Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Derivatives of Trig Functions**

With this section we’re going to start looking at the derivatives of functions other than polynomials or roots of polynomials. We’ll start this process off by taking a look at the derivatives of the six trig functions. Two of the derivatives will be derived. The remaining four are left to the reader and will follow similar proofs for the two given here.

Before we actually get into the derivatives of the trig functions we need to give a couple of limits that will show up in the derivation of two of the derivatives.

**Fact**

\[
\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0
\]

See the [Proof of Trig Limits](#) section of the Extras chapter to see the proof of these two limits.

Before we start differentiating trig functions let’s work a quick set of limit problems that this fact now allows us to do.

**Example 1** Evaluate each of the following limits.

(a) \( \lim_{\theta \to 0} \frac{\sin \theta}{6\theta} \)  

(b) \( \lim_{x \to 0} \frac{\sin(6x)}{x} \)  

(c) \( \lim_{x \to 0} \frac{x}{\sin(7x)} \)  

(d) \( \lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)} \)  

(e) \( \lim_{x \to 4} \frac{\sin(x - 4)}{x - 4} \)  

(f) \( \lim_{z \to 0} \frac{\cos(2z) - 1}{z} \)

**Solution**

(a) \( \lim_{\theta \to 0} \frac{\sin \theta}{6\theta} \)

There really isn’t a whole lot to this limit. In fact, it’s only here to contrast with the next example so you can see the difference in how these work. In this case since there is only a 6 in the denominator we’ll just factor this out and then use the fact.

\[
\lim_{\theta \to 0} \frac{\sin \theta}{6\theta} = \frac{1}{6} \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{1}{6} (1) = \frac{1}{6}
\]

[Return to Problems]
(b) \( \lim_{x \to 0} \frac{\sin(6x)}{x} \)

Now, in this case we can’t factor the 6 out of the sine so we’re stuck with it there and we’ll need to figure out a way to deal with it. To do this problem we need to notice that in the fact the argument of the sine is the same as the denominator (i.e. both \( \theta \)’s). So we need to get both of the argument of the sine and the denominator to be the same. We can do this by multiplying the numerator and the denominator by 6 as follows.

\[
\lim_{x \to 0} \frac{\sin(6x)}{x} = \lim_{x \to 0} \frac{6 \sin(6x)}{6x} = 6 \lim_{x \to 0} \frac{\sin(6x)}{6x}
\]

Note that we factored the 6 in the numerator out of the limit. At this point, while it may not look like it, we can use the fact above to finish the limit.

To see that we can use the fact on this limit let’s do a change of variables. A change of variables is really just a renaming of portions of the problem to make something look more like something we know how to deal with. They can’t always be done, but sometimes, such as this case, they can simplify the problem. The change of variables here is to let \( \theta = 6x \) and then notice that as \( x \to 0 \) we also have \( \theta \to 6(0) = 0 \). When doing a change of variables in a limit we need to change all the \( x \)’s into \( \theta \)’s and that includes the one in the limit.

Doing the change of variables on this limit gives,

\[
\lim_{x \to 0} \frac{\sin(6x)}{x} = 6 \lim_{x \to 0} \frac{\sin(6x)}{6x} \quad \text{let } \theta = 6x
\]

\[
= 6 \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta}
\]

\[
= 6(1)
\]

\[
= 6
\]

And there we are. Note that we didn’t really need to do a change of variables here. All we really need to notice is that the argument of the sine is the same as the denominator and then we can use the fact. A change of variables, in this case, is really only needed to make it clear that the fact does work.

(e) \( \lim_{x \to 0} \frac{x}{\sin(7x)} \)

In this case we appear to have a small problem in that the function we’re taking the limit of here is upside down compared to that in the fact. This is not the problem it appears to be once we notice that,
\[
\frac{x}{\sin(7x)} = \frac{1}{\sin(7x)}
\]

and then all we need to do is recall a nice property of limits that allows us to do,
\[
\lim_{x \to 0} \frac{x}{\sin(7x)} = \lim_{x \to 0} \frac{1}{\sin(7x)}
\]
\[
= \lim_{x \to 0} \frac{1}{\sin(7x)}
\]
\[
= \frac{1}{\lim_{x \to 0} \sin(7x)}
\]
\[
= \frac{1}{1}
\]
\[
= \frac{1}{7}
\]

