Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Interpretations of the Derivative**

Before moving on to the section where we learn how to compute derivatives by avoiding the limits we were evaluating in the previous section we need to take a quick look at some of the interpretations of the derivative. All of these interpretations arise from recalling how our definition of the derivative came about. The definition came about by noticing that all the problems that we worked in the first section on limits required us to evaluate the same limit.

**Rate of Change**

The first interpretation of a derivative is rate of change. This was not the first problem that we looked at in the limit chapter, but it is the most important interpretation of the derivative. If \( f(x) \) represents a quantity at any \( x \) then the derivative \( f'(a) \) represents the instantaneous rate of change of \( f(x) \) at \( x = a \).

**Example 1**

Suppose that the amount of water in a holding tank at \( t \) minutes is given by \( V(t) = 2t^2 - 16t + 35 \). Determine each of the following.

(a) Is the volume of water in the tank increasing or decreasing at \( t = 1 \) minute?  
   [Solution]

(b) Is the volume of water in the tank increasing or decreasing at \( t = 5 \) minutes?  
   [Solution]

(c) Is the volume of water in the tank changing faster at \( t = 1 \) or \( t = 5 \) minutes?  
   [Solution]

(d) Is the volume of water in the tank ever not changing? If so, when?  
   [Solution]

**Solution**

In the solution to this example we will use both notations for the derivative just to get you familiar with the different notations.

We are going to need the rate of change of the volume to answer these questions. This means that we will need the derivative of this function since that will give us a formula for the rate of change at any time \( t \). Now, notice that the function giving the volume of water in the tank is the same function that we saw in Example 1 in the last section except the letters have changed. The change in letters between the function in this example versus the function in the example from the last section won’t affect the work and so we can just use the answer from that example with an appropriate change in letters.

The derivative is.

\[
V'(t) = 4t - 16 \quad \text{OR} \quad \frac{dV}{dt} = 4t - 16
\]
Recall from our work in the first limits section that we determined that if the rate of change was positive then the quantity was increasing and if the rate of change was negative then the quantity was decreasing.

We can now work the problem.

(a) Is the volume of water in the tank increasing or decreasing at $t = 1$ minute?
In this case all that we need is the rate of change of the volume at $t = 1$ or,

$$V'(1) = -12 \quad \text{OR} \quad \frac{dV}{dt}igg|_{t=1} = -12$$

So, at $t = 1$ the rate of change is negative and so the volume must be decreasing at this time.

(b) Is the volume of water in the tank increasing or decreasing at $t = 5$ minutes?
Again, we will need the rate of change at $t = 5$.

$$V'(5) = 4 \quad \text{OR} \quad \frac{dV}{dt}igg|_{t=5} = 4$$

In this case the rate of change is positive and so the volume must be increasing at $t = 5$.

(c) Is the volume of water in the tank changing faster at $t = 1$ or $t = 5$ minutes?
To answer this question all that we look at is the size of the rate of change and we don’t worry about the sign of the rate of change. All that we need to know here is that the larger the number the faster the rate of change. So, in this case the volume is changing faster at $t = 1$ than at $t = 5$.

(d) Is the volume of water in the tank ever not changing? If so, when?
The volume will not be changing if it has a rate of change of zero. In order to have a rate of change of zero this means that the derivative must be zero. So, to answer this question we will then need to solve

$$V'(t) = 0 \quad \text{OR} \quad \frac{dV}{dt} = 0$$

This is easy enough to do.

$$4t - 16 = 0 \quad \Rightarrow \quad t = 4$$

So at $t = 4$ the volume isn’t changing. Note that all this is saying is that for a brief instant the volume isn’t changing. It doesn’t say that at this point the volume will quit changing permanently.
If we go back to our answers from parts (a) and (b) we can get an idea about what is going on. At \( t = 1 \) the volume is decreasing and at \( t = 5 \) the volume is increasing. So at some point in time the volume needs to switch from decreasing to increasing. That time is \( t = 4 \).

This is the time in which the volume goes from decreasing to increasing and so for the briefest instant in time the volume will quit changing as it changes from decreasing to increasing.

Note that one of the more common mistakes that students make in these kinds of problems is to try and determine increasing/decreasing from the function values rather than the derivatives. In this case if we took the function values at \( t = 0, \ t = 1 \) and \( t = 5 \) we would get,

\[
V(0) = 35 \quad V(1) = 21 \quad V(5) = 5
\]

Clearly as we go from \( t = 0 \) to \( t = 1 \) the volume has decreased. This might lead us to decide that AT \( t = 1 \) the volume is decreasing. However, we just can't say that. All we can say is that between \( t = 0 \) and \( t = 1 \) the volume has decreased at some point in time. The only way to know what is happening right at \( t = 1 \) is to compute \( V'(1) \) and look at its sign to determine increasing/decreasing. In this case \( V'(1) \) is negative and so the volume really is decreasing at \( t = 1 \).

