Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don’t have good Calculus I skills, and you are constantly getting stuck on the Calculus I portion of the problem, you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone; it is simply an acknowledgment that you can’t just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general. You will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you’ve got it memorized doesn’t mean that you can use it. You’ll need to be able to look at an integral and realize that integration by parts can be used (which isn’t always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you’ll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

So, with all that out of the way let me also get a couple of warnings out of the way to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus II many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often
don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Series - Special Series

In this section we are going to take a brief look at three special series. Actually, special may not be the correct term. All three have been named which makes them special in some way, however the main reason that we’re going to look at two of them in this section is that they are the only types of series that we’ll be looking at for which we will be able to get actual values for the series. The third type is divergent and so won’t have a value to worry about.

In general, determining the value of a series is very difficult and outside of these two kinds of series that we’ll look at in this section we will not be determining the value of series in this chapter.

So, let’s get started.

Geometric Series

A geometric series is any series that can be written in the form,

\[ \sum_{n=1}^{\infty} ar^{n-1} \]

or, with an index shift the geometric series will often be written as,

\[ \sum_{n=0}^{\infty} ar^n \]

These are identical series and will have identical values, provided they converge of course.

If we start with the first form it can be shown that the partial sums are,

\[ S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r} \]

The series will converge provided the partial sums form a convergent sequence, so let’s take the limit of the partial sums.

\[ \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( \frac{a}{1-r} - \frac{ar^n}{1-r} \right) \]

\[ = \lim_{n \to \infty} \frac{a}{1-r} - \lim_{n \to \infty} \frac{ar^n}{1-r} \]

\[ = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n \to \infty} r^n \]

Now, from Theorem 3 from the Sequences section we know that the limit above will exist and be finite provided \(-1 < r \leq 1\). However, note that we can’t let \( r = 1 \) since this will give division by zero. Therefore, this will exist and be finite provided \(-1 < r < 1\) and in this case the limit is zero and so we get,

\[ \lim_{n \to \infty} S_n = \frac{a}{1-r} \]
Therefore, a geometric series will converge if \(-1 < r < 1\), which is usually written \(|r| < 1\), its value is,

\[
\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}
\]

Note that in using this formula we’ll need to make sure that we are in the correct form. In other words, if the series starts at \(n = 0\) then the exponent on the \(r\) must be \(n\). Likewise if the series starts at \(n = 1\) then the exponent on the \(r\) must be \(n - 1\).

**Example 1** Determine if the following series converge or diverge. If they converge give the value of the series.

(a) \[\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}\] [Solution]

(b) \[\sum_{n=0}^{\infty} (-4)^{3n} 5^{n-1}\] [Solution]

**Solution**

(a) \[\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}\]

This series doesn’t really look like a geometric series. However, notice that both parts of the series term are numbers raised to a power. This means that it can be put into the form of a geometric series. We will just need to decide which form is the correct form. Since the series starts at \(n = 1\) we will want the exponents on the numbers to be \(n - 1\).

It will be fairly easy to get this into the correct form. Let’s first rewrite things slightly. One of the \(n\)'s in the exponent has a negative in front of it and that can’t be there in the geometric form. So, let’s first get rid of that.

\[
\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} 9^{-(n-2)} 4^{n+1} = \sum_{n=1}^{\infty} 4^{n+1} 9^{n-2}
\]

Now let’s get the correct exponent on each of the numbers. This can be done using simple exponent properties.

\[
\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} \frac{4^{n+1}}{9^{n-2}} = \sum_{n=1}^{\infty} \frac{4^{n-1}}{9^{n-1}}
\]

Now, rewrite the term a little.

\[
\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} 16(9) \frac{4^{n-1}}{9^{n-1}} = \sum_{n=1}^{\infty} 144 \left(\frac{4}{9}\right)^{n-1}
\]

So, this is a geometric series with \(a = 144\) and \(r = \frac{4}{9} < 1\). Therefore, since \(|r| < 1\) we know the series will converge and its value will be,
\[ \sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = \frac{144}{4} - \frac{9}{9} = 9(144) = \frac{1296}{5} \]

(b) \[ \sum_{n=0}^{\infty} \left( -\frac{4}{5} \right)^{3n} \]

Again, this doesn’t look like a geometric series, but it can be put into the correct form. In this case the series starts at \( n = 0 \) so we’ll need the exponents to be \( n \) on the terms. Note that this means we’re going to need to rewrite the exponent on the numerator a little

\[ \sum_{n=0}^{\infty} \left( \frac{-4}{5} \right)^{3n} = \sum_{n=0}^{\infty} \left( \frac{(-4)^3}{5} \right)^{n} = \sum_{n=0}^{\infty} 5 \left( \frac{-64}{5} \right)^{n} \]

So, we’ve got it into the correct form and we can see that \( a = 5 \) and \( r = -\frac{64}{5} \). Also note that \( \vert r \vert \geq 1 \) and so this series diverges.

