Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this. The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don’t have good Calculus I skills, and you are constantly getting stuck on the Calculus I portion of the problem, you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone; it is simply an acknowledgment that you can’t just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general. You will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you’ve got it memorized doesn’t mean that you can use it. You’ll need to be able to look at an integral and realize that integration by parts can be used (which isn’t always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you’ll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

So, with all that out of the way let me also get a couple of warnings out of the way to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus II many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often
don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Calculus with Vector Functions

In this section we need to talk briefly about limits, derivatives and integrals of vector functions. As you will see, these behave in a fairly predictable manner. We will be doing all of the work in \( \mathbb{R}^3 \) but we can naturally extend the formulas/work in this section to \( \mathbb{R}^n \) \( (i.e. n\)-dimensional space).

Let’s start with limits. Here is the limit of a vector function.

\[
\lim_{t \to a} \mathbf{r}(t) = \left\{ \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\} = \left\{ \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\} = \lim_{t \to a} f(t) \mathbf{i} + \lim_{t \to a} g(t) \mathbf{j} + \lim_{t \to a} h(t) \mathbf{k}
\]

So, all that we do is take the limit of each of the component’s functions and leave it as a vector.

**Example 1** Compute \( \lim_{t \to 1} \mathbf{r}(t) \) where \( \mathbf{r}(t) = \left\langle t^3, \frac{\sin(3t-3)}{t-1}, e^{2t} \right\rangle \).

**Solution**

There really isn’t all that much to do here.

\[
\lim_{t \to 1} \mathbf{r}(t) = \left\langle \lim_{t \to 1} t^3, \lim_{t \to 1} \frac{\sin(3t-3)}{t-1}, \lim_{t \to 1} e^{2t} \right\rangle = \left\langle \lim_{t \to 1} t^3, \lim_{t \to 1} \frac{3 \cos(3t-3)}{1}, \lim_{t \to 1} e^{2t} \right\rangle = \left\langle 1, 3, e^2 \right\rangle
\]

Notice that we had to use L’Hospital’s Rule on the \( y \) component.

Now let’s take care of derivatives and after seeing how limits work it shouldn’t be too surprising that we have the following for derivatives.

\[
\mathbf{r}'(t) = \left\langle f'(t), g'(t), h'(t) \right\rangle = f'(t) \mathbf{i} + g'(t) \mathbf{j} + h'(t) \mathbf{k}
\]

**Example 2** Compute \( \mathbf{r}'(t) \) for \( \mathbf{r}(t) = t^6 \mathbf{i} + \sin(2t) \mathbf{j} - \ln(t+1) \mathbf{k} \).

**Solution**

There really isn’t too much to this problem other than taking the derivatives.

\[
\mathbf{r}'(t) = 6t^5 \mathbf{i} + 2 \cos(2t) \mathbf{j} - \frac{1}{t+1} \mathbf{k}
\]

Most of the basic facts that we know about derivatives still hold however, just to make it clear here are some facts about derivatives of vector functions.
Facts

\[
\frac{d}{dt}(\vec{u} + \vec{v}) = \vec{u}' + \vec{v}'
\]

\[
(c\vec{u})' = c\vec{u}'
\]

\[
\frac{d}{dt}(f(t)\vec{u}(t)) = f'(t)\vec{u}(t) + f(t)\vec{u}'
\]

\[
\frac{d}{dt}(\vec{u} \cdot \vec{v}) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'
\]

\[
\frac{d}{dt}(\vec{u} \times \vec{v}) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'
\]

\[
\frac{d}{dt}(f(t)) = f'(t)\vec{u}'(f(t))
\]

There is also one quick definition that we should get out of the way so that we can use it when we need to.

A **smooth curve** is any curve for which \( \vec{r}'(t) \) is continuous and \( \vec{r}''(t) \neq 0 \) for any \( t \) except possibly at the endpoints. A helix is a smooth curve, for example.

Finally, we need to discuss integrals of vector functions. Using both limits and derivatives as a guide it shouldn’t be too surprising that we also have the following for integration for indefinite integrals

\[
\int \vec{r}(t) = \left( \int f(t) dt, \int g(t) dt, \int h(t) dt \right) + \vec{c}
\]

\[
\int \vec{r}(t) = \int f(t) dt \vec{i} + \int g(t) dt \vec{j} + \int h(t) dt \vec{k} + \vec{c}
\]

and the following for definite integrals.

\[
\int_{a}^{b} \vec{r}(t) dt = \left( \int_{a}^{b} f(t) dt, \int_{a}^{b} g(t) dt, \int_{a}^{b} h(t) dt \right)
\]

\[
\int_{a}^{b} \vec{r}(t) dt = \int_{a}^{b} f(t) dt \vec{i} + \int_{a}^{b} g(t) dt \vec{j} + \int_{a}^{b} h(t) dt \vec{k}
\]

With the indefinite integrals we put in a constant of integration to make sure that it was clear that the constant in this case needs to be a vector instead of a regular constant.

Also, for the definite integrals we will sometimes write it as follows,

\[
\int_{a}^{b} \vec{r}(t) dt = \left( \int_{a}^{b} f(t) dt, \int_{a}^{b} g(t) dt, \int_{a}^{b} h(t) dt \right)_{a}^{b}
\]

\[
\int_{a}^{b} \vec{r}(t) dt = \left( \int_{a}^{b} f(t) dt \vec{i} + \int_{a}^{b} g(t) dt \vec{j} + \int_{a}^{b} h(t) dt \vec{k} \right)_{a}^{b}
\]

In other words, we will do the indefinite integral and then do the evaluation of the vector as a whole instead of on a component by component basis.
Example 3  Compute $\int \vec{r}(t)\,dt$ for $\vec{r}(t) = \langle \sin(t), 6t, 4t \rangle$.

Solution
All we need to do is integrate each of the components and be done with it.

$$\int \vec{r}(t)\,dt = \langle -\cos(t), 6t, 2t^2 \rangle + \vec{c}$$

Example 4  Compute $\int_0^1 \vec{r}(t)\,dt$ for $\vec{r}(t) = \langle \sin(t), 6, 4 \rangle$.

Solution
In this case all that we need to do is reuse the result from the previous example and then do the evaluation.

$$\int_0^1 \vec{r}(t)\,dt = \left(\langle -\cos(t), 6t, 2t^2 \rangle \right)_0^1$$

$$= \langle -\cos(1), 6, 2 \rangle - \langle -1, 0, 0 \rangle$$

$$= \langle 1 - \cos(1), 6, 2 \rangle$$