Preface

Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

**Chain Rule**

1. Differentiate \( f(x) = (6x^2 + 7x)^4 \).

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the exponent of 4 on the parenthesis while the inside function is the polynomial that is being raised to the power. The derivative is then,

\[
  f'(x) = 4(6x^2 + 7x)^3 (12x + 7) = 4(12x + 7)(6x^2 + 7x)^3
\]

2. Differentiate \( g(t) = (4t^2 - 3t + 2)^2 \).
Calculus I

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the exponent of -2 on the parenthesis while the inside function is the polynomial that is being raised to the power. The derivative is then,

$$g'(t) = -2 \left( 4t^2 - 3t + 2 \right)^3 (8t - 3) = -2(8t - 3)(4t^2 - 3t + 2)^3$$

3. Differentiate $y = \sqrt[3]{1 - 8z}$. 

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For this problem, after converting the root to a fractional exponent, the outside function is (hopefully) clearly the exponent of $\frac{1}{3}$ while the inside function is the polynomial that is being raised to the power (or the polynomial inside the root – depending upon how you want to think about it). The derivative is then,

$$y = (1 - 8z)^{\frac{1}{3}} \Rightarrow \frac{dy}{dz} = \frac{1}{3} (1 - 8z)^{-\frac{2}{3}} (-8) = -\frac{8}{3} (1 - 8z)^{-\frac{2}{3}}$$

4. Differentiate $R(w) = \csc(7w)$. 

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the trig function and the inside function is the stuff inside of the trig function. The derivative is then,
In dealing with functions like cosecant (or secant for that matter) be careful to make sure that the inside function gets substituted into both terms of the derivative of the outside function. One of the more common mistakes with this kind of problem is to only substitute the $7w$ into only the cosecant or only the cotangent instead of both as it should be.

5. Differentiate $G(x) = 2 \sin(3x + \tan(x))$.

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the sine function and the inside function is the stuff inside of the trig function. The derivative is then,

$$G'(x) = 2 \left(3 + \sec^2(x)\right) \cos(3x + \tan(x))$$

6. Differentiate $h(u) = \tan(4 + 10u)$.

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the trig function and the inside function is the stuff inside of the trig function. The derivative is then,

$$h'(u) = 10 \sec^2(4 + 10u)$$

7. Differentiate $f(t) = 5 + e^{4t+7}$.
Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

Note that we only need to use the Chain Rule on the second term as we can differentiate the first term without the Chain Rule.

Now, recall that for exponential functions outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$f'(t) = (4 + 7t^6)e^{4t^7}$$

8. Differentiate $g(x) = e^{1-\cos(x)}$.

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For exponential functions remember that the outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$g'(x) = \sin(x)e^{1-\cos(x)}$$

9. Differentiate $H(z) = 2^{1-6z}$.

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For exponential functions remember that the outside function is the exponential function itself and the inside function is the exponent. The derivative is then,

$$H'(z) = -6(2^{1-6z})\ln(2)$$
10. Differentiate \( u(t) = \tan^{-1}(3t-1) \).

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the inverse tangent and the inside function is the stuff inside of the inverse tangent. The derivative is then,

\[
\frac{3}{(3t-1)^2 + 1}
\]

11. Differentiate \( F(y) = \ln\left(1 - 5y^2 + y^3\right) \).

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside" functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the logarithm and the inside function is the stuff inside of the logarithm. The derivative is then,

\[
F'(y) = \frac{1}{1 - 5y^2 + y^3} \left(-10y + 3y^2\right) = \frac{-10y + 3y^2}{1 - 5y^2 + y^3}
\]

With logarithm problems remember that after differentiating the logarithm (i.e. the outside function) you need to substitute the inside function into the derivative. So, instead of getting just, \( \frac{1}{y} \), we get the following (i.e. we plugged the inside function into the derivative),

\[
\frac{1}{1 - 5y^2 + y^3}
\]

Then, we can’t forget of course to multiply by the derivative of the inside function.
12. Differentiate $V(x) = \ln(\sin(x) - \cot(x))$.

