Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

Differentiation Formulas

1. Find the derivative of \( f(x) = 6x^3 - 9x + 4 \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

\[
\frac{d}{dx} f(x) = 18x^2 - 9
\]

2. Find the derivative of \( y = 2t^4 - 10t^2 + 13t \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.
3. Find the derivative of \( g(z) = 4z^7 - 3z^7 + 9z \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

\[
g'(z) = 28z^6 + 21z^{-8} + 9
\]

4. Find the derivative of \( h(y) = y^4 - 9y^{-3} + 8y^{-2} + 12 \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

\[
h'(y) = -4y^{-5} + 27y^{-4} - 16y^{-3}
\]

5. Find the derivative of \( y = \sqrt[4]{x} + 8\sqrt[3]{x} - 2\sqrt{x} \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that you’ll need to convert the roots to fractional exponents before you start taking the derivative. Here is the rewritten function.

\[
y = x^{\frac{1}{4}} + 8x^{\frac{1}{3}} - 2x^{\frac{1}{2}}
\]

The derivative is,

\[
\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + \frac{8}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-\frac{3}{4}}
\]
6. Find the derivative of \( f(x) = 10 \sqrt[4]{x^3} - \sqrt{x^7} + 6 \sqrt[3]{x^8} - 3 \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that you’ll need to convert the roots to fractional exponents before you start taking the derivative. Here is the rewritten function.

\[
F(x) = 10 \left( x^3 \right)^{\frac{1}{4}} - \left( x^7 \right)^{\frac{1}{2}} + 6 \left( x^8 \right)^{\frac{1}{3}} - 3 = 10x^\frac{3}{4} - x^\frac{7}{2} + 6x^\frac{8}{3} - 3
\]

The derivative is,

\[
f'(x) = 10 \left( \frac{3}{5} \right) x^{-\frac{2}{5}} - \frac{7}{2} x^\frac{5}{2} + 6 \left( \frac{8}{3} \right) x^\frac{5}{3} = 6x^{-\frac{2}{5}} - \frac{7}{2} x^\frac{5}{2} + 16x^\frac{5}{3}
\]

7. Find the derivative of \( f(t) = \frac{4}{t^6} - \frac{1}{6t^3} + \frac{8}{t^5} \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that you’ll need to rewrite the terms so that each of the \( t \)'s are in the numerator with negative exponents before taking the derivative. Here is the rewritten function.

\[
f(t) = 4t^{-6} - \frac{1}{6} t^{-3} + 8t^{-5}
\]

The derivative is,

\[
f'(t) = -4t^{-7} + \frac{1}{2} t^{-4} - 40t^{-6}
\]
8. Find the derivative of \( R(z) = \frac{6}{\sqrt{z^3}} + \frac{1}{8z^4} - \frac{1}{3z^{10}} \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that you’ll need to rewrite the terms so that each of the \( z \)'s are in the numerator with negative exponents and rewrite the root as a fractional exponent before taking the derivative. Here is the rewritten function.

\[
R(z) = 6z^{-\frac{3}{2}} + \frac{1}{8}z^{-4} - \frac{1}{3}z^{-10}
\]

The derivative is,

\[
R'(z) = 6\left(-\frac{3}{2}\right)z^{-\frac{5}{2}} + \frac{1}{8}(-4)z^{-5} - \frac{1}{3}(-10)z^{-11}
= -9z^{-\frac{5}{2}} - \frac{1}{2}z^{-5} + \frac{10}{3}z^{-11}
\]

9. Find the derivative of \( z = x(3x^2 - 9) \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we’ll first need to multiply the function out before we take the derivative. Here is the rewritten function.

\[
z = 3x^3 - 9x
\]

The derivative is,

\[
\frac{dz}{dx} = 9x^2 - 9
\]

10. Find the derivative of \( g(y) = (y - 4)(2y + y^2) \).

Solution
There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we’ll first need to multiply the function out before we take the derivative. Here is the rewritten function.