Now, if we'd plugged into the function rather than the derivative we would have gotten the correct answer for \( t = 1 \) even though our reasoning would have been wrong. It’s important to not let this give you the idea that this will always be the case. It just happened to work out in the case of \( t = 1 \).

To see that this won’t always work let’s now look at \( t = 5 \). If we plug \( t = 1 \) and \( t = 5 \) into the volume we can see that again as we go from \( t = 1 \) to \( t = 5 \) the volume has decreased. Again, however all this says is that the volume HAS decreased somewhere between \( t = 1 \) and \( t = 5 \). It does NOT say that the volume is decreasing at \( t = 5 \). The only way to know what is going on right at \( t = 5 \) is to compute \( V'(5) \) and in this case \( V'(5) \) is positive and so the volume is actually increasing at \( t = 5 \).

So, be careful. When asked to determine if a function is increasing or decreasing at a point make sure and look at the derivative. It is the only sure way to get the correct answer. We are not looking to determine is the function has increased/decreased by the time we reach a particular point. We are looking to determine if the function is increasing/decreasing at that point in question.
Slope of Tangent Line
This is the next major interpretation of the derivative. The slope of the tangent line to $f(x)$ at $x = a$ is $f'(a)$. The tangent line then is given by,

$$ y = f(a) + f'(a)(x - a) $$

**Example 2** Find the tangent line to the following function at $z = 3$.

$$ R(z) = \sqrt{5z} - 8 $$

**Solution**
We first need the derivative of the function and we found that in Example 3 in the last section. The derivative is,

$$ R'(z) = \frac{5}{2\sqrt{5z} - 8} $$

Now all that we need is the function value and derivative (for the slope) at $z = 3$.

$$ R(3) = \sqrt{17} \quad m = R'(3) = \frac{5}{2\sqrt{17}} $$

The tangent line is then,

$$ y = \sqrt{17} + \frac{5}{2\sqrt{17}}(z - 3) $$

Velocity
Recall that this can be thought of as a special case of the rate of change interpretation. If the position of an object is given by $f(t)$ after $t$ units of time the velocity of the object at $t = a$ is given by $f'(a)$.

**Example 3** Suppose that the position of an object after $t$ hours is given by,

$$ g(t) = \frac{t}{t+1} $$

Answer both of the following about this object.

(a) Is the object moving to the right or the left at $t = 10$ hours?  
(b) Does the object ever stop moving?  

**Solution**
Once again we need the derivative and we found that in Example 2 in the last section. The derivative is,

$$ g'(t) = \frac{1}{(t+1)^2} $$

(a) Is the object moving to the right or the left at $t = 10$ hours?

To determine if the object is moving to the right (velocity is positive) or left (velocity is
negative) we need the derivative at \( t = 10 \).

\[
g'(10) = \frac{1}{121}
\]

So the velocity at \( t = 10 \) is positive and so the object is moving to the right at \( t = 10 \).

(b) Does the object ever stop moving?
The object will stop moving if the velocity is ever zero. However, note that the only way a rational expression will ever be zero is if the numerator is zero. Since the numerator of the derivative (and hence the speed) is a constant it can’t be zero.

Therefore, the object will never stop moving.

In fact, we can say a little more here. The object will always be moving to the right since the velocity is always positive.

We’ve seen three major interpretations of the derivative here. You will need to remember these, especially the rate of change, as they will show up continually throughout this course.

Before we leave this section let’s work one more example that encompasses some of the ideas discussed here and is just a nice example to work.

**Example 4** Below is the sketch of a function \( f(x) \). Sketch the graph of the derivative of this function, \( f'(x) \).

![Graph of a function and its derivative](image)

**Solution**
At first glance this seems to an all but impossible task. However, if you have some basic knowledge of the interpretations of the derivative you can get a sketch of the derivative. It will not be a perfect sketch for the most part, but you should be able to get most of the basic features.
of the derivative in the sketch.

Let’s start off with the following sketch of the function with a couple of additions.