Hint: Recall that with Chain Rule problems you need to identify the “inside” and “outside” functions and then apply the chain rule.

Solution

For this problem the outside function is (hopefully) clearly the logarithm and the inside function is the stuff inside of the logarithm. The derivative is then,

$$V(x) = \frac{1}{\sin(x) - \cot(x)} \left( \cos(x) + \csc^2(x) \right) = \frac{\cos(x) + \csc^2(x)}{\sin(x) - \cot(x)}$$

With logarithm problems remember that after differentiating the logarithm (i.e. the outside function) you need to substitute the inside function into the derivative. So, instead of getting just, 

$$\frac{1}{x}$$

we get the following (i.e. we plugged the inside function into the derivative),

$$\frac{1}{\sin(x) - \cot(x)}$$

Then, we can’t forget of course to multiply by the derivative of the inside function.

13. Differentiate $h(z) = \sin(z^6) + \sin^6(z)$.

Hint: Don’t get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem each term will require a separate application of the Chain Rule and don’t forget that,

$$\sin^6(z) = \left[\sin(z)\right]^6$$
So, in the first term the outside function is the sine function, while the sine function is the inside function in the second term. The derivative is then,

\[
\frac{d}{dz}(z^5 \cos(z^6) + 6 \sin(z) \cos(z)) = 6z^5 \cos(z^6) + 6 \sin(z) \cos(z)
\]

14. Differentiate \( S(w) = \sqrt{7w} + e^{-w} \).

Hint : Don’t get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem each term will require a separate application of the Chain Rule and make sure you are careful with parenthesis in dealing with the root in the first term.

The derivative is then,

\[
S'(w) = \frac{1}{2} \left(7w\right)^{-\frac{1}{2}} e^{-w} = \frac{7}{2} \left(7w\right)^{-\frac{1}{2}} e^{-w}
\]

15. Differentiate \( g(z) = 3z^7 - \sin(z^2 + 6) \).

Hint : Don’t get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem the first term requires no Chain Rule and the second term will require the Chain Rule. The derivative is then,

\[
g'(z) = 21z^6 - 2 \cos(z^2 + 6)
\]
16. Differentiate \( f(x) = \ln(\sin(x)) - (x^4 - 3x)^{10} \).

Hint: Don’t get too locked into problems only requiring a single use of the Chain Rule. Sometimes separate terms will require different applications of the Chain Rule, or maybe only one of the terms will require the Chain Rule.

Solution

For this problem each term will require a separate application of the Chain Rule. The derivative is then,

\[
\frac{f'(x)}{\sin(x)} = -10\left(4x^3 - 3\right)\left(x^4 - 3x\right)^9 = \cot(x) - 10\left(4x^3 - 3\right)\left(x^4 - 3x\right)^9
\]

17. Differentiate \( h(t) = t^6\sqrt{5t^2 - t} \).

Hint: Don’t forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you’ll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we’ll need to do the Product Rule to start off the derivative. In the process we’ll need to use the Chain Rule when we differentiate the second term.

The derivative is then,

\[
h'(t) = 6t^5\left(5t^2 - t\right)^{1/2} + t^6\left(1/2\right)\left(5t^2 - t\right)^{-1/2}(10t - 1) = 6t^5\left(5t^2 - t\right)^{1/2} + \frac{1}{2}t^6(10t-1)\left(5t^2 - t\right)^{-1/2}
\]

18. Differentiate \( q(t) = t^3 \ln(t^5) \).
Calculus I

Hint: Don’t forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you’ll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we’ll need to do the Product Rule to start off the derivative. In the process we’ll need to use the Chain Rule when we differentiate the second term.

The derivative is then,

\[ q'(t) = 2t \ln(t^2) + t^2 \left( \frac{5t^4}{t^5} \right) = 2t \ln(t^2) + 5t \]

19. Differentiate \( g(w) = \cos(3w) \sec(1-w) \).

Hint: Don’t forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you’ll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we’ll need to do the Product Rule to start off the derivative. In the process we’ll need to use the Chain Rule when we differentiate each term.