\[ g(y) = y^3 - 2y^2 - 8y \]

The derivative is,

\[ g'(y) = 3y^2 - 4y - 8 \]

11. Find the derivative of \( h(x) = \frac{4x^3 - 7x + 8}{x} \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we’ll first need to divide the function out and simplify before we take the derivative. Here is the rewritten function.

\[ h(x) = \frac{4x^3}{x} - \frac{7x}{x} + \frac{8}{x} = 4x^2 - 7 + 8x^{-1} \]

The derivative is,

\[ h'(x) = 8x - 8x^{-2} \]

12. Find the derivative of \( f(y) = \frac{y^5 - 5y^3 + 2y}{y^3} \).

Solution

There isn’t much to do here other than take the derivative using the rules we discussed in this section.

Remember that in order to do this derivative we’ll first need to divide the function out and simplify before we take the derivative. Here is the rewritten function.
The derivative is, 

\[ f'(y) = 2y - 4y^3 \]

13. Determine where, if anywhere, the function \( f(x) = x^3 + 9x^2 - 48x + 2 \) is not changing.

Hint: Recall the various interpretations of the derivative. One of them is exactly what we need to do this problem.

Solution

Step 1
Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. So, the function won’t be changing if its rate of change is zero and so all we need to do is find the derivative and set it equal to zero to determine where the rate of change is zero and hence the function will not be changing.

First the derivative, and we’ll do a little factoring while we are at it.

\[ f'(x) = 3x^2 + 18x - 48 = 3(x^2 + 6x - 16) = 3(x + 8)(x - 2) \]

Step 2
Now all that we need to do is set this equation to zero and solve.

\[ f'(x) = 0 \quad \Rightarrow \quad 3(x + 8)(x - 2) = 0 \]

We can easily see from this that the derivative will be zero at \( x = -8 \) and \( x = 2 \). The function therefore not be changing at,

\[ x = -8 \quad \text{and} \quad x = 2 \]

14. Determine where, if anywhere, the function \( y = 2z^4 - z^3 - 3z^2 \) is not changing.

Hint: Recall the various interpretations of the derivative. One of them is exactly what we need to do this problem.
Solution

Step 1
Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. So, the function won’t be changing if its rate of change is zero and so all we need to do is find the derivative and set it equal to zero to determine where the rate of change is zero and hence the function will not be changing.

First the derivative, and we’ll do a little factoring while we are at it.

\[
\frac{dy}{dz} = 8z^3 - 3z^2 - 6z = z(8z^2 - 3z - 6)
\]

Step 2
Now all that we need to do is set this equation to zero and solve.

\[
\frac{dy}{dz} = 0
\]

\[
z(8z^2 - 3z - 6) = 0 \quad \rightarrow \quad z = 0, \quad 8z^2 - 3z - 6 = 0
\]

We can easily see from this that the derivative will be zero at \( z = 0 \), however, because the quadratic doesn’t factor we’ll need to use the quadratic formula to determine where, if anywhere, that will be zero.

\[
z = \frac{3 \pm \sqrt{(-3)^2 - 4(8)(-6)}}{2(8)} = \frac{3 \pm \sqrt{201}}{16}
\]

The function therefore not be changing at,

\[
\begin{align*}
&z = 0 \quad z = \frac{3 + \sqrt{201}}{16} = 1.07359 \\
&z = \frac{3 - \sqrt{201}}{16} = -0.69859
\end{align*}
\]

15. Find the tangent line to \( g(x) = \frac{16}{x} - 4\sqrt{x} \) at \( x = 4 \).

Hint : Recall the various interpretations of the derivative. One of them will help us do this problem.

Solution

Step 1
Recall that one of the interpretations of the derivative is that it gives slope of the tangent line to the graph of the function.

So, we’ll need the derivative of the function. However before doing that we’ll need to do a little rewrite. Here is that work as well as the derivative.

\[ g(x) = 16x^{-1} - 4x^{\frac{1}{2}} \quad \Rightarrow \quad g'(x) = -16x^{-2} - 2x^{-\frac{1}{2}} = -\frac{16}{x^2} - \frac{2}{\sqrt{x}} \]

Note that we rewrote the derivative back into rational expressions with roots to help with the evaluation.

