Preface

Here are the solutions to the practice problems for my Calculus II notes. Some solutions will have more or less detail than other solutions. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Arc Length with Vector Functions

1. Determine the length of \( \mathbf{r}(t) = (3 - 4t)i + 6t \mathbf{j} - (9 + 2t)k \) from \(-6 \leq t \leq 8\).

Step 1
We first need the magnitude of the derivative of the vector function. This is,

\[
\mathbf{r}'(t) = -4i + 6 \mathbf{j} - 2k
\]

\[
\|\mathbf{r}'(t)\| = \sqrt{16 + 36 + 4} = \sqrt{56} = 2\sqrt{14}
\]

Step 2
The length of the curve is then,

\[
L = \int_{-6}^{8} 2\sqrt{14} \, dt = 2\sqrt{14}\left[ t \right]_{-6}^{8} = 28\sqrt{14}
\]

2. Determine the length of \( \mathbf{r}(t) = \left( \frac{1}{3}t^3, 4t, \sqrt{2t^2} \right) \) from \(0 \leq t \leq 2\).

Step 1
We first need the magnitude of the derivative of the vector function. This is,
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\[ \vec{r}'(t) = \left\langle t^2, 4\sqrt{2}t \right\rangle \]

\[ \left\| \vec{r}'(t) \right\| = \sqrt{t^4 + 16 + 8t^2} = \sqrt{t^4 + 8t^2 + 16} = \sqrt{(t^2 + 4)^2} = t^2 + 4 \]

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Step 2
The length of the curve is then,

\[ L = \int_0^2 r^2 + 4 \, dt = \left( \frac{1}{12} t^3 + 4t \right) \bigg|_0^2 = \frac{17}{6} \]

Note that if we’d not simplified the magnitude this would have been a very difficult integral to compute!

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3. Find the arc length function for \( \vec{r}(t) = \left\langle t^2, 2t^3, 1-t^3 \right\rangle \).

Step 1
We first need the magnitude of the derivative of the vector function. This is,

\[ \vec{r}'(t) = \left\langle 2t, 6t^2, -3t^2 \right\rangle \]

\[ \left\| \vec{r}'(t) \right\| = \sqrt{4t^2 + 36t^4 + 9t^4} = \sqrt{t^2 \left( 4 + 45t^2 \right)} = t\sqrt{4 + 45t^2} = \sqrt{t} \sqrt{4 + 45t^2} \]

For these kinds of problems make sure to simplify the magnitude as much as you can. It can mean the difference between a really simple problem and an incredibly difficult problem.

Note as well that because we are assuming that we are starting at \( t = 0 \) for this kind of problem it is safe to assume that \( t \geq 0 \) and so \( \sqrt{t^2} = |t| = t \).

Step 2
The arc length function is then,

\[ s(t) = \int_0^t u\sqrt{4 + 45u^2} \, du = \frac{1}{135} \left[ 4 + 45t^2 \right]^{3/2} \bigg|_0^t = \frac{1}{135} \left[ (4 + 45t^2)^{3/2} - 8 \right] \]

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4. Find the arc length function for \( \vec{r}(t) = \left\langle 4t, -2t, \sqrt{5} t^2 \right\rangle \).
Step 1
We first need the magnitude of the derivative of the vector function. This is,
\[ \vec{r}'(t) = \left( 4, -2, 2\sqrt{t} \right) \]
\[ \|\vec{r}'(t)\| = \sqrt{16 + 4 + 20t^2} = \sqrt{20 + 20t^2} = \sqrt{20(1 + t^2)} = 2\sqrt{5}\sqrt{1 + t^2} \]

Step 2
The arc length function is then,
\[ s(t) = \int_{0}^{t} 2\sqrt{5}\sqrt{1 + u^2} \, du \]

