Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don’t have good Calculus I skills, and you are constantly getting stuck on the Calculus I portion of the problem, you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone; it is simply an acknowledgment that you can’t just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general. You will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you’ve got it memorized doesn’t mean that you can use it. You’ll need to be able to look at an integral and realize that integration by parts can be used (which isn’t always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you’ll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

So, with all that out of the way let me also get a couple of warnings out of the way to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus II many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often
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don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Comparison Test / Limit Comparison Test

In the previous section we saw how to relate a series to an improper integral to determine the convergence of a series. While the integral test is a nice test, it does force us to do improper integrals which aren’t always easy and in some cases may be impossible to determine the convergence of.

For instance consider the following series.

\[
\sum_{n=0}^{\infty} \frac{1}{3^n + n}
\]

In order to use the Integral Test we would have to integrate

\[
\int_0^{\infty} \frac{1}{3^x + x} \, dx
\]

and I’m not even sure if it’s possible to do this integral. Nicely enough for us there is another test that we can use on this series that will be much easier to use.

First, let’s note that the series terms are positive. As with the Integral Test that will be important in this section. Next let’s note that we must have \( x > 0 \) since we are integrating on the interval \( 0 \leq x < \infty \). Likewise, regardless of the value of \( x \) we will always have \( 3^x > 0 \). So, if we drop the \( x \) from the denominator the denominator will get smaller and hence the whole fraction will get larger. So,

\[
\frac{1}{3^n + n} < \frac{1}{3^n}
\]

Now,

\[
\sum_{n=0}^{\infty} \frac{1}{3^n}
\]

is a geometric series and we know that since \(|r| = \frac{1}{3} < 1\) the series will converge and its value will be,

\[
\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}
\]

Now, if we go back to our original series and write down the partial sums we get,

\[
S_n = \sum_{i=0}^{n} \frac{1}{3^i + i}
\]

Since all the terms are positive adding a new term will only make the number larger and so the sequence of partial sums must be an increasing sequence.

\[
S_n = \sum_{i=0}^{n} \frac{1}{3^i + i} < \sum_{i=0}^{n+1} \frac{1}{3^i + i} = S_{n+1}
\]

Then since,
and because the terms in these two sequences are positive we can also say that,

\[ s_n = \sum_{i=0}^{n} \frac{1}{3^i + i} < \sum_{i=0}^{n} \frac{1}{3^i} < \sum_{i=0}^{\infty} \frac{1}{3^i} = \frac{3}{2} \quad \Rightarrow \quad s_n < \frac{3}{2} \]

Therefore, the sequence of partial sums is also a bounded sequence. Then from the second section on sequences we know that a monotonic and bounded sequence is also convergent.

So, the sequence of partial sums of our series is a convergent sequence. This means that the series itself,

\[ \sum_{n=0}^{\infty} \frac{1}{3^n + n} \]

is also convergent.

So, what did we do here? We found a series whose terms were always larger than the original series terms and this new series was also convergent. Then since the original series terms were positive (very important) this meant that the original series was also convergent.

To show that a series (with only positive terms) was divergent we could go through a similar argument and find a new divergent series whose terms are always smaller than the original series. In this case the original series would have to take a value larger than the new series. However, since the new series is divergent its value will be infinite. This means that the original series must also be infinite and hence divergent.

We can summarize all this in the following test.

**Comparison Test**

Suppose that we have two series \( \sum a_n \) and \( \sum b_n \) with \( a_n, b_n \geq 0 \) for all \( n \) and \( a_n \leq b_n \) for all \( n \). Then,

1. If \( \sum b_n \) is convergent then so is \( \sum a_n \).
2. If \( \sum a_n \) is divergent then so is \( \sum b_n \).

In other words, we have two series of positive terms and the terms of one of the series is always larger than the terms of the other series. Then if the larger series is convergent the smaller series must also be convergent. Likewise, if the smaller series is divergent then the larger series must also be divergent. Note as well that in order to apply this test we need both series to start at the same place.

A formal proof of this test is at the end of this section.

Do not misuse this test. Just because the smaller of the two series converges does not say anything about the larger series. The larger series may still diverge. Likewise, just because we know that the larger of two series diverges we can’t say that the smaller series will also diverge! Be very careful in using this test.
Recall that we had a similar test for improper integrals back when we were looking at integration techniques. So, if you could use the comparison test for improper integrals you can use the comparison test for series as they are pretty much the same idea.

Note as well that the requirement that \( a_n, b_n \geq 0 \) and \( a_n \leq b_n \) really only need to be true eventually. In other words, if a couple of the first terms are negative or \( a_n \not\leq b_n \) for a couple of the first few terms we’re okay. As long as we eventually reach a point where \( a_n, b_n \geq 0 \) and \( a_n \leq b_n \) for all sufficiently large \( n \) the test will work.