Back in the Series – Basics section we talked about stripping out terms from a series, but didn’t really provide any examples of how this idea could be used in practice. We can now do some examples.

**Example 2** Use the results from the previous example to determine the value of the following series.

(a) \[ \sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1} \] [Solution]

(b) \[ \sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1} \] [Solution]

**Solution**

(a) \[ \sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1} \]

In this case we could just acknowledge that this is a geometric series that starts at \( n = 0 \) and so we could put it into the correct form and be done with it. However, this does provide us with a nice example of how to use the idea of stripping out terms to our advantage.

Let’s notice that if we strip out the first term from this series we arrive at,

\[ \sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1} = 9^2 4^1 + \sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = 324 + \sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} \]

From the previous example we know the value of the new series that arises here and so the value of the series in this example is,
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$$\sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1} = 324 + \frac{1296}{5} = \frac{2916}{5}$$

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(b) $$\sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1}$$

In this case we can’t strip out terms from the given series to arrive at the series used in the previous example. However, we can start with the series used in the previous example and strip terms out of it to get the series in this example. So, let’s do that. We will strip out the first two terms from the series we looked at in the previous example.

$$\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} = 9^1 4^2 + 9^0 4^3 + \sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1} = 208 + \sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1}$$

We can now use the value of the series from the previous example to get the value of this series.

$$\sum_{n=3}^{\infty} 9^{-n+2} 4^{n+1} = \sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1} - 208 = \frac{1296}{5} - 208 = \frac{256}{5}$$

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Notice that we didn’t discuss the convergence of either of the series in the above example. Here’s why. Consider the following series written in two separate ways (i.e. we stripped out a couple of terms from it).

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \sum_{n=3}^{\infty} a_n$$

Let’s suppose that we know $$\sum_{n=3}^{\infty} a_n$$ is a convergent series. This means that it’s got a finite value and adding three finite terms onto this will not change that fact. So the value of $$\sum_{n=0}^{\infty} a_n$$ is also finite and so is convergent.

Likewise, suppose that $$\sum_{n=0}^{\infty} a_n$$ is convergent. In this case if we subtract three finite values from this value we will remain finite and arrive at the value of $$\sum_{n=3}^{\infty} a_n$$. This is now a finite value and so this series will also be convergent.

In other words, if we have two series and they differ only by the presence, or absence, of a finite number of finite terms they will either both be convergent or they will both be divergent. The difference of a few terms one way or the other will not change the convergence of a series. This is an important idea and we will use it several times in the following sections to simplify some of the tests that we’ll be looking at.
Telescoping Series
It’s now time to look at the second of the three series in this section. In this portion we are going to look at a series that is called a telescoping series. The name in this case comes from what happens with the partial sums and is best shown in an example.

Example 3  Determine if the following series converges or diverges. If it converges find its value.
\[ \sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2} \]

Solution
We first need the partial sums for this series.
\[ s_n = \sum_{i=0}^{n} \frac{1}{i^2 + 3i + 2} \]

Now, let’s notice that we can use partial fractions on the series term to get,
\[ \frac{1}{i^2 + 3i + 2} = \frac{1}{(i+2)(i+1)} = \frac{1}{i+1} - \frac{1}{i+2} \]

I’ll leave the details of the partial fractions to you. By now you should be fairly adept at this since we spent a fair amount of time doing partial fractions back in the Integration Techniques chapter. If you need a refresher you should go back and review that section.

So, what does this do for us? Well, let’s start writing out the terms of the general partial sum for this series using the partial fraction form.
\[ s_n = \sum_{i=0}^{n} \left( \frac{1}{i+1} - \frac{1}{i+2} \right) \]

\[ = \left( \frac{1}{1} - \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{3} \right) + \left( \frac{1}{3} - \frac{1}{4} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+1} \right) + \left( \frac{1}{n+1} - \frac{1}{n+2} \right) \]

\[ = 1 - \frac{1}{n+2} \]

Notice that every term except the first and last term canceled out. This is the origin of the name telescoping series.

This also means that we can determine the convergence of this series by taking the limit of the partial sums.
\[ \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 1 - \frac{1}{n+2} \right) = 1 \]

The sequence of partial sums is convergent and so the series is convergent and has a value of
\[ \sum_{n=0}^{\infty} \frac{1}{n^2 + 3n + 2} = 1 \]

In telescoping series be careful to not assume that successive terms will be the ones that cancel. Consider the following example.
Example 4  Determine if the following series converges or diverges. If it converges find its value.

