Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**IVP’s With Step Functions**

In this section we will use Laplace transforms to solve IVP’s which contain Heaviside functions in the forcing function. This is where Laplace transform really starts to come into its own as a solution method.

To work these problems we’ll just need to remember the following two formulas,

\[ \mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s) \quad \text{where } F(s) = \mathcal{L}\{f(t)\} \]

\[ \mathcal{L}^{-1}\{e^{-cs}F(s)\} = u_c(t)f(t-c) \quad \text{where } f(t) = \mathcal{L}^{-1}\{F(s)\} \]

In other words, we will always need to remember that in order to take the transform of a function that involves a Heaviside we’ve got to make sure the function has been properly shifted.

Let’s work an example.

**Example 1** Solve the following IVP.

\[ y'' - y' + 5y = 4 + u_2(t)e^{4-2t}, \quad y(0) = 2 \quad y'(0) = -1 \]

**Solution**

First let’s rewrite the forcing function to make sure that it’s being shifted correctly and to identify the function that is actually being shifted.

\[ y'' - y' + 5y = 4 + u_2(t)e^{-2(t-2)} \]

So, it is being shifted correctly and the function that is being shifted is \( e^{-2t} \). Taking the Laplace transform of everything and plugging in the initial conditions gives,

\[ s^2Y(s) - sy(0) - y'(0) - \left(sY(s) - y(0)\right) + 5Y(s) = \frac{4}{s} + \frac{e^{-2s}}{s + 2} \]

\[ \left( s^2 + 5 \right)Y(s) - 2s + 3 = \frac{4}{s} + \frac{e^{-2s}}{s + 2} \]

Now solve for \( Y(s) \).

\[ \left( s^2 + 5 \right)Y(s) = \frac{4}{s} + \frac{e^{-2s}}{s + 2} + 2s - 3 \]

\[ \left( s^2 - s + 5 \right)Y(s) = \frac{2s^2 - 3s + 4}{s} + \frac{e^{-2s}}{s + 2} \]

\[ Y(s) = \frac{2s^2 - 3s + 4}{s(s^2 - s + 5)} + \frac{e^{-2s}}{(s + 2)(s^2 - s + 5)} \]

\[ Y(s) = F(s) + e^{-2t}G(s) \]

Notice that we combined a couple of terms to simplify things a little. Now we need to partial fraction \( F(s) \) and \( G(s) \). We’ll leave it to you to check the details of the partial fractions.
We now need to do the inverse transforms on each of these. We’ll start with \( F(s) \).

\[
F(s) = \frac{2s^2 - 3s + 4}{s(s^2 - s + 5)} = \frac{4}{5} \left( \frac{4}{s} + \frac{6s - 11}{s^2 - s + 5} \right)
\]

\[
G(s) = \frac{1}{(s + 2)(s^2 - s + 5)} = \frac{1}{11} \left( \frac{1}{s + 2} - \frac{s - 3}{s^2 - s + 5} \right)
\]

We now need to do the inverse transforms on each of these. We’ll start with \( F(s) \).

\[
F(s) = \frac{1}{5} \left( \frac{4}{s} + \frac{6(s - \frac{1}{2}) - 11}{(s - \frac{1}{2})^2 + \frac{19}{4}} \right)
\]

\[
f(t) = \frac{1}{5} \left( 4 + 6e^{\frac{t}{2}} \cos \left( \frac{\sqrt{19}}{2} t \right) - \frac{16}{\sqrt{19}} e^{\frac{t}{2}} \sin \left( \frac{\sqrt{19}}{2} t \right) \right)
\]

Now \( G(s) \).

\[
G(s) = \frac{1}{11} \left( \frac{1}{s + 2} - \frac{s - \frac{1}{2}}{(s - \frac{1}{2})^2 + \frac{19}{4}} \right)
\]

\[
g(t) = \frac{1}{11} \left( e^{-2t} - e^{\frac{t}{2}} \cos \left( \frac{\sqrt{19}}{2} t \right) + \frac{5}{\sqrt{19}} e^{\frac{t}{2}} \sin \left( \frac{\sqrt{19}}{2} t \right) \right)
\]

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

\[
Y(s) = F(s) + e^{-2s} G(s)
\]

\[
y(t) = f(t) + u_z(t) g(t - 2)
\]

where \( f(t) \) and \( g(t) \) are the functions shown above.

There is can be a fair amount of work involved in solving differential equations that involve Heaviside functions.

