Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Integrals Involving Quadratics

1. Evaluate the integral \( \int \frac{7}{w^2 + 3w + 3} \, dw \).

Step 1
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

\[
\int \frac{7}{w^2 + 3w + 3} \, dw = \int \frac{7}{(w + \frac{3}{2})^2 + \frac{3}{4}} \, dw
\]

Step 2
From this we can see that the following substitution should work for us.

\[ u = w + \frac{3}{2} \quad \Rightarrow \quad du = dw \]

Doing the substitution gives,

\[
\int \frac{7}{w^2 + 3w + 3} \, dw = \int \frac{7}{u^2 + \frac{3}{4}} \, du
\]

Step 3
This integral can be done with the formula given at the start of this section.
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\[ \int \frac{7}{w^2 + 3w + 3} \, dw = \frac{14}{\sqrt{3}} \tan^{-1} \left( \frac{2u}{\sqrt{3}} \right) + c = \frac{14}{\sqrt{3}} \tan^{-1} \left( \frac{2w + 3}{\sqrt{3}} \right) + c \]

Don’t forget to back substitute in for \( u \)!

2. Evaluate the integral \( \int \frac{10x}{4x^2 - 8x + 9} \, dx \).

Step 1
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

\[ \int \frac{10x}{4x^2 - 8x + 9} \, dx = \int \frac{10x}{4(x-1)^2 + 5} \, dx \]

Step 2
From this we can see that the following substitution should work for us.

\[ u = x - 1 \quad \Rightarrow \quad du = dx \quad & \quad x = u + 1 \]

Doing the substitution gives,

\[ \int \frac{10x}{4x^2 - 8x + 9} \, dx = \int \frac{10(u+1)}{4u^2 + 5} \, du \]

Step 3
We can quickly do this integral if we split it up as follows,

\[ \int \frac{10x}{4x^2 - 8x + 9} \, dx = \int \frac{10u}{4u^2 + 5} \, du + \int \frac{10}{4u^2 + 5} \, du = \int \frac{10u}{4u^2 + 5} \, du + \frac{5}{2} \int \frac{1}{u^2 + \frac{5}{4}} \, du \]

After a quick rewrite of the second integral we can see that we can do the first with the substitution \( v = 4u^2 + 5 \) and the second is an inverse trig integral we can evaluate using the formula given at the start of the notes for this section.

\[ \int \frac{10x}{4x^2 - 8x + 9} \, dx = \frac{5}{4} \left| v \right| + \frac{5}{2} \left( \frac{2}{\sqrt{5}} \right) \tan^{-1} \left( \frac{2u}{\sqrt{5}} \right) + c \]

\[ = \frac{5}{4} \ln \left| 4u^2 + 5 \right| + \sqrt{5} \tan^{-1} \left( \frac{2u}{\sqrt{5}} \right) + c \]

\[ = \frac{5}{4} \ln \left| 4(x-1)^2 + 5 \right| + \sqrt{5} \tan^{-1} \left( \frac{2x-2}{\sqrt{5}} \right) + c \]

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http://tutorial.math.lamar.edu/terms.aspx
Don’t forget to back substitute in for $u$!

3. Evaluate the integral

$$
\int \frac{2t + 9}{\left( t^2 - 14t + 46 \right)^{\frac{3}{2}}} \, dt
$$

Step 1
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

$$
\int \frac{2t + 9}{\left( t^2 - 14t + 46 \right)^{\frac{3}{2}}} \, dt = \int \frac{2t + 9}{\left( (t - 7)^2 - 3 \right)^{\frac{3}{2}}} \, dt
$$

Step 2
From this we can see that the following substitution should work for us.

$$
u = t - 7 \quad \Rightarrow \quad du = dt \quad \& \quad t = u + 7$$

Doing the substitution gives,

$$
\int \frac{2t + 9}{\left( t^2 - 14t + 46 \right)^{\frac{3}{2}}} \, dt = \int \frac{2(u + 7) + 9}{(u^2 - 3)^{\frac{3}{2}}} \, du = \int \frac{2u + 23}{(u^2 - 3)^{\frac{3}{2}}} \, du
$$