Step 2
Next we need to evaluate the function and derivative at \( x = 4 \).

\[ g(4) = \frac{16}{4} - 4\sqrt{4} = -4 \quad \quad g'(4) = -\frac{16}{4^2} - \frac{2}{\sqrt{4}} - 2 \]

Step 3
Now all that we need to do is write down the equation of the tangent line.

\[ y = g(4) + g'(4)(x - 4) = -4 - 2(x - 4) \quad \rightarrow \quad y = -2x + 4 \]

16. Find the tangent line to \( f(x) = 7x^4 + 8x^6 + 2x \) at \( x = -1 \).

Hint: Recall the various interpretations of the derivative. One of them will help us do this problem.

Solution

Step 1
Recall that one of the interpretations of the derivative is that it gives slope of the tangent line to the graph of the function.

So, we’ll need the derivative of the function.

\[ f''(x) = 28x^3 - 48x^7 + 2 = 28x^3 - \frac{48}{x^7} + 2 \]

Note that we rewrote the derivative back into rational expressions help a little with the evaluation.

Step 2
Next we need to evaluate the function and derivative at \( x = -1 \).

\[
f(-1) = 7 + 8 - 2 = 13 \quad \quad \quad f'(-1) = -28 + 48 + 2 = 22
\]

Step 3

Now all that we need to do is write down the equation of the tangent line.

\[
y = f(-1) + f'(-1)(x + 1) = 13 + 22(x + 1) \quad \Rightarrow \quad y = 22x + 35
\]

17. The position of an object at any time \( t \) is given by \( s(t) = 3t^4 - 40t^3 + 126t^2 - 9 \).

(a) Determine the velocity of the object at any time \( t \).

(b) Does the object ever stop changing?

(c) When is the object moving to the right and when is the object moving to the left?

Solution

Hint : Recall the various interpretations of the derivative. One of them is exactly what we need for this part.

(a) Determine the velocity of the object at any time \( t \).

Recall that one of the interpretations of the derivative is that it gives the velocity of an object if we know the position function of the object.

We’ve been given the position function of the object and so all we need to do is find its derivative and we’ll have the velocity of the object at any time \( t \).

The velocity of the object is then,

\[
s'(t) = 12t^3 - 120t^2 + 252t = 12t(t - 3)(t - 7)
\]

Note that the derivative was factored for later parts and doesn’t really need to be done in general.

Hint : If the object isn’t moving what is the velocity?

(b) Does the object ever stop changing?

The object will not be moving if the velocity is ever zero and so all we need to do is set the derivative equal to zero and solve.

\[
s'(t) = 0 \quad \Rightarrow \quad 12t(t - 3)(t - 7) = 0
\]
From this it is pretty easy to see that the derivative will be zero, and hence the object will not be moving, at
\[ t = 0 \quad t = 3 \quad t = 7 \]

Hint: How does the direction (right vs. left) of movement relate to the sign (positive or negative) of the derivative?

(c) When is the object moving to the right and when is the object moving to the left?

To answer this part all we need to know is where the derivative is positive (and hence the object is moving to the right) or negative (and hence the object is moving to the left). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.

\[
\begin{align*}
\text{s}'(-1) &= -384 & \text{s}'(1) &= 144 & \text{s}'(4) &= -144 & \text{s}'(8) &= 480 \\
\text{s}'(t) &< 0 & \text{s}'(t) &> 0 & \text{s}'(t) &< 0 & \text{s}'(t) &> 0
\end{align*}
\]

\[-2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9\]

From this we get the following right/left movement information.

Moving Right: \(0 < t < 3, \quad 7 < t < \infty\)

Moving Left: \(-\infty < t < 0, \quad 3 < t < 7\)

Note that depending upon your interpretation of \(t\) as time you may or may not have included the interval \(-\infty < t < 0\) in the “Moving Left” portion.