The derivative is then,

\[ g'(w) = -\sin(3w)(3)\sec(1-w) + \cos(3w)\sec(1-w)\tan(1-w)(-1) \]
\[ = -3\sin(3w)\sec(1-w) - \cos(3w)\sec(1-w)\tan(1-w) \]

20. Differentiate \( y = \frac{\sin(3t)}{1+t^2} \).

Hint: Don’t forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you’ll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.
Solution

For this problem we’ll need to do the Quotient Rule to start off the derivative. In the process we’ll need to use the Chain Rule when we differentiate the numerator.

The derivative is then,

\[
\frac{dy}{dt} = \frac{3\cos(3t)(1+t^2) - \sin(3t)(2t)}{(1+t^2)^2} = \frac{3\cos(3t)(1+t^2) - 2t\sin(3t)}{(1+t^2)^2}
\]

21. Differentiate \( K(x) = \frac{1+e^{-2x}}{x + \tan(12x)} \).

Hint: Don’t forget the Product and Quotient Rule. Sometimes, in the process of doing the Product or Quotient Rule you’ll need to use the Chain Rule when differentiating one or both of the terms in the product or quotient.

Solution

For this problem we’ll need to do the Quotient Rule to start off the derivative. In the process we’ll need to use the Chain Rule when we differentiate both the numerator and the denominator.

The derivative is then,

\[
K'(x) = \frac{-2e^{-2x}(x + \tan(12x)) - (1 + 2x)(1 + 12\sec^2(12x))}{(x + \tan(12x))^2}
\]

22. Differentiate \( f(x) = \cos(x^2e^x) \).

Hint: Don’t forget the Product and Quotient Rule. Sometimes, in the process of using the Chain Rule, you’ll also need the Product and/or Quotient Rule.

Solution

For this problem we’ll start off using the Chain Rule, however when we differentiate the inside function we’ll need to do the Product Rule.
The derivative is then,

\[ f'(x) = -\left(2xe^x + x^2 e^x\right)\sin\left(x^2 e^x\right) \]

23. Differentiate \( z = \sqrt{5x + \tan(4x)} \).

Hint: Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we’ll step though the derivative process to make each use clear.

Here is the first step of the derivative and we’ll need to use the Chain Rule in this step.

\[
z = \left(5x + \tan(4x)\right)^{\frac{1}{2}}
\]

\[
\frac{dz}{dx} = \frac{1}{2} \left(5x + \tan(4x)\right)^{-\frac{1}{2}} \frac{d}{dx} \left(5x + \tan(4x)\right)
\]

Step 2

In this step we can see that we’ll need to use the Chain Rule on the second term.

The derivative is then,

\[
\frac{dz}{dx} = \frac{1}{2} \left(5x + \tan(4x)\right)^{-\frac{1}{2}} \left(5 + 4 \sec^2(4x)\right)
\]

In this step we were using the Chain Rule on the second term and so when multiplying by the derivative of the inside function we only multiply the second term by the derivative of the inside function and not both terms.

24. Differentiate \( f(t) = (e^{-6t} + \sin(2 - t))^3 \).
Hint: Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we’ll step though the derivative process to make each use clear.

Here is the first step of the derivative and we’ll need to use the Chain Rule in this step.

$$f'(t) = 3\left(e^{-6t} + \sin(2 - t)\right)^2 \frac{d}{dt}\left(e^{-6t} + \sin(2 - t)\right)$$

Step 2

In this step we can see that we’ll need to use the Chain Rule on each of the terms.

The derivative is then,

$$f'(t) = 3\left(e^{-6t} + \sin(2 - t)\right)^2 \left(-6e^{-6t} - \cos(2 - t)\right)$$

25. Differentiate \( g(x) = \left(\ln(x^2 + 1) - \tan^{-1}(6x)\right)^{10} \).

Hint: Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we’ll step though the derivative process to make each use clear.