Step 3
Next we’ll need to split the integral up as follows,

$$
\int \frac{2t + 9}{\left( t^2 - 14t + 46 \right)^{\frac{3}{2}}} \, dt = \int \frac{2u}{(u^2 - 3)^{\frac{3}{2}}} \, du + \int \frac{23}{(u^2 - 3)^{\frac{3}{2}}} \, du
$$

The first integral can be done with the substitution $v = u^2 - 3$ and the second integral will require the trig substitution $u = \sqrt{3} \sec \theta$. Here is the substitution work.
\[
\int \frac{2t + 9}{(t^2 - 14t + 46)^{3/2}} \, dt = \int v^{-2/3} \, dv + \int \frac{23}{(3\sec^2 \theta - 3)^{3/2}} (\sqrt{3} \sec \theta \tan \theta) \, d\theta \\
= \int v^{-2/3} \, dv + \int \frac{23\sqrt{3} \sec \theta \tan \theta}{(3\tan^2 \theta)^{3/2}} \, d\theta \\
= \int v^{-2/3} \, dv + \int \frac{23\sec \theta}{9\tan \theta} \, d\theta \\
= \int v^{-2/3} \, dv + \frac{23}{9} \int \cos^3 \theta \, d\theta
\]

Now, for the second integral, don’t forget the manipulations we often need to do so we can do these kinds of integrals. If you need some practice on these kinds of integrals go back to the practice problems for the second section of this chapter and work some of them.

Here is the rest of the integration process for this problem.

\[
\int \frac{2t + 9}{(t^2 - 14t + 46)^{3/2}} \, dt = \int v^{-2/3} \, dv + \frac{23}{9} \int \frac{1 - \sin^2 \theta}{\sin^4 \theta} \cos \theta \, d\theta \quad w = \sin \theta
\]

\[
= \int v^{-2/3} \, dv + \frac{23}{9} \int w^{-4} - w^{-2} \, dw
\]

\[
= -\frac{2}{3} v^{-2/3} + \frac{23}{9} \left[-\frac{1}{3} (\sin \theta)^{-3} + (\sin \theta)^{-1}\right] + c
\]

Step 4
We now need to do quite a bit of back substitution to get the answer back into \( t \)'s. Let’s start with the result of the second integration. Converting the \( \theta \)'s back to \( u \)'s will require a quick right triangle.

From the substitution we have,

\[
\sec \theta = \frac{u}{\sqrt{3}} \quad \left(= \frac{\text{hyp}}{\text{adj}}\right)
\]

From the right triangle we get,

\[
\sin \theta = \frac{\sqrt{u^2 - 3}}{u}
\]

Plugging this into the integral above gives,

\[
\int \frac{2t + 9}{(t^2 - 14t + 46)^{3/2}} \, dt = -\frac{2}{3\left(u^2 - 3\right)^{3/2}} - \frac{23u^3}{27\left(u^2 - 3\right)^{3/2}} + \frac{23u}{9\sqrt{u^2 - 3}} + c
\]
Note that we also back substituted for the $v$ in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the $u$.

\[
\int \frac{2t + 9}{(t^2 - 14t + 46)^{\frac{3}{2}}} \, dt = -\frac{2}{3((t-7)^2 - 3)^{\frac{3}{2}}} - \frac{23(t-7)^3}{27((t-7)^2 - 3)^{\frac{3}{2}}} + \frac{23(t-7)}{9\sqrt{(t-7)^2 - 3}} + c
\]

We’ll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don’t get into too much of a hurry when working them.