To see why this is true let’s suppose that the series start at \( n = k \) and that the conditions of the test are only true for \( n \geq N + 1 \) and for \( k \leq n \leq N \) at least one of the conditions is not true. If we then look at \( \sum a_n \) (the same thing could be done for \( \sum b_n \)) we get,

\[
\sum_{n=k}^{\infty} a_n = \sum_{n=k}^{N} a_n + \sum_{n=N+1}^{\infty} a_n
\]

The first series is nothing more than a finite sum (no matter how large \( N \) is) of finite terms and so will be finite. So the original series will be convergent/divergent only if the second infinite series on the right is convergent/divergent and the test can be done on the second series as it satisfies the conditions of the test.

Let’s take a look at some examples.

**Example 1** Determine if the following series is convergent or divergent.

\[
\sum_{n=1}^{\infty} \frac{n}{n^2 - \cos^2 (n)}
\]

**Solution**

Since the cosine term in the denominator doesn’t get too large we can assume that the series terms will behave like,

\[
\frac{n}{n^2} = \frac{1}{n}
\]

which, as a series, will diverge. So, from this we can guess that the series will probably diverge and so we’ll need to find a smaller series that will also diverge.

Recall that from the comparison test with improper integrals that we determined that we can make a fraction smaller by either making the numerator smaller or the denominator larger. In this case the two terms in the denominator are both positive. So, if we drop the cosine term we will in fact be making the denominator larger since we will no longer be subtracting off a positive quantity. Therefore,

\[
\frac{n}{n^2} > \frac{1}{n}
\]

Then, since
\[ \sum_{n=1}^{\infty} \frac{1}{n} \]
diverges (it’s harmonic or the p-series test) by the Comparison Test our original series must also diverge.

**Example 2** Determine if the following series converges or diverges.
\[ \sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4 + 5} \]

**Solution**
In this case the “+2” and the “+5” don’t really add anything to the series and so the series terms should behave pretty much like
\[ \frac{n^2}{n^4} = \frac{1}{n^2} \]
which will converge as a series. Therefore, we can guess that the original series will converge and we will need to find a larger series which also converges.

This means that we’ll either have to make the numerator larger or the denominator smaller. We can make the denominator smaller by dropping the “+5”. Doing this gives,
\[ \frac{n^2 + 2}{n^4 + 5} < \frac{n^2 + 2}{n^4} \]

At this point, notice that we can’t drop the “+2” from the numerator since this would make the term smaller and that’s not what we want. However, this is actually the furthest that we need to go. Let’s take a look at the following series.
\[ \sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4 + 5} = \sum_{n=1}^{\infty} \frac{n^2}{n^4} + \sum_{n=1}^{\infty} \frac{2}{n^4} \]
\[ = \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{2}{n^4} \]

As shown, we can write the series as a sum of two series and both of these series are convergent by the p-series test. Therefore, since each of these series are convergent we know that the sum,
\[ \sum_{n=1}^{\infty} \frac{n^2 + 2}{n^4} \]
is also a convergent series. Recall that the sum of two convergent series will also be convergent.

Now, since the terms of this series are larger than the terms of the original series we know that the original series must also be convergent by the Comparison Test.

The comparison test is a nice test that allows us to do problems that either we couldn’t have done with the integral test or at the best would have been very difficult to do with the integral test. That doesn’t mean that it doesn’t have problems of its own.

Consider the following series.
\[ \sum_{n=0}^{\infty} \frac{1}{3^n - n} \]
This is not much different from the first series that we looked at. The original series converged because the $3^n$ gets very large very fast and will be significantly larger than the $n$. Therefore, the $n$ doesn’t really affect the convergence of the series in that case. The fact that we are now subtracting the $n$ off instead of adding the $n$ on really shouldn’t change the convergence. We can say this because the $3^n$ gets very large very fast and the fact that we’re subtracting $n$ off won’t really change the size of this term for all sufficiently large values of $n$.

So, we would expect this series to converge. However, the comparison test won’t work with this series. To use the comparison test on this series we would need to find a larger series that we could easily determine the convergence of. In this case we can’t do what we did with the original series. If we drop the $n$ we will make the denominator larger (since the $n$ was subtracted off) and so the fraction will get smaller and just like when we looked at the comparison test for improper integrals knowing that the smaller of two series converges does not mean that the larger of the two will also converge.

So, we will need something else to do help us determine the convergence of this series. The following variant of the comparison test will allow us to determine the convergence of this series.

**Limit Comparison Test**

Suppose that we have two series $\sum a_n$ and $\sum b_n$ with $a_n \geq 0$, $b_n > 0$ for all $n$. Define,

$$ c = \lim_{n \to \infty} \frac{a_n}{b_n} $$

If $c$ is positive (i.e. $c > 0$) and is finite (i.e. $c < \infty$) then either both series converge or both series diverge.

