

**Fact**

Given the two Laplace transforms \( F(s) \) and \( G(s) \) then

\[
 \mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}
\]

for any constants \( a \) and \( b \).

So, we take the inverse transform of the individual transforms, put any constants back in and then add or subtract the results back up.

Let’s take a look at a couple of fairly simple inverse transforms.

**Example 1** Find the inverse transform of each of the following.

(a) \( F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3} \)  [Solution]

(b) \( H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^3} \)  [Solution]

(c) \( F(s) = \frac{6s}{s^2+25} + \frac{3}{s^2+25} \)  [Solution]

(d) \( G(s) = \frac{8}{3s^2+12} + \frac{3}{s^2-49} \)  [Solution]

**Solution**

I’ve always felt that the key to doing inverse transforms is to look at the denominator and try to identify what you’ve got based on that. If there is only one entry in the table that has that particular denominator, the next step is to make sure the numerator is correctly set up for the inverse transform process. If it isn’t, correct it (this is always easy to do) and then take the inverse transform.

If there is more than one entry in the table has a particular denominator, then the numerators of each will be different, so go up to the numerator and see which one you’ve got. If you need to correct the numerator to get it into the correct form and then take the inverse transform.

So, with this advice in mind let’s see if we can take some inverse transforms.
(a) \( F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3} \)

From the denominator of the first term it looks like the first term is just a constant. The correct numerator for this term is a “1” so we’ll just factor the 6 out before taking the inverse transform. The second term appears to be an exponential with \( a = 8 \) and the numerator is exactly what it needs to be. The third term also appears to be an exponential, only this time \( a = 3 \) and we’ll need to factor the 4 out before taking the inverse transforms.

So, with a little more detail than we’ll usually put into these,

\[
F(t) = 6(1) - e^{8t} + 4(e^{3t}) = 6e^{8t} + 4e^{3t}
\]

(b) \( H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^3} \)

The first term in this case looks like an exponential with \( a = -2 \) and we’ll need to factor out the 19. Be careful with negative signs in these problems, it’s very easy to lose track of them.

The second term almost looks like an exponential, except that it’s got a 3s instead of just an s in the denominator. It is an exponential, but in this case we’ll need to factor a 3 out of the denominator before taking the inverse transform.

The denominator of the third term appears to be \( \#3 \) in the table with \( n = 4 \). The numerator however, is not correct for this. There is currently a 7 in the numerator and we need a 4! = 24 in the numerator. This is very easy to fix. Whenever a numerator is off by a multiplicative constant, as in this case, all we need to do is put the constant that we need in the numerator. We will just need to remember to take it back out by dividing by the same constant.

So, let’s first rewrite the transform.

\[
H(s) = \frac{19}{s-(-2)} - \frac{1}{3(s-\frac{5}{3})} + \frac{7}{s^{4+1}}
\]

\[
= 19 \left( \frac{1}{s-(-2)} - \frac{1}{3 \left( s - \frac{5}{3} \right)} + \frac{7}{4! \cdot s^{4+1}} \right)
\]

So, what did we do here? We factored the 19 out of the first term. We factored the 3 out of the denominator of the second term since it can’t be there for the inverse transform and in the third term we factored everything out of the numerator except the 4! since that is the portion that we need in the numerator for the inverse transform process.

Let’s now take the inverse transform.

\[
h(t) = 19e^{-2t} - \frac{1}{3}e^{\frac{5}{3}t} + \frac{7}{24}t^4
\]
In this part we’ve got the same denominator in both terms and our table tells us that we’ve either
got #7 or #8. The numerators will tell us which we’ve actually got. The first one has an \( s \) in the
numerator and so this means that the first term must be #8 and we’ll need to factor the 6 out of
the numerator in this case. The second term has only a constant in the numerator and so this term
must be #7, however, in order for this to be exactly #7 we’ll need to multiply/divide a 5 in the
numerator to get it correct for the table.