4. Evaluate the integral  
\[
\int \frac{3z}{(1-4z-2z^2)^{\frac{3}{2}}} \, dz 
\]

Step 1
The first thing to do is to complete the square (we’ll leave it to you to verify the completing the square details) on the quadratic in the denominator.

\[
\int \frac{3z}{(1-4z-2z^2)^{\frac{3}{2}}} \, dz = \int \frac{3z}{(3-2(z+1)^2)^{\frac{3}{2}}} \, dz
\]

Step 2
From this we can see that the following substitution should work for us.

\[
u = z + 1 \quad \Rightarrow \quad du = dz \quad \& \quad z = u - 1
\]

Doing the substitution gives,

\[
\int \frac{3z}{(1-4z-2z^2)^{\frac{3}{2}}} \, dz = \int \frac{3(u-1)}{(3-2u^2)^{\frac{3}{2}}} \, du = \int \frac{3u-3}{(3-2u^2)^{\frac{3}{2}}} \, du
\]

Step 3
Next we’ll need to split the integral up as follows,

\[
\int \frac{3z}{(1-4z-2z^2)^{\frac{3}{2}}} \, dz = \int \frac{3u}{(3-2u^2)^{\frac{3}{2}}} \, du - \int \frac{3}{(3-2u^2)^{\frac{3}{2}}} \, du
\]
The first integral can be done with the substitution \( v = 3 - 2u^2 \) and the second integral will require the trig substitution \( u = \frac{\sqrt{3}}{\sqrt{2}} \sin \theta \). Here is the substitution work.

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = \frac{3}{4} \int v^{-2} \, dv - \frac{3}{(3 - 3\sin^2 \theta)^2} \left( \frac{\sqrt{3}}{\sqrt{2}} \cos \theta \right) d\theta
\]

\[
= \frac{3}{4} \int v^{-2} \, dv - \frac{3}{(3\cos^2 \theta)^2} \left( \frac{\sqrt{3}}{\sqrt{2}} \cos \theta \right) d\theta
\]

\[
= \frac{3}{4} \int v^{-2} \, dv - \frac{1}{\sqrt{2}} \int \sec^3 \theta \, d\theta
\]

The second integral for this problem comes down to an integral that was done in the notes for the second section of this chapter and so we’ll just use the formula derived in that section to do this integral.

Here is the rest of the integration process for this problem.

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = \frac{3}{4} v^{-1} - \frac{1}{2\sqrt{6}} \left[ \sec \theta \tan \theta + \ln \left| \sec \theta + \tan \theta \right| \right] + c
\]

Step 4
We now need to do quite a bit of back substitution to get the answer back into \( z \)'s. Let’s start with the result of the second integration. Converting the \( \theta \)'s back to \( u \)'s will require a quick right triangle.

From the substitution we have,

\[
\sin \theta = \frac{\sqrt{2} u}{\sqrt{3}} \quad \left( \frac{\text{opp}}{\text{hyp}} \right)
\]

From the right triangle we get,

\[
\tan \theta = \frac{\sqrt{2} u}{\sqrt{3 - 2u^2}} \quad \& \quad \sec \theta = \frac{\sqrt{3}}{\sqrt{3 - 2u^2}}
\]

Plugging this into the integral above gives,

\[
\int \frac{3z}{(1 - 4z - 2z^2)^2} \, dz = \frac{3}{4} v^{-1} - \frac{1}{2\sqrt{6}} \left[ \frac{\sqrt{6} u}{3 - 2u^2} + \ln \left| \frac{\sqrt{3} + \sqrt{2} u}{\sqrt{3 - 2u^2}} \right| \right] + c
\]

Note that we also back substituted for the \( v \) in the first term as well and rewrote the first term a little. Finally, all we need to do is back substitute for the \( u \).
\[
\int \frac{3z}{(1-4z-2z^2)^2} \, dz = \frac{3}{4(3-2(z+1)^2)} - \frac{z+1}{6-4(z+1)^2} = \frac{1}{2\sqrt{6}} \ln \left| \frac{\sqrt{3} + \sqrt{2}(z+1)}{\sqrt{3} - 2(z+1)^2} \right| + c
\]

We’ll leave this solution with a final note about these kinds of problems. They are often very long, messy and there are ample opportunities for mistakes so be careful with these and don’t get into too much of a hurry when working them.