Let’s take a look at another example or two.
Example 2  Solve the following IVP.
\[ y'' - y' = \cos(2t) + \cos(2t - 12)u_6(t) \quad y(0) = -4, \ y'(0) = 0 \]

Solution

Let’s rewrite the differential equation so we can identify the function that is actually being shifted.

\[ y'' - y' = \cos(2t) + \cos(2(t - 6))u_6(t) \]

So, the function that is being shifted is \( \cos(2t) \) and it is being shifted correctly. Taking the Laplace transform of everything and plugging in the initial conditions gives,

\[ s^2Y(s) - sy(0) - y'(0) - (sY(s) - y'(0)) = \frac{s}{s^2 + 4} + \frac{se^{-6s}}{s^2 + 4} \]

\[ (s^2 - s)Y(s) + 4s - 4 = \frac{s}{s^2 + 4} + \frac{se^{-6s}}{s^2 + 4} \]

Now solve for \( Y(s) \).

\[ (s^2 - s)Y(s) = \frac{s + se^{-6s}}{s^2 + 4} - 4s + 4 \]

\[ Y(s) = \frac{s(1 + e^{-6s})}{s(s - 1)(s^2 + 4)} - 4\frac{s - 1}{s(s - 1)} \]

\[ = \frac{1 + e^{-6s}}{(s - 1)(s^2 + 4)} \cdot \frac{4}{s} \]

\[ Y(s) = \left(1 + e^{-6s}\right)F(s) - \frac{4}{s} \]

Notice that we combined the first two terms to simplify things a little. Also there was some canceling going on in this one. Do not expect that to happen on a regular basis. We now need to partial fraction \( F(s) \). We’ll leave the details to you to check.

\[ F(s) = \frac{1}{(s - 1)(s^2 + 4)} = \frac{1}{5} \left( \frac{1}{s - 1} - \frac{s + 1}{s^2 + 4} \right) \]

\[ f(t) = \frac{1}{5} \left( e^t - \cos(2t) - \frac{1}{2} \sin(2t) \right) \]

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

\[ Y(s) = F(s) + e^{-6s}F(s) \frac{4}{s} - \frac{4}{s} \]

\[ y(t) = f(t) + u_6(t)f(t - 6) - 4 \]

where \( f(t) \) is given above.
Example 3  Solve the following IVP.
\[ y'' - 5y' - 14y = 9 + u_3(t) + 4(t-1)u_1(t) \quad y(0) = 0, \quad y'(0) = 10 \]

Solution
Let’s take the Laplace transform of everything and note that in the third term we are shifting \(4t\).

\[
\begin{align*}
  s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) - 14Y(s) &= \frac{9e^{-3s}}{s} + 4e^{-s} \\
  \left(s^2 - 5s - 14\right)Y(s) - 10 &= \frac{9e^{-3s}}{s} + 4e^{-s}
\end{align*}
\]

Now solve for \(Y(s)\).

\[
Y(s) = \frac{9 + e^{-3s}}{s(s - 7)(s + 2)} + \frac{4e^{-s}}{s^2(s - 7)(s + 2)} + \frac{10}{(s - 7)(s + 2)}
\]

So, we have three functions that we’ll need to partial fraction for this problem. I’ll leave it to you to check the details.

\[
F(s) = \frac{1}{s(s - 7)(s + 2)} = -\frac{1}{14} + \frac{1}{63} + \frac{1}{18} s + 2
\]

\[f(t) = -\frac{1}{14}e^{7t} + \frac{1}{18}e^{-2t} \]

\[
G(s) = \frac{1}{s^2(s - 7)(s + 2)} = \frac{5}{196} s + \frac{1}{14} s^2 + \frac{1}{441} s - 7 - \frac{1}{36} s + 2
\]

\[g(t) = \frac{5}{196} e^{7t} + \frac{1}{14} e^{-2t} \]

\[
H(s) = \frac{10}{(s - 7)(s + 2)} = \frac{10}{9} s - 7 - \frac{10}{9} s + 2
\]

\[h(t) = \frac{10}{9} e^{7t} - \frac{10}{9} e^{-2t} \]

Okay, we can now get the solution to the differential equation. Starting with the transform we get,

\[
Y(s) = 9F(s) + e^{-3s}F(s) + 4e^{-s}G(s) + H(s)
\]

\[y(t) = 9f(t) + u_3(t)f(t-3) + 4u_1(t)g(t-1) + h(t) \]

where \(f(t)\), \(g(t)\) and \(h(t)\) are given above.

Let’s work one more example.
Example 4  Solve the following IVP.

\[ y'' + 3y' + 2y = g(t), \quad y(0) = 0 \quad y'(0) = -2 \]

where,

\[
g(t) = \begin{cases} 
 2 & \text{if } t < 6 \\
 4 & \text{if } t \geq 10 \\
 t & \text{if } 6 \leq t < 10
\end{cases}
\]

Solution

The first step is to get \( g(t) \) written in terms of Heaviside functions so that we can take the transform.

\[
g(t) = 2 + (t - 2)u_6(t) + (4 - t)u_{10}(t)
\]

Now, while this is \( g(t) \) written in terms of Heaviside functions it is not yet in proper form for us to take the transform. Remember that each function must be shifted by a proper amount. So, getting things set up for the proper shifts gives us,

\[
g(t) = 2 + (t - 6 + 6 - 2)u_6(t) + (4 - (t - 10 + 10))u_{10}(t)
\]

\[
g(t) = 2 + (t - 6 + 4)u_6(t) + (-6 - (t - 10))u_{10}(t)
\]

So, for the first Heaviside it looks like \( f(t) = t + 4 \) is the function that is being shifted and for the second Heaviside it looks like \( f(t) = -6 - t \) is being shifted.