The transform becomes,

\[
F(s) = \frac{6s}{s^2 + 25} + \frac{3}{s^2 + 25}
\]

Taking the inverse transform gives,

\[
f(t) = 6 \cos(5t) + \frac{3}{5} \sin(5t)
\]

In this case the first term will be a sine once we factor a 3 out of the denominator, while the
second term appears to be a hyperbolic sine (#17). Again, be careful with the difference between
these two. Both of the terms will also need to have their numerators fixed up. Here is the
transform once we’re done rewriting it.

\[
G(s) = \frac{8}{3s^2 + 12} + \frac{3}{s^2 - 49}
\]

Notice that in the first term we took advantage of the fact that we could get the 2 in the numerator
that we needed by factoring the 8. The inverse transform is then,

\[
g(t) = \frac{4}{3} \sin(2t) + \frac{3}{7} \sinh(7t)
\]
Let’s do some slightly harder problems. These are a little more involved than the first set.

**Example 2** Find the inverse transform of each of the following.

(a) \( F(s) = \frac{6s - 5}{s^2 + 7} \) [Solution]

(b) \( F(s) = \frac{1 - 3s}{s^2 + 8s + 21} \) [Solution]

(c) \( G(s) = \frac{3s - 2}{2s^2 - 6s - 2} \) [Solution]

(d) \( H(s) = \frac{s + 7}{s^2 - 3s - 10} \) [Solution]

**Solution**

(a) \( F(s) = \frac{6s - 5}{s^2 + 7} \)

From the denominator of this one it appears that it is either a sine or a cosine. However, the numerator doesn’t match up to either of these in the table. A cosine wants just an \( s \) in the numerator with at most a multiplicative constant, while a sine wants only a constant and no \( s \) in the numerator.

We’ve got both in the numerator. This is easy to fix however. We will just split up the transform into two terms and then do inverse transforms.

\[
F(t) = 6 \cos(\sqrt{7}t) - \frac{5}{\sqrt{7}} \sin(\sqrt{7}t)
\]

Do not get too used to always getting the perfect squares in sines and cosines that we saw in the first set of examples. More often than not (at least in my class) they won’t be perfect squares!

(b) \( F(s) = \frac{1 - 3s}{s^2 + 8s + 21} \)

In this case there are no denominators in our table that look like this. We can however make the denominator look like one of the denominators in the table by completing the square on the denominator. So, let’s do that first.

\[
s^2 + 8s + 21 = s^2 + 8s + 16 - 16 + 21 = (s + 4)^2 + 5
\]

Recall that in completing the square you take half the coefficient of the \( s \), square this, and then add and subtract the result to the polynomial. After doing this the first three terms should factor as a perfect square.
So, the transform can be written as the following.

\[
F(s) = \frac{1 - 3s}{(s + 4)^2 + 5}
\]

Okay, with this rewrite it looks like we’ve got #19 and/or #20’s from our table of transforms. However, note that in order for it to be a #19 we want just a constant in the numerator and in order to be a #20 we need an \( s - a \) in the numerator. We’ve got neither of these so we’ll have to correct the numerator to get it into proper form.

In correcting the numerator always get the \( s - a \) first. This is the important part. We will also need to be careful of the 3 that sits in front of the \( s \). One way to take care of this is to break the term into two pieces, factor the 3 out of the second and then fix up the numerator of this term. This will work, however it will put three terms into our answer and there are really only two terms.

So, we will leave the transform as a single term and correct it as follows,

\[
F(s) = \frac{1 - 3(s + 4 - 4)}{(s + 4)^2 + 5} = \frac{1 - 3(s + 4) + 12}{(s + 4)^2 + 5} = \frac{-3(s + 4) + 13}{(s + 4)^2 + 5}
\]

We needed an \( s + 4 \) in the numerator, so we put that in. We just needed to make sure and take the 4 back out by subtracting it back out. Also, because of the 3 multiplying the \( s \) we needed to do all this inside a set of parenthesis. Then we partially multiplied the 3 through the second term and combined the constants. With the transform in this form, we can break it up into two transforms each of which are in the tables and so we can do inverse transforms on them,

\[
F(s) = -3 \frac{s + 4}{(s + 4)^2 + 5} + \frac{13 \sqrt{5}}{(s + 4)^2 + 5}
\]

\[
f(t) = -3e^{-4t} \cos(\sqrt{5}t) + 13 \frac{\sqrt{5}}{5} e^{-4t} \sin(\sqrt{5}t)
\]

