Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Limit Properties

The time has almost come for us to actually compute some limits. However, before we do that we will need some properties of limits that will make our life somewhat easier. So, let’s take a look at those first. The proof of some of these properties can be found in the Proof of Various Limit Properties section of the Extras chapter.

Properties

First we will assume that \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist and that \( c \) is any constant. Then,

1. \( \lim_{x \to a} [c f(x)] = c \lim_{x \to a} f(x) \)
   In other words we can “factor” a multiplicative constant out of a limit.

2. \( \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \)
   So to take the limit of a sum or difference all we need to do is take the limit of the individual parts and then put them back together with the appropriate sign. This is also not limited to two functions. This fact will work no matter how many functions we’ve got separated by “+” or “-”.

3. \( \lim_{x \to a} [f(x) g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \)
   We take the limits of products in the same way that we can take the limit of sums or differences. Just take the limit of the pieces and then put them back together. Also, as with sums or differences, this fact is not limited to just two functions.

4. \( \lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \), provided \( \lim_{x \to a} g(x) \neq 0 \)
   As noted in the statement we only need to worry about the limit in the denominator being zero when we do the limit of a quotient. If it were zero we would end up with a division by zero error and we need to avoid that.

5. \( \lim_{x \to a} [f(x)]^n = \left( \lim_{x \to a} f(x) \right)^n \), where \( n \) is any real number
   In this property \( n \) can be any real number (positive, negative, integer, fraction, irrational, zero, etc.). In the case that \( n \) is an integer this rule can be thought of as an extended case of 3.

   For example consider the case of \( n = 2 \).
\[
\lim_{x \to a} [f(x)]^2 = \lim_{x \to a} [f(x)f(x)]
\]
\[
= \lim_{x \to a} f(x) \lim_{x \to a} f(x) \quad \text{using property 3}
\]
\[
= \left[ \lim_{x \to a} f(x) \right]^2
\]

The same can be done for any integer \( n \).

6. \[ \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} \]
   
   This is just a special case of the previous example.
   
   \[
   \lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}
   \]
   
   \[
   = \left[ \lim_{x \to a} f(x) \right]^{\frac{1}{n}}
   \]
   
   \[
   = \sqrt[n]{\lim_{x \to a} f(x)}
   \]

7. \[ \lim_{x \to a} c = c, \quad c \text{ is any real number} \]
   
   In other words, the limit of a constant is just the constant. You should be able to
   
   convince yourself of this by drawing the graph of \( f(x) = c \).

8. \[ \lim_{x \to a} x = a \]
   
   As with the last one you should be able to convince yourself of this by drawing the graph
   
   of \( f(x) = x \).

9. \[ \lim_{x \to a} x^n = a^n \]
   
   This is really just a special case of property 5 using \( f(x) = x \).

Note that all these properties also hold for the two one-sided limits as well we just didn’t write

them down with one sided limits to save on space.

Let’s compute a limit or two using these properties. The next couple of examples will lead us to

some truly useful facts about limits that we will use on a continual basis.

**Example 1** Compute the value of the following limit.

\[ \lim_{x \to 2} (3x^2 + 5x - 9) \]

**Solution**

This first time through we will use only the properties above to compute the limit.
First we will use property 2 to break up the limit into three separate limits. We will then use property 1 to bring the constants out of the first two limits. Doing this gives us,
\[
\lim_{x \to -2} (3x^2 + 5x - 9) = \lim_{x \to -2} 3x^2 + \lim_{x \to -2} 5x - \lim_{x \to -2} 9 = 3 \lim_{x \to -2} x^2 + 5 \lim_{x \to -2} x - \lim_{x \to -2} 9
\]

We can now use properties 7 through 9 to actually compute the limit.
\[
\lim_{x \to -2} (3x^2 + 5x - 9) = 3 \lim_{x \to -2} x^2 + 5 \lim_{x \to -2} x - \lim_{x \to -2} 9 = 3(-2)^2 + 5(-2) - 9 = -7
\]

Now, let’s notice that if we had defined
\[
p(x) = 3x^2 + 5x - 9
\]
then the proceeding example would have been,
\[
\lim_{x \to -2} p(x) = \lim_{x \to -2} (3x^2 + 5x - 9) = 3(-2)^2 + 5(-2) - 9 = -7 = p(-2)
\]

In other words, in this case we see that the limit is the same value that we’d get by just evaluating the function at the point in question. This seems to violate one of the main concepts about limits that we’ve seen to this point.

In the previous two sections we made a big deal about the fact that limits do not care about what is happening at the point in question. They only care about what is happening around the point. So how does the previous example fit into this since it appears to violate this main idea about limits?