Here is the first step of the derivative and we’ll need to use the Chain Rule in this step.

$$g'(x) = 10\left(\ln(x^2 + 1) - \tan^{-1}(6x)\right)^9 \frac{d}{dx}\left(\ln(x^2 + 1) - \tan^{-1}(6x)\right)$$
Step 2

In this step we can see that we’ll need to use the Chain Rule on each of the terms.

The derivative is then,

\[
g'(x) = 10\left(\ln(x^2 + 1) - \tan^{-1}(6x)\right)^9 \left(\frac{2x}{x^2 + 1} - \frac{6}{36x^2 + 1}\right)
\]

26. Differentiate \( h(z) = \tan^4(z^2 + 1) \).

Hint: Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we’ll step through the derivative process to make each use clear. Also, recall that,

\[ \tan^4(x) = \left[\tan(x)\right]^4 \]

Here is the first step of the derivative and we’ll need to use the Chain Rule in this step.

\[
h'(z) = 4 \tan^3(z^2 + 1) \frac{d}{dz}\left[\tan(z^2 + 1)\right]
\]

Step 2

As we can see the derivative from the previous step will also require the Chain Rule.

The derivative is then,

\[
h'(z) = 4 \tan^3(z^2 + 1) \sec^2(z^2 + 1)(2z) = 8z \tan^3(z^2 + 1) \sec^2(z^2 + 1)
\]

27. Differentiate \( f(x) = \left(\sqrt{12x + \sin^2(3x)}\right)^{-1} \).
Hint: Sometimes the Chain Rule will need to be done multiple times before we finish taking the derivative.

Step 1

This problem will require multiple uses of the Chain Rule and so we’ll step though the derivative process to make each use clear.

Here is the first step of the derivative and we’ll need to use the Chain Rule in this step.

\[
f'(x) = -\left(\sqrt[3]{12x + \sin^2 (3x)}\right)^{-2} \frac{d}{dx} \left((12x)^{\frac{1}{3}} + \sin^2 (3x)\right)
\]

Step 2

As we can see the derivative from the previous step will also require the Chain Rule on each of the terms.

The derivative from this step is,

\[
f'(x) = -\left(\sqrt[3]{12x + \sin^2 (3x)}\right)^{-2} \left(\frac{1}{3} (12x)^{-\frac{2}{3}} (12) + 2 \sin (3x) \frac{d}{dx} (\sin (3x))\right)
\]

Step 3

The second term will again use the Chain Rule as we can see.

The derivative is then,

\[
f'(x) = -\left(\sqrt[3]{12x + \sin^2 (3x)}\right)^{-2} \left(4 (12x)^{-\frac{2}{3}} + 6 \sin (3x) \cos (3x)\right)
\]

28. Find the tangent line to \( f(x) = 4\sqrt{2x - 6e^{2-x}} \) at \( x = 2 \).

Solution

Step 1

We know that the derivative of the function will give us the slope of the tangent line so we’ll need the derivative of the function. Differentiating each term will require the Chain Rule as well.
\[ f(x) = 4(2x)^{\frac{1}{3}} - 6e^{2-x} \]
\[ f'(x) = 4 \left( \frac{1}{2} \right)(2x)^{-\frac{1}{3}} (2) - 6e^{2-x} (-1) = 4(2x)^{-\frac{1}{3}} + 6e^{2-x} = 4 \sqrt[3]{2x} + 6e^{2-x} \]

Step 2
Now all we need to do is evaluate the function and the derivative at the point in question.
\[ f(2) = 4(2) - 6e^0 = 2 \quad f'(2) = \frac{4}{2} + 6e^0 = 8 \]

Step 3
Now all that we need to do is write down the equation of the tangent line.
\[ y = f(2) + f'(2)(x-2) = 2 + 8(x-2) \quad \rightarrow \quad y = 8x - 14 \]

29. Determine where \( V(z) = z^4 (2z - 8)^3 \) is increasing and decreasing.

Solution

Step 1
We’ll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.
\[ V'(z) = 4z^3 (2z - 8)^3 + z^4 (3)(2z - 8)^2 (2) \]
\[ = 2z^3 (2z - 8)^2 \left[ 2(2z - 8) + 3z \right] = 2z^3 (2z - 8)^2 (7z - 16) \]

Note that we factored the derivative to help with the next step. In general we don’t need to do this.

