Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Derivatives of Inverse Trig Functions

In this section we are going to look at the derivatives of the inverse trig functions. In order to derive the derivatives of inverse trig functions we’ll need the formula from the last section relating the derivatives of inverse functions. If \( f(x) \) and \( g(x) \) are inverse functions then,

\[
g'(x) = \frac{1}{f'(g(x))}
\]

Recall as well that two functions are inverses if \( f(g(x)) = x \) and \( g(f(x)) = x \).

We’ll go through inverse sine, inverse cosine and inverse tangent in detail here and leave the other three to you to derive if you’d like to.

Inverse Sine
Let’s start with inverse sine. Here is the definition of the inverse sine.

\[
y = \sin^{-1} x \quad \Leftrightarrow \quad \sin y = x \quad \text{for} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}
\]

So, evaluating an inverse trig function is the same as asking what angle \( i.e. y \) did we plug into the sine function to get \( x \). The restrictions on \( y \) given above are there to make sure that we get a consistent answer out of the inverse sine. We know that there are in fact an infinite number of angles that will work and we want a consistent value when we work with inverse sine. Using the range of angles above gives all possible values of the sine function exactly once. If you’re not sure of that sketch out a unit circle and you’ll see that that range of angles (the \( y \)’s) will cover all possible values of sine.

Note as well that since \(-1 \leq \sin(y) \leq 1\) we also have \(-1 \leq x \leq 1\).

Let’s work a quick example.

**Example 1** Evaluate \( \sin^{-1}\left(\frac{1}{2}\right) \)

**Solution**
So we are really asking what angle \( y \) solves the following equation.

\[
\sin(y) = \frac{1}{2}
\]

and we are restricted to the values of \( y \) above.

From a unit circle we can quickly see that \( y = \frac{\pi}{6} \).

We have the following relationship between the inverse sine function and the sine function.
\[
\sin\left(\sin^{-1} x\right) = x \quad \sin^{-1}\left(\sin x\right) = x
\]

In other words they are inverses of each other. This means that we can use the fact above to find the derivative of inverse sine. Let’s start with,

\[
f(x) = \sin x \quad \quad \quad g(x) = \sin^{-1} x
\]

Then,

\[
g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(\sin^{-1} x)}
\]

This is not a very useful formula. Let’s see if we can get a better formula. Let’s start by recalling the definition of the inverse sine function.

\[y = \sin^{-1}(x) \quad \Rightarrow \quad x = \sin(y)\]

Using the first part of this definition the denominator in the derivative becomes,

\[
\cos(\sin^{-1} x) = \cos(y)
\]

Now, recall that

\[
\cos^2 y + \sin^2 y = 1 \quad \Rightarrow \quad \cos y = \sqrt{1-\sin^2 y}
\]

Using this, the denominator is now,

\[
\cos(\sin^{-1} x) = \cos(y) = \sqrt{1-\sin^2 y}
\]

Now, use the second part of the definition of the inverse sine function. The denominator is then,

\[
\cos(\sin^{-1} x) = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}
\]

Putting all of this together gives the following derivative.

\[
\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}
\]

**Inverse Cosine**

Now let’s take a look at the inverse cosine. Here is the definition for the inverse cosine.

\[y = \cos^{-1} x \quad \Leftrightarrow \quad \cos y = x \quad \text{for} \quad 0 \leq y \leq \pi\]

As with the inverse since we’ve got a restriction on the angles, \(y\), that we get out of the inverse cosine function. Again, if you’d like to verify this a quick sketch of a unit circle should convince you that this range will cover all possible values of cosine exactly once. Also, we also have \(-1 \leq \cos(y) \leq 1\).
Example 2 Evaluate $\cos^{-1} \left( -\frac{\sqrt{2}}{2} \right)$.

Solution

As with the inverse sine we are really just asking the following.

$$\cos y = -\frac{\sqrt{2}}{2}$$

where $y$ must meet the requirements given above. From a unit circle we can see that we must have $y = \frac{3\pi}{4}$.

The inverse cosine and cosine functions are also inverses of each other and so we have,

$$\cos \left( \cos^{-1} x \right) = x \hspace{1cm} \cos^{-1} \left( \cos x \right) = x$$

To find the derivative we’ll do the same kind of work that we did with the inverse sine above. If we start with

$$f \left( x \right) = \cos x \hspace{1cm} g \left( x \right) = \cos^{-1} x$$

then,

$$g' \left( x \right) = \frac{1}{f' \left( g \left( x \right) \right)} = \frac{1}{-\sin \left( \cos^{-1} x \right)}$$

Simplifying the denominator here is almost identical to the work we did for the inverse sine and so isn’t shown here. Upon simplifying we get the following derivative.

$$\frac{d}{dx} \left( \cos^{-1} x \right) = -\frac{1}{\sqrt{1-x^2}}$$

So, the derivative of the inverse cosine is nearly identical to the derivative of the inverse sine. The only difference is the negative sign.