The proof of this test is at the end of this section.

Note that it doesn’t really matter which series term is in the numerator for this test, we could just have easily defined $c$ as,

$$ c = \lim_{n \to \infty} \frac{b_n}{a_n} $$

and we would get the same results. To see why this is, consider the following two definitions.

$$ c = \lim_{n \to \infty} \frac{a_n}{b_n} \quad \quad \quad \quad \quad \overline{c} = \lim_{n \to \infty} \frac{b_n}{a_n} $$

Start with the first definition and rewrite it as follows, then take the limit.

$$ c = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1}{\frac{b_n}{a_n}} = \frac{1}{\lim_{n \to \infty} \frac{b_n}{a_n}} = \frac{1}{\overline{c}} $$

In other words, if $c$ is positive and finite then so is $\overline{c}$ and if $\overline{c}$ is positive and finite then so is $c$. Likewise if $\overline{c} = 0$ then $c = \infty$ and if $\overline{c} = \infty$ then $c = 0$. Both definitions will give the same results from the test so don’t worry about which series terms should be in the numerator and which should be in the denominator. Choose this to make the limit easy to compute.
Also, this really is a comparison test in some ways. If $c$ is positive and finite this is saying that both of the series terms will behave in generally the same fashion and so we can expect the series themselves to also behave in a similar fashion. If $c = 0$ or $c = \infty$ we can’t say this and so the test fails to give any information.

The limit in this test will often be written as,

$$c = \lim_{n \to \infty} a_n \cdot \frac{1}{b_n}$$

since often both terms will be fractions and this will make the limit easier to deal with.

Let’s see how this test works.

**Example 3** Determine if the following series converges or diverges.

$$\sum_{n=0}^{\infty} \frac{1}{3^n - n}$$

**Solution**

To use the limit comparison test we need to find a second series that we can determine the convergence of easily and has what we assume is the same convergence as the given series. On top of that we will need to choose the new series in such a way as to give us an easy limit to compute for $c$.

We’ve already guessed that this series converges and since it’s vaguely geometric let’s use

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

as the second series. We know that this series converges and there is a chance that since both series have the $3^n$ in it the limit won’t be too bad.

Here’s the limit.

$$c = \lim_{n \to \infty} \frac{1}{3^n} \frac{3^n - n}{1}$$

$$= \lim_{n \to \infty} 1 - \frac{n}{3^n}$$

Now, we’ll need to use L’Hospital’s Rule on the second term in order to actually evaluate this limit.

$$c = 1 - \lim_{n \to \infty} \frac{1}{3^n \ln(3)}$$

$$= 1$$

So, $c$ is positive and finite so by the Comparison Test both series must converge since

$$\sum_{n=0}^{\infty} \frac{1}{3^n}$$

converges.
Example 4 Determine if the following series converges or diverges.

\[
\sum_{n=3}^{\infty} \frac{4n^2 + n}{\sqrt[3]{n^7 + n^3}}
\]

Solution

Fractions involving only polynomials or polynomials under radicals will behave in the same way as the largest power of \(n\) will behave in the limit. So, the terms in this series should behave as,

\[
\frac{n^2}{\sqrt[3]{n^7}} = \frac{n^2}{n^{\frac{7}{3}}} = \frac{1}{n^{\frac{1}{3}}}
\]

and as a series this will diverge by the \(p\)-series test. In fact, this would make a nice choice for our second series in the limit comparison test so let’s use it.

\[
\lim_{n \to \infty} \frac{4n^2 + n}{\sqrt[3]{n^7 + n^3}} \cdot \frac{n^{\frac{1}{3}}}{1} = \lim_{n \to \infty} \frac{4n^\frac{7}{3} + n^\frac{3}{3}}{n\left(1 + \frac{1}{n^4}\right)}
\]

\[
= \lim_{n \to \infty} \frac{n^\frac{3}{3}}{\frac{4 + \frac{1}{n}}{n^\frac{1}{3} + \frac{1}{n^4}}} = \frac{4}{3} = c
\]

So, \(c\) is positive and finite and so both limits will diverge since

\[
\sum_{n=3}^{\infty} \frac{1}{n^{\frac{1}{3}}}
\]

diverges.

Finally, to see why we need \(c\) to be positive and finite (i.e. \(c \neq 0\) and \(c \neq \infty\)) consider the following two series.

\[
\sum_{n=3}^{\infty} \frac{1}{n} \quad \sum_{n=3}^{\infty} \frac{1}{n^2}
\]

The first diverges and the second converges.