(c) \( G(s) = \frac{3s - 2}{2s^2 - 6s - 2} \)

This one is similar to the last one. We just need to be careful with the completing the square however. The first thing that we should do is factor a 2 out of the denominator, then complete the square. Remember that when completing the square a coefficient of 1 on the \( s^2 \) term is needed! So, here’s the work for this transform.
\[ G(s) = \frac{3s - 2}{2(s^2 - 3s - 1)} \]
\[ = \frac{1}{2} \cdot \frac{3s - 2}{s^2 - 3s + \frac{9}{4} - \frac{9}{4}} - 1 \]
\[ = \frac{1}{2} \cdot \frac{3s - 2}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}} \]

So, it looks like we’ve got #21 and #22 with a corrected numerator. Here’s the work for that and the inverse transform.

\[ G(s) = \frac{1}{2} \cdot \frac{3\left(s - \frac{3}{2} + \frac{3}{2}\right) - 2}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}} \]
\[ = \frac{1}{2} \cdot \frac{3\left(s - \frac{3}{2}\right) + \frac{5}{2}}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}} \]
\[ = \frac{1}{2} \left( \frac{3\left(s - \frac{3}{2}\right)}{\left(s - \frac{3}{2}\right)^2 - \frac{13}{4}} + \frac{\frac{5}{2}}{\frac{\sqrt{13}}{2}} \right) \]
\[ g(t) = \frac{1}{2} \left( 3e^{\frac{\sqrt{13} t}{2}} \cosh \left( \frac{\sqrt{13} t}{2} \right) + \frac{5}{\sqrt{13}} e^{\frac{\sqrt{13} t}{2}} \sinh \left( \frac{\sqrt{13} t}{2} \right) \right) \]

In correcting the numerator of the second term, notice that I only put in the square root since we already had the “over 2” part of the fraction that we needed in the numerator.

\[ H(s) = \frac{s + 7}{s^2 - 3s - 10} \]

This one appears to be similar to the previous two, but it actually isn’t. The denominators in the previous two couldn’t be easily factored. In this case the denominator does factor and so we need to deal with it differently. Here is the transform with the factored denominator.

\[ H(s) = \frac{s + 7}{(s + 2)(s - 5)} \]

The denominator of this transform seems to suggest that we’ve got a couple of exponentials, however in order to be exponentials there can only be a single term in the denominator and no \( s \)‘s in the numerator.

To fix this we will need to do partial fractions on this transform. In this case the partial fraction decomposition will be

\[ H(s) = \frac{A}{s + 2} + \frac{B}{s - 5} \]

Don’t remember how to do partial fractions? In this example we’ll show you one way of getting
the values of the constants and after this example we’ll review how to get the correct form of the partial fraction decomposition.

Okay, so let’s get the constants. There is a method for finding the constants that will always work, however it can lead to more work than is sometimes required. Eventually, we will need that method, however in this case there is an easier way to find the constants.

Regardless of the method used, the first step is to actually add the two terms back up. This gives the following.

\[
\frac{s + 7}{(s + 2)(s - 5)} = \frac{A(s - 5) + B(s + 2)}{(s + 2)(s - 5)}
\]

Now, this needs to be true for any \( s \) that we should choose to put in. So, since the denominators are the same we just need to get the numerators equal. Therefore, set the numerators equal.

\[
s + 7 = A(s - 5) + B(s + 2)
\]

Again, this must be true for ANY value of \( s \) that we want to put in. So, let’s take advantage of that. If it must be true for any value of \( s \) then it must be true for \( s = -2 \), to pick a value at random. In this case we get,

\[
5 = A(-7) + B(0) \quad \Rightarrow A = -\frac{5}{7}
\]

We found \( A \) by appropriately picking \( s \). We can \( B \) in the same way if we chose \( s = 5 \).

\[
12 = A(0) + B(7) \quad \Rightarrow B = \frac{12}{7}
\]

This will not always work, but when it does it will usually simplify the work considerably.