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3} \]

Solution

As with the last example we’ll leave the partial fractions details to you to verify. The partial sums are,

\[ s_n = \sum_{i=1}^{n} \left( \frac{1}{i+1} - \frac{1}{i+3} \right) = \frac{1}{2} \sum_{i=1}^{n} \left( \frac{1}{i+1} - \frac{1}{i+3} \right) \]

\[ = \frac{1}{2} \left[ \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \left( \frac{1}{4} - \frac{1}{6} \right) + \cdots + \left( \frac{1}{n} - \frac{1}{n+2} \right) + \left( \frac{1}{n+1} - \frac{1}{n+3} \right) \right] \]

\[ = \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right] \]

In this case instead of successive terms canceling a term will cancel with a term that is farther down the list. The end result this time is two initial and two final terms are left. Notice as well that in order to help with the work a little we factored the \( \frac{1}{2} \) out of the series.

The limit of the partial sums is,

\[ \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{1}{2} \left( \frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{5}{12} \]

So, this series is convergent (because the partial sums form a convergent sequence) and its value is,

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3} = \frac{5}{12} \]

Note that it’s not always obvious if a series is telescoping or not until you try to get the partial sums and then see if they are in fact telescoping. There is no test that will tell us that we’ve got a telescoping series right off the bat. Also note that just because you can do partial fractions on a series term does not mean that the series will be a telescoping series. The following series, for example, is not a telescoping series despite the fact that we can partial fraction the series terms.

\[ \sum_{n=1}^{\infty} \frac{3 + 2n}{n^2 + 3n + 2} = \sum_{n=1}^{\infty} \left( \frac{1}{n+1} + \frac{1}{n+2} \right) \]

In order for a series to be a telescoping series we must get terms to cancel and all of these terms are positive and so none will cancel.

Next, we need to go back and address an issue that was first raised in the previous section. In that section we stated that the sum or difference of convergent series was also convergent and that the presence of a multiplicative constant would not affect the convergence of a series. Now that we have a few more series in hand let’s work a quick example showing that.
Example 5  Determine the value of the following series.

\[ \sum_{n=1}^{\infty} \left( \frac{4}{n^2 + 4n + 3} - 9^{-n+1} 4^{n+1} \right) \]

Solution

To get the value of this series all we need to do is rewrite it and then use the previous results.

\[
\sum_{n=1}^{\infty} \left( \frac{4}{n^2 + 4n + 3} - 9^{-n+1} 4^{n+1} \right) = \sum_{n=1}^{\infty} \frac{4}{n^2 + 4n + 3} - \sum_{n=1}^{\infty} 9^{-n+1} 4^{n+1} \\
= 4 \sum_{n=1}^{\infty} \frac{1}{n^2 + 4n + 3} - \sum_{n=1}^{\infty} 9^{-n+1} 4^{n+1} \\
= 4 \left( \frac{5}{12} \right) - \frac{1296}{5} \\
= -\frac{3863}{15}
\]

We didn’t discuss the convergence of this series because it was the sum of two convergent series and that guaranteed that the original series would also be convergent.

Harmonic Series

This is the third and final series that we’re going to look at in this section. Here is the harmonic series.

\[ \sum_{n=1}^{\infty} \frac{1}{n} \]

The harmonic series is divergent and we’ll need to wait until the next section to show that. This series is here because it’s got a name and so I wanted to put it here with the other two named series that we looked at in this section. We’re also going to use the harmonic series to illustrate a couple of ideas about divergent series that we’ve already discussed for convergent series. We’ll do that with the following example.

Example 6  Show that each of the following series are divergent.

(a) \[ \sum_{n=1}^{\infty} \frac{5}{n} \]

(b) \[ \sum_{n=4}^{\infty} \frac{1}{n} \]

Solution

(a) \[ \sum_{n=1}^{\infty} \frac{5}{n} \]

To see that this series is divergent all we need to do is use the fact that we can factor a constant out of a series as follows,

\[ \sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n} \]
Now, \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent and so five times this will still not be a finite number and so the series has to be divergent. In other words, if we multiply a divergent series by a constant it will still be divergent.

(b) \( \sum_{n=4}^{\infty} \frac{1}{n} \)

In this case we’ll start with the harmonic series and strip out the first three terms.

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \sum_{n=4}^{\infty} \frac{1}{n} \quad \Rightarrow \quad \sum_{n=4}^{\infty} \frac{1}{n} = \left( \sum_{n=1}^{\infty} \frac{1}{n} \right) - \frac{11}{6}
\]

In this case we are subtracting a finite number from a divergent series. This subtraction will not change the divergence of the series. We will either have infinity minus a finite number, which is still infinity, or a series with no value minus a finite number, which will still have no value.

Therefore, this series is divergent.

Just like with convergent series, adding/subtracting a finite number from a divergent series is not going to change the divergence of the series.

So, some general rules about the convergence/divergence of a series are now in order. Multiplying a series by a constant will not change the convergence/divergence of the series and adding or subtracting a constant from a series will not change the convergence/divergence of the series. These are nice ideas to keep in mind.