18. Determine where the function \(h(z) = 6 + 40z^3 - 5z^4 - 4z^5\) is increasing and decreasing.

Solution

Hint: Recall the various interpretations of the derivative. One of them is exactly what we need to get the problem started.
Step 1
Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. Since we are talking about where the function is increasing and decreasing we are clearly talking about the rate of change of the function.

So, we’ll need the derivative.

\[ h'(z) = 120z^2 - 20z^3 - 20z^4 = -20z^2(z + 3)(z - 2) \]

Note that the derivative was factored for later steps and doesn’t really need to be done in general.

Hint : Where is the function not changing?

Step 2
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

\[ h'(z) = 0 \implies -20z^2(z + 3)(z - 2) = 0 \]

From this it is pretty easy to see that the derivative will be zero, and hence the function will not be moving, at,

\[ z = 0 \quad z = -3 \quad z = 2 \]

Hint : How does the increasing/decreasing behavior of the function relate to the sign (positive or negative) of the derivative?

Step 3
To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.
19. Determine where the function \( R(x) = (x + 1)(x - 2)^2 \) is increasing and decreasing.

Solution

Hint : Recall the various interpretations of the derivative. One of them is exactly what we need to get the problem started.

Step 1
Recall that one of the interpretations of the derivative is that it gives the rate of change of the function. Since we are talking about where the function is increasing and decreasing we are clearly talking about the rate of change of the function.

So, we'll need the derivative. First however we'll need to multiply out the function so we can actually take the derivative. Here is the rewritten function and the derivative.

\[
R(x) = x^3 - 3x^2 + 4 \quad \quad \quad R'(x) = 3x^2 - 6x = 3x(x - 2)
\]

Note that the derivative was factored for later steps and doesn't really need to be done in general.

Hint : Where is the function not changing?

Step 2
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

\[
R'(x) = 0 \quad \Rightarrow \quad 3x(x - 2) = 0
\]

From this it is pretty easy to see that the derivative will be zero, and hence the function will not be moving, at,

\[
x = 0 \quad x = 2
\]

Hint : How does the increasing/decreasing behavior of the function relate to the sign (positive or negative) of the derivative?

Step 3
To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.

From this we get the following increasing/decreasing information.

<table>
<thead>
<tr>
<th>$R'(x) &gt; 0$</th>
<th>$R'(1) &lt; 0$</th>
<th>$R'(3) &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R'(-1) = 9$</td>
<td>$R'(1) = -3$</td>
<td>$R'(3) = 9$</td>
</tr>
</tbody>
</table>

Increasing : $-\infty < x < 0$, $2 < z < \infty$
Decreasing : $0 < z < 2$

20. Determine where, if anywhere, the tangent line to $f(x) = x^3 - 5x^2 + x$ is parallel to the line $y = 4x + 23$.

Solution

Step 1
The first thing that we’ll need of course is the slope of the tangent line. So, all we need to do is take the derivative of the function.

$$f'(x) = 3x^2 - 10x + 1$$

Hint : What is the relationship between the slope of two parallel lines?

Step 2
Two lines that are parallel will have the same slope and so all we need to do is determine where the slope of the tangent line will be 4, the slope of the given line. In other words, we’ll need to solve,
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\[ f''(x) = 4 \quad \rightarrow \quad 3x^2 - 10x + 1 = 4 \quad \rightarrow \quad 3x^2 - 10x - 3 = 0 \]

This quadratic doesn’t factor and so a quick use of the quadratic formula will solve this for us.

\[ x = \frac{10 \pm \sqrt{136}}{6} = \frac{10 \pm 2\sqrt{34}}{6} = \frac{5 \pm \sqrt{34}}{3} \]

So, the tangent line will be parallel to \( y = 4x + 23 \) at,

\[
\begin{array}{c|c}
\text{ } & \text{ } \\
\hline
-0.276984 & 3.61032 \\
\hline
\end{array}
\]