Now compute each of the following limits.

\[
\lim_{n \to \infty} \frac{1}{n} - \frac{n^2}{n} = \lim_{n \to \infty} n = \infty \quad \lim_{n \to \infty} \frac{1}{n^2} - \frac{1}{n} = \lim_{n \to \infty} \frac{1}{n} = 0
\]

In the first case the limit from the limit comparison test yields \(c = \infty\) and in the second case the limit yields \(c = 0\). Clearly, both series do not have the same convergence.

Note however, that just because we get \(c = 0\) or \(c = \infty\) doesn’t mean that the series will have the opposite convergence. To see this consider the series,
Both of these series converge and here are the two possible limits that the limit comparison test uses.

\[
\lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{n^3}{n^2} = \lim_{n \to \infty} n = \infty
\]

So, even though both series had the same convergence we got both \( c = 0 \) and \( c = \infty \).

The point of all of this is to remind us that if we get \( c = 0 \) or \( c = \infty \) from the limit comparison test we will know that we have chosen the second series incorrectly and we’ll need to find a different choice in order to get any information about the convergence of the series.

We’ll close out this section with proofs of the two tests.

**Proof of Comparison Test**

The test statement did not specify where each series should start. We only need to require that they start at the same place so to help with the proof we’ll assume that the series start at \( n = 1 \). If the series don’t start at \( n = 1 \) the proof can be redone in exactly the same manner or you could use an index shift to start the series at \( n = 1 \) and then this proof will apply.

We’ll start off with the partial sums of each series.

\[
s_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad t_n = \sum_{i=1}^{n} b_i
\]

Let’s notice a couple of nice facts about these two partial sums. First, because \( a_n, b_n \geq 0 \) we know that,

\[
s_n \leq s_n + a_{n+1} = \sum_{i=1}^{n} a_i + a_{n+1} = \sum_{i=1}^{n+1} a_i = s_{n+1} \quad \Rightarrow \quad s_n \leq s_{n+1}
\]

\[
t_n \leq t_n + b_{n+1} = \sum_{i=1}^{n} b_i + b_{n+1} = \sum_{i=1}^{n+1} b_i = t_{n+1} \quad \Rightarrow \quad t_n \leq t_{n+1}
\]

So, both partial sums form increasing sequences.

Also, because \( a_n \leq b_n \) for all \( n \) we know that we must have \( s_n \leq t_n \) for all \( n \).

With these preliminary facts out of the way we can proceed with the proof of the test itself.

Let’s start out by assuming that \( \sum_{n=1}^{\infty} b_n \) is a convergent series. Since \( b_n \geq 0 \) we know that,

\[
t_n = \sum_{i=1}^{n} b_i \leq \sum_{i=1}^{\infty} b_i
\]
However, we also have established that \( s_n \leq t_n \) for all \( n \) and so for all \( n \) we also have,

\[
s_n \leq \sum_{i=1}^{\infty} b_i
\]

Finally since \( \sum_{n=1}^{\infty} b_n \) is a convergent series it must have a finite value and so the partial sums, \( s_n \) are bounded above. Therefore, from the second section on sequences we know that a monotonic and bounded sequence is also convergent and so \( \{s_n\}_{n=1}^{\infty} \) is a convergent sequence and so \( \sum_{n=1}^{\infty} a_n \) is convergent.

Next, let’s assume that \( \sum_{n=1}^{\infty} a_n \) is divergent. Because \( a_n \geq 0 \) we then know that we must have \( s_n \to \infty \) as \( n \to \infty \). However, we also know that for all \( n \) we have \( s_n \leq t_n \) and therefore we also know that \( t_n \to \infty \) as \( n \to \infty \).

So, \( \{t_n\}_{n=1}^{\infty} \) is a divergent sequence and so \( \sum_{n=1}^{\infty} b_n \) is divergent.

\[
\square
\]

**Proof of Limit Comparison Test**

Because \( 0 < c < \infty \) we can find two positive and finite numbers, \( m \) and \( M \), such that \( m < c < M \).

Now, because \( c = \lim_{n \to \infty} \frac{a_n}{b_n} \) we know that for large enough \( n \) the quotient \( \frac{a_n}{b_n} \) must be close to \( c \) and so there must be a positive integer \( N \) such that if \( n > N \) we also have,

\[
m < \frac{a_n}{b_n} < M
\]

Multiplying through by \( b_n \) gives,

\[
mb_n < a_n < Mb_n
\]

provided \( n > N \).

Now, if \( \sum b_n \) diverges then so does \( \sum mb_n \) and so since \( mb_n < a_n \) for all sufficiently large \( n \) by the Comparison Test \( \sum a_n \) also diverges.

Likewise, if \( \sum b_n \) converges then so does \( \sum Mb_n \) and since \( a_n < Mb_n \) for all sufficiently large \( n \) by the Comparison Test \( \sum a_n \) also converges.

\[
\square
\]