Now take the Laplace transform of everything and plug in the initial conditions.

\[
s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{2}{s} + e^{-6s} \left( \frac{1}{s^2 + 4} \right) - e^{-10s} \left( \frac{1}{s^2 + 6} \right)
\]

\[
\left( s^2 + 3s + 2 \right) Y(s) + 2 = \frac{2}{s} + e^{-6s} \left( \frac{1}{s^2 + 4} \right) - e^{-10s} \left( \frac{1}{s^2 + 6} \right)
\]

Solve for \( Y(s) \).

\[
\left( s^2 + 3s + 2 \right) Y(s) = \frac{2}{s} + e^{-6s} \left( \frac{1}{s^2 + 4} \right) - e^{-10s} \left( \frac{1}{s^2 + 6} \right) - 2
\]

\[
\left( s^2 + 3s + 2 \right) Y(s) = \frac{2 + 4e^{-6s} - 6e^{-10s}}{s} + \frac{e^{-6s} - e^{-10s}}{s^2} - 2
\]

\[
Y(s) = \frac{2 + 4e^{-6s} - 6e^{-10s}}{s(s + 1)(s + 2)} + \frac{e^{-6s} - e^{-10s}}{s^2(s + 1)(s + 2)} - \frac{2}{(s+1)(s+2)}
\]

\[
Y(s) = \left( 2 + 4e^{-6s} - 6e^{-10s} \right) F(s) + (e^{6s} - e^{-10s}) G(s) - H(s)
\]

Now, in the solving process we simplified things into as few terms as possible. Even doing this, it looks like we’ll still need to do three partial fractions.

I’ll leave the details of the partial fractioning to you to verify. The partial fraction form and...
inverse transform of each of these are.

\[
F(s) = \frac{1}{s(s+1)(s+2)} = \frac{1}{s} - \frac{1}{s+1} + \frac{1}{s+2}
\]

\[
f(t) = \frac{1}{2} e^{-t} + \frac{1}{2} e^{-2t}
\]

\[
G(s) = \frac{1}{s^2(s+1)(s+2)} = \frac{3}{s} + \frac{1}{s^2} + \frac{1}{s+1} - \frac{1}{4} \frac{1}{s+2}
\]

\[
g(t) = -\frac{3}{4} + \frac{1}{2} t + e^{-t} - \frac{1}{4} e^{-2t}
\]

\[
H(s) = \frac{2}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{2}{s+2}
\]

\[
h(t) = 2e^{-t} - 2e^{-2t}
\]

Putting this all back together is going to be a little messy. First rewrite the transform a little to make the inverse transform process possible.

\[
Y(s) = 2F(s) + e^{-6s} \left(4F(s) + G(s)\right) - e^{-10s} \left(6F(s) + G(s)\right) - H(s)
\]

Now, taking the inverse transform of all the pieces gives us the final solution to the IVP.

\[
y(t) = 2f(t) - h(t) + u_6(t) \left(4f(t-6) + g(t-6)\right) - u_{10}(t) \left(6f(t-10) + g(t-10)\right)
\]

where \(f(t), g(t),\) and \(h(t)\) are defined above.

So, the answer to this example is a little messy to write down, but overall the work here wasn’t too terribly bad.

Before proceeding with the next section let’s see how we would have had to solve this IVP if we hadn’t had Laplace transforms. To solve this IVP we would have had to solve three separate IVP’s. One for each portion of \(g(t)\). Here is a list of the IVP’s that we would have had to solve.

1. \(0 < t < 6\)

\[
y'' + 3y' + 2y = 2, \quad y(0) = 0, \quad y'(0) = -2
\]

The solution to this IVP, with some work, can be made to look like,

\[
y_1(t) = 2f(t) - h(t)
\]

2. \(6 \leq t < 10\)

\[
y'' + 3y' + 2y = t, \quad y(6) = y_1(6), \quad y'(6) = y_1'(6)
\]

where, \(y_1(t)\) is the solution to the first IVP. The solution to this IVP, with some work, can be made to look like,

\[
y_2(t) = 2f(t) - h(t) + 4f(t-6) + g(t-6)
\]
3. \( t \geq 10 \)

\[
y'' + 3y' + 2y = 4, \quad y(10) = y_2(10) \quad y'(10) = y'_2(10)
\]

where, \( y_2(t) \) is the solution to the second IVP. The solution to this IVP, with some work, can be made to look like,

\[
y_3(t) = 2f(t) - h(t) + 4f(t - 6) + g(t - 6) - 6f(t - 10) - g(t - 10)
\]

There is a considerable amount of work required to solve all three of these and in each of these the forcing function is not that complicated. Using Laplace transforms saved us a fair amount of work.