Inverse Tangent

Here is the definition of the inverse tangent.

$$y = \tan^{-1} x \quad \Leftrightarrow \quad \tan y = x \quad \text{for} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Again, we have a restriction on $y$, but notice that we can’t let $y$ be either of the two endpoints in the restriction above since tangent isn’t even defined at those two points. To convince yourself that this range will cover all possible values of tangent do a quick sketch of the tangent function and we can see that in this range we do indeed cover all possible values of tangent. Also, in this case there are no restrictions on $x$ because tangent can take on all possible values.
Example 3 Evaluate $\tan^{-1} 1$

Solution
Here we are asking,

$$\tan y = 1$$

where $y$ satisfies the restrictions given above. From a unit circle we can see that $y = \frac{\pi}{4}$.

Because there is no restriction on $x$ we can ask for the limits of the inverse tangent function as $x$ goes to plus or minus infinity. To do this we'll need the graph of the inverse tangent function. This is shown below.

From this graph we can see that

$$\lim_{x \to \pm\infty} \tan^{-1} x = \pm \frac{\pi}{2}$$

The tangent and inverse tangent functions are inverse functions so,

$$\tan \left( \tan^{-1} x \right) = x \quad \quad \tan^{-1} \left( \tan x \right) = x$$

Therefore to find the derivative of the inverse tangent function we can start with

$$f(x) = \tan x \quad \quad g(x) = \tan^{-1} x$$

We then have,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sec^2 \left( \tan^{-1} x \right)}$$

Simplifying the denominator is similar to the inverse sine, but different enough to warrant showing the details. We'll start with the definition of the inverse tangent.

$$y = \tan^{-1} x \quad \Rightarrow \quad \tan y = x$$

The denominator is then,

$$\sec^2 \left( \tan^{-1} x \right) = \sec^2 y$$
Now, if we start with the fact that
\[ \cos^2 y + \sin^2 y = 1 \]
and divide every term by \( \cos^2 y \) we will get,
\[ 1 + \tan^2 y = \sec^2 y \]

The denominator is then,
\[ \sec^2 \left( \tan^{-1} x \right) = \sec^2 y = 1 + \tan^2 y \]

Finally using the second portion of the definition of the inverse tangent function gives us,
\[ \sec^2 \left( \tan^{-1} x \right) = 1 + \tan^2 y = 1 + x^2 \]

The derivative of the inverse tangent is then,
\[ \frac{d}{dx} \left( \tan^{-1} x \right) = \frac{1}{1 + x^2} \]

There are three more inverse trig functions but the three shown here the most common ones. Formulas for the remaining three could be derived by a similar process as we did those above. Here are the derivatives of all six inverse trig functions.

\[
\begin{align*}
\frac{d}{dx} \left( \sin^{-1} x \right) &= \frac{1}{\sqrt{1 - x^2}} & \frac{d}{dx} \left( \cos^{-1} x \right) &= -\frac{1}{\sqrt{1 - x^2}} \\
\frac{d}{dx} \left( \tan^{-1} x \right) &= \frac{1}{1 + x^2} & \frac{d}{dx} \left( \cot^{-1} x \right) &= -\frac{1}{1 + x^2} \\
\frac{d}{dx} \left( \sec^{-1} x \right) &= \frac{1}{|x|\sqrt{x^2 - 1}} & \frac{d}{dx} \left( \csc^{-1} x \right) &= -\frac{1}{|x|\sqrt{x^2 - 1}}
\end{align*}
\]

We should probably now do a couple of quick derivatives here before moving on to the next section.

**Example 4** Differentiate the following functions.

(a) \( f(t) = 4 \cos^{-1} (t) - 10 \tan^{-1} (t) \)

(b) \( y = \sqrt{z} \sin^{-1} (z) \)

**Solution**

(a) Not much to do with this one other than differentiate each term.

\[
f'(t) = -\frac{4}{\sqrt{1-t^2}} - 10 \frac{1}{1+t^2}
\]
(b) Don’t forget to convert the radical to fractional exponents before using the product rule.

\[ y' = \frac{1}{2} z^{-\frac{3}{2}} \sin^{-1}(z) + \frac{\sqrt{z}}{\sqrt{1 - z^2}} \]

Alternate Notation

There is some alternate notation that is used on occasion to denote the inverse trig functions. This notation is,

\[ \sin^{-1} x = \arcsin x \quad \cos^{-1} x = \arccos x \]
\[ \tan^{-1} x = \arctan x \quad \cot^{-1} x = \arccot x \]
\[ \sec^{-1} x = \arcsec x \quad \csc^{-1} x = \arccsc x \]