Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don’t have good Calculus I skills and you are constantly getting stuck on the Calculus I portion of the problem you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone it is simply an acknowledgement that you can’t just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general and you will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you’ve got it memorized doesn’t mean that you can use it. You’ll need to be able to look at an integral and realize that integration by parts can be used (which isn’t always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you’ll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

So, with all that out of the way let me also get a couple of warnings out of the way to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus II many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often
don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Root Test
This is the last test for series convergence that we’re going to be looking at. As with the Ratio Test this test will also tell whether a series is absolutely convergent or not rather than simple convergence.

Root Test
Suppose that we have the series \( \sum a_n \). Define,

\[
L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \left| a_n \right|^{\frac{1}{n}}
\]

Then,
1. if \( L < 1 \) the series is absolutely convergent (and hence convergent).
2. if \( L > 1 \) the series is divergent.
3. if \( L = 1 \) the series may be divergent, conditionally convergent, or absolutely convergent.

A proof of this test is at the end of the section.

As with the ratio test, if we get \( L = 1 \) the root test will tell us nothing and we’ll need to use another test to determine the convergence of the series. Also note that if \( L = 1 \) in the Ratio Test then the Root Test will also give \( L = 1 \).

We will also need the following fact in some of these problems.

Fact
\[
\lim_{n \to \infty} \frac{1}{n^n} = 1
\]

Let’s take a look at a couple of examples.

Example 1 Determine if the following series is convergent or divergent.

\[
\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}
\]

Solution
There really isn’t much to these problems other than computing the limit and then using the root test. Here is the limit for this problem.

\[
L = \lim_{n \to \infty} \left| \frac{n^n}{3^{1+2n}} \right|^{\frac{1}{n}} = \lim_{n \to \infty} \frac{1}{3^{\frac{1}{n+2}}} = \frac{\infty}{3^{\frac{1}{2}}} = \infty > 1
\]

So, by the Root Test this series is divergent.

Example 2 Determine if the following series is convergent or divergent.

\[
\sum_{n=0}^{\infty} \left( \frac{5n - 3n^3}{7n^3 + 2} \right)^n
\]

Solution
Again, there isn’t too much to this series.
\[
L = \lim_{n \to \infty} \left| \frac{5n - 3n^3}{7n^3 + 2} \right|^{\frac{1}{n}} = \lim_{n \to \infty} \frac{5n - 3n^3}{7n^3 + 2} = \frac{-3}{7} = \frac{3}{7} < 1
\]

Therefore, by the Root Test this series converges absolutely and hence converges.

Note that we had to keep the absolute value bars on the fraction until we’d taken the limit to get the sign correct.

**Example 3** Determine if the following series is convergent or divergent.

\[
\sum_{n=3}^{\infty} \frac{(-12)^n}{n}
\]

**Solution**

Here’s the limit for this series.

\[
L = \lim_{n \to \infty} \left| \frac{(-12)^n}{n} \right|^{\frac{1}{n}} = \lim_{n \to \infty} \frac{12}{n} = \frac{12}{1} = 12 > 1
\]

After using the fact from above we can see that the Root Test tells us that this series is divergent.

**Proof of Root Test**

First note that we can assume without loss of generality that the series will start at \( n = 1 \) as we’ve done for all our series test proofs. Also note that this proof is very similar to the proof of the Ratio Test.

Let’s start off the proof here by assuming that \( L < 1 \) and we’ll need to show that \( \sum a_n \) is absolutely convergent. To do this let’s first note that because \( L < 1 \) there is some number \( r \) such that \( L < r < 1 \).

Now, recall that,

\[
L = \lim_{n \to \infty} \left| a_n \right| = \lim_{n \to \infty} \left| a_n \right|^{\frac{1}{n}}
\]

and because we also have chosen \( r \) such that \( L < r \) there is some \( N \) such that if \( n \geq N \) we will have,

\[
\left| a_n \right|^{\frac{1}{n}} < r \quad \Rightarrow \quad \left| a_n \right| < r^n
\]

Now the series

\[
\sum_{n=0}^{\infty} r^n
\]

is a geometric series and because \( 0 < r < 1 \) we in fact know that it is a convergent series. Also because \( \left| a_n \right| < r^n \) \( n \geq N \) by the Comparison test the series

\[
\sum_{n=N}^{\infty} \left| a_n \right|
\]
is convergent. However since,
\[
\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{N-1} |a_n| + \sum_{n=N}^{\infty} |a_n|
\]
we know that \(\sum_{n=1}^{\infty} |a_n|\) is also convergent since the first term on the right is a finite sum of finite terms and hence finite. Therefore \(\sum_{n=1}^{\infty} a_n\) is absolutely convergent.

Next, we need to assume that \(L > 1\) and we’ll need to show that \(\sum a_n\) is divergent. Recalling that,
\[
L = \lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} |a_n|^{\frac{1}{n}}
\]
and because \(L > 1\) we know that there must be some \(N\) such that if \(n \geq N\) we will have,
\[
|a_n|^{\frac{1}{n}} > 1 \quad \Rightarrow \quad |a_n| > 1^n = 1
\]
However, if \(|a_n| > 1\) for all \(n \geq N\) then we know that,
\[
\lim_{n \to \infty} |a_n| \neq 0
\]
This in turn means that,
\[
\lim_{n \to \infty} a_n \neq 0
\]
Therefore, by the Divergence Test \(\sum a_n\) is divergent.

Finally, we need to assume that \(L = 1\) and show that we could get a series that has any of the three possibilities. To do this we just need a series for each case. We’ll leave the details of checking to you but all three of the following series have \(L = 1\) and each one exhibits one of the possibilities.

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{absolutely convergent}
\]

\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \quad \text{conditionally convergent}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{divergent}
\]