Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Center of Mass

1. Find the center of mass for the region bounded by \( y = 4 - x^2 \) that is in the first quadrant.

Step 1
Let’s start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step…..).

We’ll also need the area of this region so let’s find that first.

\[
A = \int_{0}^{2} (4 - x^2) \, dx = \left( 4x - \frac{1}{3} x^3 \right)_{0}^{2} = \frac{16}{3}
\]

Step 2
Next we need to compute the two moments. We didn’t include the density in the computations below because it will only cancel out in the final step.
\[ M_x = \int_0^2 \frac{1}{2} (4 - x^2)^2 \, dx = \int_0^2 \frac{1}{2} (16 - 8x^2 + x^4) \, dx = \frac{1}{2} \left( 16x - \frac{8}{3} x^3 + \frac{1}{5} x^5 \right)_0^2 = \frac{12\pi}{15} \]
\[ M_y = \int_0^2 x (4 - x^2) \, dx = \int_0^2 4x - x^3 \, dx = \left( 2x^2 - \frac{1}{4} x^4 \right)_0^2 = 4 \]

Step 3
Finally the coordinates of the center of mass is,

\[ \bar{x} = \frac{M_y}{M} = \frac{\rho(4)}{\rho(\frac{16\pi}{3})} = \frac{3}{4} \quad \bar{y} = \frac{M_x}{M} = \frac{\rho(\frac{12\pi}{15})}{\rho(\frac{16\pi}{3})} = \frac{8}{5} \]

The center of mass is then: \( \left( \frac{3}{4}, \frac{8}{5} \right) \).

2. Find the center of mass for the region bounded by \( y = 3 - e^{-x} \), the \( x \)-axis, \( x = 2 \) and the \( y \)-axis.

Step 1
Let’s start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step……).

![Sketch of the region](image)

We’ll also need the area of this region so let’s find that first.

\[ A = \int_0^2 3 - e^{-x} \, dx = \left( 3x + e^{-x} \right)_0^2 = 5 + e^{-2} \]

Step 2
Next we need to compute the two moments. We didn’t include the density in the computations below because it will only cancel out in the final step.

\[ M_x = \int_0^2 \frac{1}{2} (3 - e^{-x})^2 \, dx = \int_0^2 \frac{1}{2} (9 - 6e^{-x} + e^{-2x}) \, dx = \frac{1}{2} \left( 9x + 6e^{-x} - \frac{1}{2} e^{-2x} \right)_0^2 = \frac{25}{4} + 3e^{-2} - \frac{1}{4} e^{-4} \]
\[ M_y = \int_0^2 x (3 - e^{-x}) \, dx = \int_0^2 3x - xe^{-x} \, dx = \left( \frac{3}{2} x^2 - xe^{-x} - e^{-x} \right)_0^2 = 5 + 3e^{-2} \]
For the second term in the $M_y$ integration we used the following integration by parts.

$$\int x e^{-x} \, dx \quad u = x \quad du = dx \quad dv = e^{-x} \, dx \quad v = -e^{-x}$$

$$\int x e^{-x} \, dx = -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} = -(xe^{-x} + e^{-x})$$

The minus sign here canceled with the minus sign that was in front of the term in the full integral.

Make sure you don’t forget integration by parts! It is a fairly common integration technique for these kinds of problems.

Step 3
Finally the coordinates of the center of mass is,

$$\bar{x} = \frac{M_y}{M} = \frac{\rho(5+3e^{-2})}{\rho(5+e^{-2})} = 1.05271$$

$$\bar{y} = \frac{M_x}{M} = \frac{\rho\left(\frac{25}{4} + 3e^{-2} - \frac{1}{2}e^{-4}\right)}{\rho(5+e^{-2})} = 1.29523$$

The center of mass is then: $(1.05271, 1.29523)$.

3. Find the center of mass for the triangle with vertices $(0, 0)$, $(-4, 2)$ and $(0, 6)$.

Step 1
Let’s start out with a quick sketch of the region, with the center of mass indicated by the dot (the coordinates of this dot are of course to be determined in the final step…..).

We’ll leave it to you verify the equations of the upper and lower leg of the triangle.

We’ll also need the area of this region so let’s find that first.

$$A = \int_{-4}^{0} (x + 6) - (-\frac{1}{2}x) \, dx = \int_{-4}^{0} \frac{3}{2}x + 6 \, dx = \left(\frac{3}{4}x^2 + 6x\right)\bigg|_{-4}^{0} = 12$$

Step 2
Next we need to compute the two moments. We didn’t include the density in the computations below because it will only cancel out in the final step.

\[
M_x = \int_{-4}^{0} \left[ \left( x + 6 \right)^2 - \left( -\frac{1}{2} x \right)^2 \right] dx = \int_{-4}^{0} \frac{3}{8} x^2 + 6x + 18 \, dx - \frac{1}{8} x^3 + 3x^2 + 18x \right]_{-4}^{0} = 32
\]

\[
M_y = \int_{-4}^{0} x \left( (x + 6) - \left( -\frac{1}{2} x \right) \right) dx = \int_{-4}^{0} \frac{1}{8} x^2 + 6x \, dx - \frac{1}{2} x^3 + 3x^2 \right]_{-4}^{0} = -16
\]

Step 3
Finally the coordinates of the center of mass is,

\[
\bar{x} = \frac{M_y}{M} = \frac{\rho(-16)}{\rho(12)} = -\frac{4}{3}
\]

\[
\bar{y} = \frac{M_x}{M} = \frac{\rho(32)}{\rho(12)} = \frac{8}{3}
\]

The center of mass is then \((-\frac{4}{3}, \frac{8}{3})\).