Despite appearances the limit still doesn’t care about what the function is doing at \( x = -2 \). In this case the function that we’ve got is simply “nice enough” so that what is happening around the point is exactly the same as what is happening at the point. Eventually we will formalize up just what is meant by “nice enough”. At this point let’s not worry too much about what “nice enough” is. Let’s just take advantage of the fact that some functions will be “nice enough”, whatever that means.

The function in the last example was a polynomial. It turns out that all polynomials are “nice enough” so that what is happening around the point is exactly the same as what is happening at the point. This leads to the following fact.
Fact
If \( p(x) \) is a polynomial then,
\[
\lim_{x \to a} p(x) = p(a)
\]

By the end of this section we will generalize this out considerably to most of the functions that we’ll be seeing throughout this course.

Let’s take a look at another example.

**Example 2** Evaluate the following limit.
\[
\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1}
\]

**Solution**
First notice that we can use property 4) to write the limit as,
\[
\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{\lim_{z \to 1} 6 - 3z + 10z^2}{\lim_{z \to 1} -2z^4 + 7z^3 + 1}
\]

Well, actually we should be a little careful. We can do that provided the limit of the denominator isn’t zero. As we will see however, it isn’t in this case so we’re okay.

Now, both the numerator and denominator are polynomials so we can use the fact above to compute the limits of the numerator and the denominator and hence the limit itself.
\[
\lim_{z \to 1} \frac{6 - 3z + 10z^2}{-2z^4 + 7z^3 + 1} = \frac{6 - 3(1) + 10(1)^2}{-2(1)^4 + 7(1)^3 + 1}
\]
\[
= \frac{13}{6}
\]

Notice that the limit of the denominator wasn’t zero and so our use of property 4 was legitimate.

Notice in this last example that again all we really did was evaluate the function at the point in question. So it appears that there is a fairly large class of functions for which this can be done. Let’s generalize the fact from above a little.

Fact
Provided \( f(x) \) is “nice enough” we have,
\[
\lim_{x \to a} f(x) = f(a)
\]
Again, we will formalize up just what we mean by “nice enough” eventually. At this point all we want to do is worry about which functions are “nice enough”. Some functions are “nice enough” for all $x$ while others will only be “nice enough” for certain values of $x$. It will all depend on the function.

As noted in the statement, this fact also holds for the two one-sided limits as well as the normal limit.

Here is a list of some of the more common functions that are “nice enough”.

- Polynomials are nice enough for all $x$’s.
- If $f(x) = \frac{p(x)}{q(x)}$ then $f(x)$ will be nice enough provided both $p(x)$ and $q(x)$ are nice enough and if we don’t get division by zero at the point we’re evaluating at.
- $\cos(x), \sin(x)$ are nice enough for all $x$’s
- $\sec(x), \tan(x)$ are nice enough provided $x \neq \ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$ In other words secant and tangent are nice enough everywhere cosine isn’t zero. To see why recall that these are both really rational functions and that cosine is in the denominator of both then go back up and look at the second bullet above.
- $\csc(x), \cot(x)$ are nice enough provided $x \neq \ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots$ In other words cosecant and cotangent are nice enough everywhere sine isn’t zero.
- $\sqrt[n]{x}$ is nice enough for all $x$ if $n$ is odd.
- $\sqrt[n]{x}$ is nice enough for $x \geq 0$ if $n$ is even. Here we require $x \geq 0$ to avoid having to deal with complex values.
- $a^x, e^x$ are nice enough for all $x$’s.
- $\log_b x, \ln x$ are nice enough for $x > 0$. Remember we can only plug positive numbers into logarithms and not zero or negative numbers.
- Any sum, difference or product of the above functions will also be nice enough. Quotients will be nice enough provided we don’t get division by zero upon evaluating the limit.

The last bullet is important. This means that for any combination of these functions all we need to do is evaluate the function at the point in question, making sure that none of the restrictions are violated. This means that we can now do a large number of limits.

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**Example 3** Evaluate the following limit.

$$\lim_{x \to 3} \left(-\sqrt[3]{x} + \frac{e^x}{1+\ln(x)} + \sin(x)\cos(x)\right)$$

**Solution**

This is a combination of several of the functions listed above and none of the restrictions are violated so all we need to do is plug in $x = 3$ into the function to get the limit.
\[
\lim_{{x \to 3}} \left( -\sqrt[3]{{x}} + \frac{e^{x}}{1 + \ln(x)} + \sin(x) \cos(x) \right) = -\sqrt[3]{{3}} + \frac{e^{3}}{1 + \ln(3)} + \sin(3) \cos(3)
= 8.185427271
\]

Not a very pretty answer, but we can now do the limit.