So, with these constants the transform becomes,

\[
H(s) = -\frac{5}{7}\frac{1}{s + 2} + \frac{12}{7}\frac{1}{s - 5}
\]

We can now easily do the inverse transform to get,

\[
h(t) = -\frac{5}{7}e^{-2t} + \frac{12}{7}e^{5t}
\]

The last part of this example needed partial fractions to get the inverse transform. When we finally get back to differential equations and we start using Laplace transforms to solve them, you will quickly come to understand that partial fractions are a fact of life in these problems. Almost every problem will require partial fractions to one degree or another.

Note that we could have done the last part of this example as we had done the previous two parts. If we had we would have gotten hyperbolic functions. However, recalling the definition of the hyperbolic functions we could have written the result in the form we got from the way we worked our problem. However, most students have a better feel for exponentials than they do for hyperbolic functions and so it’s usually best to just use partial fractions and get the answer in
terms of exponentials. It may be a little more work, but it will give a nicer (and easier to work with) form of the answer.

Be warned that in my class I’ve got a rule that if the denominator can be factored with integer coefficients then it must be.

So, let’s remind you how to get the correct partial fraction decomposition. The first step is to factor the denominator as much as possible. Then for each term in the denominator we will use the following table to get a term or terms for our partial fraction decomposition.

<table>
<thead>
<tr>
<th>Factor in denominator</th>
<th>Term in partial fraction decomposition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ax + b$</td>
<td>$\frac{A}{ax + b}$</td>
</tr>
<tr>
<td>$(ax + b)^k$</td>
<td>$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_k}{(ax + b)^k}$</td>
</tr>
<tr>
<td>$ax^2 + bx + c$</td>
<td>$\frac{Ax + B}{ax^2 + bx + c}$</td>
</tr>
<tr>
<td>$(ax^2 + bx + c)^k$</td>
<td>$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$</td>
</tr>
</tbody>
</table>

Notice that the first and third cases are really special cases of the second and fourth cases respectively.

So, let’s do a couple more examples to remind you how to do partial fractions.

**Example 3** Find the inverse transform of each of the following.

(a) $G(s) = \frac{86s - 78}{(s+3)(s-4)(5s-1)}$ [Solution]

(b) $F(s) = \frac{2 - 5s}{(s-6)(s^2 + 11)}$ [Solution]

(c) $G(s) = \frac{25}{s^3(s^2 + 4s + 5)}$ [Solution]

**Solution**

(a) $G(s) = \frac{86s - 78}{(s+3)(s-4)(5s-1)}$

Here’s the partial fraction decomposition for this part.

$$G(s) = \frac{A}{s+3} + \frac{B}{s-4} + \frac{C}{5s-1}$$

Now, this time we won’t go into quite the detail as we did in the last example. We are after the
numerator of the partial fraction decomposition and this is usually easy enough to do in our heads. Therefore, we will go straight to setting numerators equal.

\[ 86s - 78 = A(s - 4)(5s - 1) + B(s + 3)(5s - 1) + C(s + 3)(s - 4) \]

As with the last example, we can easily get the constants by correctly picking values of \( s \).

\[
\begin{align*}
  s &= -3 \quad &-336 = A(-7)(-16) \quad &\implies A = -3 \\
  s &= \frac{1}{5} \quad &- \frac{304}{5} = C\left(\frac{16}{5}\right)\left(-\frac{19}{5}\right) \quad &\implies C = 5 \\
  s &= 4 \quad &266 = B(7)(19) \quad &\implies B = 2
\end{align*}
\]

So, the partial fraction decomposition for this transform is,

\[ G(s) = -\frac{\frac{3}{s+3} + \frac{2}{s-4} + \frac{5}{5s-1}}{s} \]

Now, in order to actually take the inverse transform we will need to factor a 5 out of the denominator of the last term. The corrected transform as well as its inverse transform is.

\[ G(s) = -\frac{\frac{3}{s+3} + \frac{2}{s-4} + \frac{1}{s-\frac{1}{5}}}{s} \]

\[ g(t) = -3e^{-3t} + 2e^{4t} + e^{\frac{t}{5}} \]

(b) \[ F(s) = \frac{2 - 5s}{(s - 6)(s^2 + 11)} \]

So, for the first time we’ve got a quadratic in the denominator. Here’s the decomposition for this part.

\[ F(s) = \frac{A}{s - 6} + \frac{Bs + C}{s^2 + 11} \]

Setting numerators equal gives,

\[ 2 - 5s = A(s^2 + 11) + (Bs + C)(s - 6) \]

Okay, in this case we could use \( s = 6 \) to quickly find \( A \), but that’s all it would give. In this case we will need to go the “long” way around to getting the constants. Note that this way will always work, but is sometimes more work than is required.