Notice that at $x = -3$, $x = -1$, $x = 2$ and $x = 4$ the tangent line to the function is horizontal. This means that the slope of the tangent line must be zero. Now, we know that the slope of the tangent line at a particular point is also the value of the derivative of the function at that point. Therefore, we now know that,

$$f''(-3) = 0 \quad f''(-1) = 0 \quad f''(2) = 0 \quad f''(4) = 0$$

This is a good starting point for us. It gives us a few points on the graph of the derivative. It also breaks the domain of the function up into regions where the function is increasing and decreasing. We know, from our discussions above, that if the function is increasing at a point then the derivative must be positive at that point. Likewise, we know that if the function is decreasing at a point then the derivative must be negative at that point.

We can now give the following information about the derivative.

$$x < -3 \quad f''(x) < 0$$
$$-3 < x < -1 \quad f''(x) > 0$$
$$-1 < x < 2 \quad f''(x) < 0$$
$$2 < x < 4 \quad f''(x) < 0$$
$$x > 4 \quad f''(x) > 0$$

Remember that we are giving the signs of the derivatives here and these are solely a function of whether the function is increasing or decreasing. The sign of the function itself is completely immaterial here and will not in any way effect the sign of the derivative.

This may still seem like we don’t have enough information to get a sketch, but we can get a little
bit more information about the derivative from the graph of the function. In the range \( x < -3 \) we know that the derivative must be negative, however we can also see that the derivative needs to be increasing in this range. It is negative here until we reach \( x = -3 \) and at this point the derivative must be zero. The only way for the derivative to be negative to the left of \( x = -3 \) and zero at \( x = -3 \) is for the derivative to increase as we increase \( x \) towards \( x = -3 \).

Now, in the range \( -3 < x < -1 \) we know that the derivative must be zero at the endpoints and positive in between the two endpoints. Directly to the right of \( x = -3 \) the derivative must also be increasing (because it starts at zero and then goes positive – therefore it must be increasing). So, the derivative in this range must start out increasing and must eventually get back to zero at \( x = -1 \). So, at some point in this interval the derivative must start decreasing before it reaches \( x = -1 \). Now, we have to be careful here because this is just general behavior here at the two endpoints. We won’t know where the derivative goes from increasing to decreasing and it may well change between increasing and decreasing several times before we reach \( x = -1 \). All we can really say is that immediately to the right of \( x = -3 \) the derivative will be increasing and immediately to the left of \( x = -1 \) the derivative will be decreasing.

Next, for the ranges \( -1 < x < 2 \) and \( 2 < x < 4 \) we know the derivative will be zero at the endpoints and negative in between. Also, following the type of reasoning given above we can see in each of these ranges that the derivative will be decreasing just to the right of the left hand endpoint and increasing just to the left of the right hand endpoint.

Finally, in the last region \( x > 4 \) we know that the derivative is zero at \( x = 4 \) and positive to the right of \( x = 4 \). Once again, following the reasoning above, the derivative must also be increasing in this range.

Putting all of this material together (and always taking the simplest choices for increasing and/or decreasing information) gives us the following sketch for the derivative.

![Sketch of the derivative](image.png)

Note that this was done with the actual derivative and so is in fact accurate. Any sketch you do
will probably not look quite the same. The “humps” in each of the regions may be at different places and/or different heights for example. Also, note that we left off the vertical scale because given the information that we’ve got at this point there was no real way to know this information.

That doesn’t mean however that we can’t get some ideas of specific points on the derivative other than where we know the derivative to be zero. To see this let’s check out the following graph of the function (not the derivative, but the function).

At $x = -2$ and $x = 3$ we’ve sketched in a couple of tangent lines. We can use the basic rise/run slope concept to estimate the value of the derivative at these points.

Let’s start at $x = 3$. We’ve got two points on the line here. We can see that each seem to be about one-quarter of the way off the grid line. So, taking that into account and the fact that we go through one complete grid we can see that the slope of the tangent line, and hence the derivative, is approximately -1.5.

At $x = -2$ it looks like (with some heavy estimation) that the second point is about 6.5 grids above the first point and so the slope of the tangent line here, and hence the derivative, is approximately 6.5.

Here is the sketch of the derivative with the vertical scale included and from this we can see that in fact our estimates are pretty close to reality.
Note that this idea of estimating values of derivatives can be a tricky process and does require a fair amount of (possible bad) approximations so while it can be used, you need to be careful with it.

We’ll close out this section by noting that while we’re not going to include an example here we could also use the graph of the derivative to give us a sketch of the function itself. In fact, in the next chapter where we discuss some applications of the derivative we will be looking using information the derivative gives us to sketch the graph of a function.