With a little rewriting we can see that we do in fact end up needing to do a limit like the one we did in the previous part. So, let’s do the limit here and this time we won’t bother with a change of variable to help us out. All we need to do is multiply the numerator and denominator of the fraction in the denominator by 7 to get things set up to use the fact. Here is the work for this limit.
\[
\lim_{x \to 0} \frac{x}{\sin(7x)} = \lim_{x \to 0} \frac{1}{7 \sin(7x)}
\]
\[
= \frac{1}{7} \lim_{x \to 0} \frac{\sin(7x)}{7x}
\]
\[
= \frac{1}{7} \lim_{x \to 0} \frac{\sin(7x)}{7x}
\]
\[
= \frac{1}{7}
\]

(d) \(\lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)}\)

This limit looks nothing like the limit in the fact, however it can be thought of as a combination of the previous two parts by doing a little rewriting. First, we’ll split the fraction up as follows,
\[
\lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)} = \lim_{t \to 0} \frac{\sin(3t)}{8t} \cdot \frac{1}{\sin(8t)}
\]

Now, the fact wants a \(t\) in the denominator of the first and in the numerator of the second. This is
easy enough to do if we multiply the whole thing by \( \frac{1}{t} \) (which is just one after all and so won’t change the problem) and then do a little rearranging as follows,

\[
\lim_{t \to 0} \frac{\sin(3t)}{\sin(8t)} = \lim_{t \to 0} \frac{\sin(3t)}{1} \frac{1}{\sin(8t)} \frac{t}{t} = \lim_{t \to 0} \frac{\sin(3t)}{t} \frac{t}{\sin(8t)} = \left( \lim_{t \to 0} \frac{\sin(3t)}{t} \right) \left( \lim_{t \to 0} \frac{t}{\sin(8t)} \right)
\]

At this point we can see that this really is two limits that we’ve seen before. Here is the work for each of these and notice on the second limit that we’re going to work it a little differently than we did in the previous part. This time we’re going to notice that it doesn’t really matter whether the sine is in the numerator or the denominator as long as the argument of the sine is the same as what’s in the numerator the limit is still one.

Here is the work for this limit.

\[
\lim_{t \to 0} \frac{3\sin(3t)}{8t} = \left( \lim_{t \to 0} \frac{3\sin(3t)}{3t} \right) \left( \lim_{t \to 0} \frac{8t}{8\sin(8t)} \right) = \left( \lim_{t \to 0} \frac{\sin(3t)}{3t} \right) \left( \frac{1}{8} \lim_{t \to 0} \frac{8t}{\sin(8t)} \right) = \left( \frac{1}{8} \right) = \frac{3}{8}
\]

(e) \( \lim_{x \to 4} \frac{\sin(x - 4)}{x - 4} \)

This limit almost looks the same as that in the fact in the sense that the argument of the sine is the same as what is in the denominator. However, notice that, in the limit, \( x \) is going to 4 and not 0 as the fact requires. However, with a change of variables we can see that this limit is in fact set to use the fact above regardless.

So, let \( \theta = x - 4 \) and then notice that as \( x \to 4 \) we have \( \theta \to 0 \). Therefore, after doing the change of variable the limit becomes,

\[
\lim_{x \to 4} \frac{\sin(x - 4)}{x - 4} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1
\]
(f) \[ \lim_{z \to 0} \frac{\cos(2z) - 1}{z} \]

The previous parts of this example all used the sine portion of the fact. However, we could just have easily used the cosine portion so here is a quick example using the cosine portion to illustrate this. We’ll not put in much explanation here as this really does work in the same manner as the sine portion.

\[
\lim_{z \to 0} \frac{\cos(2z) - 1}{z} = \lim_{z \to 0} \frac{2(\cos(2z) - 1)}{2z} = 2\lim_{z \to 0} \frac{\cos(2z) - 1}{2z} = 2(0) = 0
\]

All that is required to use the fact is that the argument of the cosine is the same as the denominator.

Okay, now that we’ve gotten this set of limit examples out of the way let’s get back to the main point of this section, differentiating trig functions.

We’ll start with finding the derivative of the sine function. To do this we will need to use the definition of the derivative. It’s been a while since we’ve had to use this, but sometimes there just isn’t anything we can do about it. Here is the definition of the derivative for the sine function.

\[
\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x + h) - \sin(x)}{h}
\]

Since we can’t just plug in \( h = 0 \) to evaluate the limit we will need to use the following trig formula on the first sine in the numerator.

\[ \sin(x + h) = \sin(x)\cos(h) + \cos(x)\sin(h) \]

Doing this gives us,

\[
\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} = \lim_{h \to 0} \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} = \lim_{h \to 0} \sin(x)\frac{\cos(h) - 1}{h} + \lim_{h \to 0} \cos(x)\frac{\sin(h)}{h}
\]
As you can see upon using the trig formula we can combine the first and third term and then factor a sine out of that. We can then break up the fraction into two pieces, both of which can be dealt with separately.