The “long” way is to completely multiply out the right side and collect like terms.

\[ 2 - 5s = A(s^2 + 11) + (Bs + C)(s - 6) \]

\[ = As^2 + 11A + Bs^2 - 6B + Cs - 6C \]

\[ = \left(A + B\right)s^2 + (-6B + C)s + 11A - 6C \]

In order for these two to be equal the coefficients of the \( s^2 \), \( s \) and the constants must all be equal.
So, setting coefficients equal gives the following system of equations that can be solved.

\[ \begin{align*}
   s^2: & \quad A + B = 0 \\
   s^1: & \quad -6B + C = -5 \\
   s^0: & \quad 11A - 6C = 2
\end{align*} \]

\[ \Rightarrow \quad A = -\frac{28}{47}, \quad B = \frac{28}{47}, \quad C = -\frac{67}{47} \]

Notice that I used \( s^0 \) to denote the constants. This is habit on my part and isn’t really required, it’s just what I’m used to doing. Also, the coefficients are fairly messy fractions in this case. Get used to that. They will often be like this when we get back into solving differential equations.

There is a way to make our life a little easier as well with this. Since all of the fractions have a denominator of 47 we’ll factor that out as we plug them back into the decomposition. This will make dealing with them much easier. The partial fraction decomposition is then,

\[ F(s) = \frac{1}{47} \left( -\frac{28}{s-6} + \frac{28s-67}{s^2+11} \right) \]

\[ = \frac{1}{47} \left( -\frac{28}{s-6} + \frac{28s}{s^2+11} - \frac{67\sqrt{11}}{s^2+11} \right) \]

The inverse transform is then.

\[ f(t) = \frac{1}{47} \left( -28e^{6t} + 28\cos(\sqrt{11}t) - \frac{67}{\sqrt{11}}\sin(\sqrt{11}t) \right) \]

(c) \( G(s) = \frac{25}{s^3(s^2+4s+5)} \)

With this last part do not get excited about the \( s^3 \). We can think of this term as

\[ s^3 = (s-0)^3 \]

and it becomes a linear term to a power. So, the partial fraction decomposition is

\[ G(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds + E}{s^2 + 4s + 5} \]

Setting numerators equal and multiplying out gives.

\[ 25 = As^2 \left( s^2 + 4s + 5 \right) + Bs \left( s^2 + 4s + 5 \right) + C \left( s^2 + 4s + 5 \right) + (Ds + E)s^3 \]

\[ = (A + D)s^4 + (4A + B + E)s^3 + (5A + 4B + C)s^2 + (5B + 4C)s + 5C \]

Setting coefficients equal gives the following system.
This system looks messy, but it’s easier to solve than it might look. First we get $C$ for free from the last equation. We can then use the fourth equation to find $B$. The third equation will then give $A$, etc.

When plugging into the decomposition we’ll get everything with a denominator of 5, then factor that out as we did in the previous part in order to make things easier to deal with.

$$G(s) = \frac{1}{5} \left( \frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11s + 24}{s^2 + 4s + 5} \right)$$

Note that we also factored a minus sign out of the last two terms. To complete this part we’ll need to complete the square on the later term and fix up a couple of numerators. Here’s that work.

$$G(s) = \frac{1}{5} \left( \frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11(s + 2 - 2) + 24}{(s + 2)^2 + 1} \right)$$

$$= \frac{1}{5} \left( \frac{11}{s} - \frac{20}{s^2} + \frac{25}{s^3} - \frac{11(s + 2)}{(s + 2)^2 + 1} - \frac{2}{(s + 2)^2 + 1} \right)$$

The inverse transform is then.

$$g(t) = \frac{1}{5} \left( 11 - 20t + \frac{25}{2} t^2 - 11e^{-2t}\cos(t) - 2e^{-2t}\sin(t) \right)$$

So, one final time. Partial fractions are a fact of life when using Laplace transforms to solve differential equations. Make sure that you can deal with them.