Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Convolution Integrals

On occasion we will run across transforms of the form,

\[ H(s) = F(s)G(s) \]

that can’t be dealt with easily using partial fractions. We would like a way to take the inverse transform of such a transform. We can use a convolution integral to do this.

Convolution Integral

If \( f(t) \) and \( g(t) \) are piecewise continuous function on \([0, \infty)\) then the convolution integral of \( f(t) \) and \( g(t) \) is,

\[
(f * g)(t) = \int_{0}^{t} f(t - \tau) g(\tau) \, d\tau
\]

A nice property of convolution integrals is.

\[
(f * g)(t) = (g * f)(t)
\]

Or,

\[
\int_{0}^{t} f(t - \tau) g(\tau) \, d\tau = \int_{0}^{t} f(\tau) g(t - \tau) \, d\tau
\]

The following fact will allow us to take the inverse transforms of a product of transforms.

Fact

\[ \mathcal{L}\{f * g\} = F(s)G(s) \quad \mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) \]

Let’s work a quick example to see how this can be used.

**Example 1** Use a convolution integral to find the inverse transform of the following transform.

\[ H(s) = \frac{1}{(s^2 + a^2)^2} \]

**Solution**

First note that we could use #11 from out table to do this one so that will be a nice check against our work here.

Now, since we are going to use a convolution integral here we will need to write it as a product whose terms are easy to find the inverse transforms of. This is easy to do in this case.

\[ H(s) = \left(\frac{1}{s^2 + a^2}\right) \left(\frac{1}{s^2 + a^2}\right) \]

So, in this case we have,

\[ F(s) = G(s) = \frac{1}{s^2 + a^2} \quad \Rightarrow \quad f(t) = g(t) = \frac{1}{a} \sin(at) \]

Using a convolution integral \( h(t) \) is,
\[ h(t) = (f \ast g)(t) \]
\[ = \frac{1}{a^2} \int_0^t \sin(at - a\tau) \sin(\alpha \tau) d\tau \]
\[ = \frac{1}{2a^2} \left[ \sin(at) - at \cos(at) \right] \]

This is exactly what we would have gotten by using #11 from the table.

Convolution integrals are very useful in the following kinds of problems.

**Example 2** Solve the following IVP

\[ 4y'' + y = g(t), \quad y(0) = 3, \quad y'(0) = -7 \]

**Solution**

First, notice that the forcing function in this case has not been specified. Prior to this section we would not have been able to get a solution to this IVP. With convolution integrals we will be able to get a solution to this kind of IVP. The solution will be in terms of \(g(t)\) but it will be a solution.

Take the Laplace transform of all the terms and plug in the initial conditions.

\[ 4 \left( s^2 Y(s) - sy(0) - y'(0) \right) + Y(s) = G(s) \]
\[ \left( 4s^2 + 1 \right) Y(s) - 12s + 28 = G(s) \]

Notice here that all we could do for the forcing function was to write down \(G(s)\) for its transform. Now, solve for \(Y(s)\).

\[ \left( 4s^2 + 1 \right) Y(s) = G(s) + 12s - 28 \]
\[ Y(s) = \frac{12s - 28}{4 \left( s^2 + \frac{1}{4} \right)} + \frac{G(s)}{4 \left( s^2 + \frac{1}{4} \right)} \]

We factored out a 4 from the denominator in preparation for the inverse transform process. To take inverse transforms we’ll need to split up the first term and we’ll also rewrite the second term a little.

\[ Y(s) = \frac{12s - 28}{4 \left( s^2 + \frac{1}{4} \right)} + \frac{G(s)}{4 \left( s^2 + \frac{1}{4} \right)} \]
\[ = \frac{3s}{s^2 + \frac{1}{4}} - \frac{7}{s^2 + \frac{1}{4}} + \frac{1}{4} G(s) \frac{\frac{2}{s^2 + \frac{1}{4}}}{s^2 + \frac{1}{4}} \]

Now, the first two terms are easy to inverse transform. We’ll need to use a convolution integral on the last term. The two functions that we will be using are,

\[ g(t) = 2 \sin \left( \frac{t}{2} \right) \]

Taking the inverse transform gives us,
\[ y(t) = 3 \cos \left( \frac{t}{2} \right) - 14 \sin \left( \frac{t}{2} \right) + \frac{1}{2} \int_{0}^{t} \sin \left( \frac{\tau}{2} \right) g(t - \tau) \, d\tau \]

So, once we decide on a \( g(t) \) all we need to do is to an integral and we’ll have the solution.

As this last example has shown, using convolution integrals will allow us to solve IVP’s with general forcing functions. This could be very convenient in cases where we have a variety of possible forcing functions and don’t which one we’re going to use. With a convolution integral all that we need to do in these cases is solve the IVP once then go back and evaluate an integral for each possible \( g(t) \). This will save us the work of having to solve the IVP for each and every \( g(t) \).