Now, both of the limits here are limits as \( h \) approaches zero. In the first limit we have a \( \sin(x) \) and in the second limit we have a \( \cos(x) \). Both of these are only functions of \( x \) only and as \( h \) moves in towards zero this has no affect on the value of \( x \). Therefore, as far as the limits are concerned, these two functions are constants and can be factored out of their respective limits. Doing this gives,

\[
\frac{d}{dx} \sin(x) = \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h}
\]

At this point all we need to do is use the limits in the fact above to finish out this problem.

\[
\frac{d}{dx} \sin(x) = \sin(x)(0) + \cos(x)(1) = \cos(x)
\]

Differentiating cosine is done in a similar fashion. It will require a different trig formula, but other than that is an almost identical proof. The details will be left to you. When done with the proof you should get,

\[
\frac{d}{dx} \cos(x) = -\sin(x)
\]

With these two out of the way the remaining four are fairly simple to get. All the remaining four trig functions can be defined in terms of sine and cosine and these definitions, along with appropriate derivative rules, can be used to get their derivatives.

Let’s take a look at tangent. Tangent is defined as,

\[
\tan(x) = \frac{\sin(x)}{\cos(x)}
\]

Now that we have the derivatives of sine and cosine all that we need to do is use the quotient rule on this. Let’s do that.

\[
\frac{d}{dx} \tan(x) = \frac{d}{dx} \left( \frac{\sin(x)}{\cos(x)} \right)
\]

\[
= \frac{\cos(x) \cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2}
\]

\[
= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}
\]

\[
= \frac{1}{\cos^2(x)}
\]
Now, recall that \( \cos^2(x) + \sin^2(x) = 1 \) and if we also recall the definition of secant in terms of cosine we arrive at,

\[
\frac{d}{dx} \left( \tan(x) \right) = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)
\]

The remaining three trig functions are also quotients involving sine and/or cosine and so can be differentiated in a similar manner. We'll leave the details to you. Here are the derivatives of all six of the trig functions.

**Derivatives of the six trig functions**

\[
\begin{align*}
\frac{d}{dx} (\sin(x)) &= \cos(x) \\
\frac{d}{dx} (\cos(x)) &= -\sin(x) \\
\frac{d}{dx} (\tan(x)) &= \sec^2(x) \\
\frac{d}{dx} (\cot(x)) &= -\csc^2(x) \\
\frac{d}{dx} (\sec(x)) &= \sec(x) \tan(x) \\
\frac{d}{dx} (\csc(x)) &= -\csc(x) \cot(x)
\end{align*}
\]

At this point we should work some examples.

**Example 2** Differentiate each of the following functions.

(a) \( g(x) = 3 \sec(x) - 10 \cot(x) \)  
(b) \( h(w) = 3w^{-4} - w^2 \tan(w) \)  
(c) \( y = 5 \sin(x) \cos(x) + 4 \csc(x) \)  
(d) \( P(t) = \frac{\sin(t)}{3 - 2 \cos(t)} \)

**Solution**

(a) \( g(x) = 3 \sec(x) - 10 \cot(x) \)

There really isn’t a whole lot to this problem. We’ll just differentiate each term using the formulas from above.

\[
g'(x) = 3 \sec(x) \tan(x) - 10 \left( -\csc^2(x) \right)
\]

\[
= 3 \sec(x) \tan(x) + 10 \csc^2(x)
\]

(b) \( h(w) = 3w^{-4} - w^2 \tan(w) \)

In this part we will need to use the product rule on the second term and note that we really will need the product rule here. There is no other way to do this derivative unlike what we saw when
we first looked at the product rule. When we first looked at the product rule the only functions we knew how to differentiate were polynomials and in those cases all we really needed to do was multiply them out and we could take the derivative without the product rule. We are now getting into the point where we will be forced to do the product rule at times regardless of whether or not we want to.

We will also need to be careful with the minus sign in front of the second term and make sure that it gets dealt with properly. There are two ways to deal with this. One way it to make sure that you use a set of parenthesis as follows,

\[
\left(2w \tan(w) + w^2 \sec^2(w)\right)
\]

Because the second term is being subtracted off of the first term then the whole derivative of the second term must also be subtracted off of the derivative of the first term. The parenthesis make this idea clear.

A potentially easier way to do this is to think of the minus sign as part of the first function in the product. Or, in other words the two functions in the product, using this idea, are \(-w^2\) and \(\tan(w)\). Doing this gives,

\[
\left(-12w^5 - 2w \tan(w) - w^2 \sec^2(w)\right)
\]

So, regardless of how you approach this problem you will get the same derivative.