Do not always expect these integrals to be “simple” integrals. They will often require techniques more involved than just a standard Calculus I substitution. In this case we need the following trig substitution.
\[ u = \tan \theta \quad du = \sec^2 \theta \, d\theta \quad \sqrt{1 + u^2} = \sqrt{1 + \tan^2 \theta} = \sqrt{\sec^2 \theta} = \sec \theta \]
The limits of the integral become,
\[ u = 0 : 0 = \tan \theta \quad \theta = 0 \quad u = t > 0 : t = \tan \theta \quad \theta = \tan^{-1}(t) \]

Now, as noted we know that \( t > 0 \) and so we can safely assume that from the \( u = t \) limit we will get \( 0 < \theta < \frac{\pi}{2} \). This in turn means that we will always be in the first quadrant and we know that secant is positive in the first quadrant. Therefore we can remove the absolute values bars on the secant above.

The arc length function is now,
\[ s(t) = \int_{0}^{\tan^{-1}(t)} 2\sqrt{5}\sec^3 \theta \, d\theta = \sqrt{5} \left( \sec \theta \tan \theta + \ln |\sec \theta + \tan \theta| \right)_{0}^{\tan^{-1}(t)} \]
\[ = \sqrt{5} \left[ \sec(\tan^{-1}(t)) \tan(\tan^{-1}(t)) + \ln |\sec(\tan^{-1}(t)) + \tan(\tan^{-1}(t))| \right] \]

Now we know that \( \tan(\tan^{-1}(t)) = t \) so that will simplify our answer a little. Let’s take a look at the secant term to see if we can simplify that as well. First, from our limit work recall that \( \theta = \tan^{-1}(t) \). Or with a little rewrite we have,
\[ \tan \theta = t = \frac{\text{opposite}}{\text{adjacent}} \]

Construct a right triangle with opposite side being \( t \) and the adjacent side being 1. The hypotenuse is then \( \sqrt{t^2 + 1} \). This in turn means that \( \sec \theta = \sqrt{t^2 + 1} \). So,
\[
\sec\left(\tan^{-1}(t)\right) = \sec(\theta) = \sqrt{t^2 + 1}
\]

With this simplification our arc length function is then,
\[
s(t) = \sqrt{5}\left[t\sqrt{t^2 + 1} + \ln\sqrt{t^2 + 1 + t}\right]
\]

There was some slightly unpleasant simplification here but once we did that we got a much nicer arc length function.

5. Determine where on the curve given by \( \vec{r}(t) = \{t^3, 2t^3, 1-t^3\} \) we are after traveling a distance of 20.

Step 1
From Problem 3 above we know that the arc length function for this vector function is,
\[
s(t) = \frac{1}{135}\left[(4 + 45t^2)^{\frac{3}{2}} - 8\right]
\]

We need to solve this for \( t \). Doing this gives,
\[
(4 + 45t^2)^{\frac{3}{2}} - 8 = 135s
\]
\[
(4 + 45t^2)^{\frac{3}{2}} = 135s + 8
\]
\[
4 + 45t^2 = (135s + 8)^{\frac{2}{3}}
\]
\[
t^2 = \frac{1}{45}\left[(135s + 8)^{\frac{2}{3}} - 4\right] \quad \Rightarrow \quad t = \frac{1}{\sqrt[3]{45}}\left[(135s + 8)^{\frac{2}{3}} - 4\right]
\]

Note that we only used the positive \( t \) after taking the root because the implicit assumption from the arc length function is that \( t \) is positive.

Step 2
We could use this to reparameterize the vector function however that would lead to a particularly unpleasant function in this case.

The key here is to simply realize that what we are being asked to compute is the value of the reparameterized vector function, \( \vec{r}(t(s)) \) when \( s = 20 \). Or, in other words, we want to compute \( \vec{r}(t(20)) \).

So, first,
\[ t(20) = \sqrt{\frac{1}{18} \left[ (135(20) + 8)^{\frac{2}{3}} - 4 \right]} = 2.05633 \]

Our position after traveling a distance of 20 is then,

\[ \vec{r}(t(20)) = \vec{r}(2.05633) = \langle 4.22849, 17.39035, -7.69518 \rangle \]