(e) \(y = 5 \sin(x) \cos(x) + 4 \csc(x)\)

As with the previous part we’ll need to use the product rule on the first term. We will also think of the 5 as part of the first function in the product to make sure we deal with it correctly. Alternatively, you could make use of a set of parenthesis to make sure the 5 gets dealt with properly. Either way will work, but we’ll stick with thinking of the 5 as part of the first term in the product. Here’s the derivative of this function.

\[
y' = 5 \cos(x) \cos(x) + 5 \sin(x)(-\sin(x)) - 4 \csc(x) \cot(x)
\]

\[
= 5 \cos^2(x) - 5 \sin^2(x) - 4 \csc(x) \cot(x)
\]

(d) \(P(t) = \frac{\sin(t)}{3 - 2 \cos(t)}\)

In this part we’ll need to use the quotient rule to take the derivative.
\[ P'(t) = \frac{\cos(t)(3 - 2\cos(t)) - \sin(t)(2\sin(t))}{(3 - 2\cos(t))^2} \]
\[ = \frac{3\cos(t) - 2\cos^2(t) - 2\sin^2(t)}{(3 - 2\cos(t))^2} \]

Be careful with the signs when differentiating the denominator. The negative sign we get from differentiating the cosine will cancel against the negative sign that is already there.

This appears to be done, but there is actually a fair amount of simplification that can yet be done. To do this we need to factor out a “-2” from the last two terms in the numerator and the make use of the fact that \( \cos^2(\theta) + \sin^2(\theta) = 1 \).

\[ P'(t) = \frac{3\cos(t) - 2\left(\cos^2(t) + \sin^2(t)\right)}{(3 - 2\cos(t))^2} \]
\[ = \frac{3\cos(t) - 2}{(3 - 2\cos(t))^2} \]

As a final problem here let’s not forget that we still have our standard interpretations to derivatives.

**Example 3** Suppose that the amount of money in a bank account is given by

\[ P(t) = 500 + 100\cos(t) - 150\sin(t) \]

where \( t \) is in years. During the first 10 years in which the account is open when is the amount of money in the account increasing?

**Solution**

To determine when the amount of money is increasing we need to determine when the rate of change is positive. Since we know that the rate of change is given by the derivative that is the first thing that we need to find.

\[ P'(t) = -100\sin(t) - 150\cos(t) \]

Now, we need to determine where in the first 10 years this will be positive. This is equivalent to asking where in the interval \([0, 10]\) is the derivative positive. Recall that both sine and cosine are continuous functions and so the derivative is also a continuous function. The [Intermediate Value Theorem](https://en.wikipedia.org/wiki/Intermediate_value_theorem) then tells us that the derivative can only change sign if it first goes through zero.

So, we need to solve the following equation.
\[-100 \sin(t) - 150 \cos(t) = 0\]
\[100 \sin(t) = -150 \cos(t)\]
\[\frac{\sin(t)}{\cos(t)} = -1.5\]
\[\tan(t) = -1.5\]

The solution to this equation is,
\[t = 2.1588 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots\]
\[t = 5.3004 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots\]

If you don’t recall how to solve trig equations go back and take a look at the sections on solving trig equations in the Review chapter.

We are only interested in those solutions that fall in the range \([0, 10]\). Plugging in values of \(n\) into the solutions above we see that the values we need are,
\[t = 2.1588\]
\[t = 5.3004\]
\[t = 2.1588 + 2\pi = 8.4420\]

So, much like solving polynomial inequalities all that we need to do is sketch in a number line and add in these points. These points will divide the number line into regions in which the derivative must always be the same sign. All that we need to do then is choose a test point from each region to determine the sign of the derivative in that region.

Here is the number line with all the information on it.

\[
\begin{array}{c}
P'(1) = -165.2 & P'(4) = 173.7 & P'(7) = -178.8 & P'(9) = 95.5 \\
P'(t) < 0 & P'(t) > 0 & P'(t) < 0 & P'(t) > 0 \\
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11
\end{array}
\]

So, it looks like the amount of money in the bank account will be increasing during the following intervals.
\[2.1588 < t < 5.3004 \quad 8.4420 < t < 10\]

Note that we can’t say anything about what is happening after \(t = 10\) since we haven’t done any work for \(t\)’s after that point.

In this section we saw how to differentiate trig functions. We also saw in the last example that our interpretations of the derivative are still valid so we can’t forget those.
Also, it is important that we be able to solve trig equations as this is something that will arise off and on in this course. It is also important that we can do the kinds of number lines that we used in the last example to determine where a function is positive and where a function is negative. This is something that we will be doing on occasion in both this chapter and the next.