Step 2
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve. In this case it’s pretty easy to spot where the derivative will be zero.
\[ 2z^3 (2z - 8)^2 (7z - 16) = 0 \quad \Rightarrow \quad z = 0, \quad z = 4, \quad z = \frac{16}{7} = 2.2857 \]

Step 3
To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is
zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.

\[
\begin{array}{c|c|c|c|c}
\mathcal{V}'(-1) &= 4600 & \mathcal{V}'(1) &= -648 & \mathcal{V}'(3) &= 1080 & \mathcal{V}'(5) &= 19000 \\
\mathcal{V}'(x) &> 0 & \mathcal{V}'(x) &< 0 & \mathcal{V}'(x) &> 0 & \mathcal{V}'(x) &> 0 \\
\end{array}
\]

From this we get the following increasing/decreasing information.

| Increasing : \(-\infty < x < 0, \ \frac{16}{7} < x < 4, \ 4 < x < \infty\) |
| Decreasing : \(0 < x < \frac{16}{7}\) |

30. The position of an object is given by \(s(t) = \sin(3t) - 2t + 4\). Determine where in the interval \([0, 3]\) the object is moving to the right and moving to the left.

Solution

Step 1
We’ll first need the derivative because we know that the derivative will give us the velocity of the object. Here is the derivative.

\[s'(t) = 3\cos(3t) - 2\]

Step 2
Next, we need to know where the object stops moving and so all we need to do is set the derivative equal to zero and solve.

\[3\cos(3t) - 2 = 0 \quad \Rightarrow \quad \cos(3t) = \frac{2}{3}\]

A quick calculator computation tells us that,
Recalling our work in the Review chapter and a quick check on a unit circle we can see that either $3t = -0.8411$ or $3t = 2\pi - 0.8411 = 5.4421$ can be used for the second angle. Note that either will work, but we’ll use the second simply because it is the positive angle.

Putting all of this together and dividing by 3 we can see that the derivative will be zero at,

$$3t = 0.8411 + 2\pi n \quad \text{and} \quad 3t = 5.4421 + 2\pi n \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$t = 0.2804 + \frac{2\pi n}{3} \quad \text{and} \quad t = 1.8140 + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Finally, all we need to do is plug in some $n$’s to determine which solutions fall in the interval we are working on, $[0, 6]$.

$$n = 0: \quad t = 0.2804 \quad t = 1.8140$$

$$n = 1: \quad t = 2.3748 \quad t = 3.9084$$

So, in the interval $[0, 6]$, the object stops moving at the following three points.

$$t = 0.2804, \ 1.8140, \ 2.3748$$

Step 3
To get the answer to this problem all we need to know is where the derivative is positive (and hence the object is moving to the right) or negative (and hence the object is moving to the left). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.

From this we get the following moving right/moving left information.
Note that because we’ve only looked at what is happening in the interval \([0,3]\) we can’t say anything about the moving right/moving left behavior of the object outside of this interval.

31. Determine where \(A(t) = t^2e^{5-t}\) is increasing and decreasing.

Solution

Step 1
We’ll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

\[
A'(t) = 2te^{5-t} - t^2e^{5-t} = te^{5-t}(2-t)
\]

Note that we factored the derivative to help with the next step. In general we don’t need to do this.

Step 2
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve. In this case it’s pretty easy to spot where the derivative will be zero.

\[te^{5-t}(2-t) = 0 \Rightarrow t = 0, t = 2\]

Step 3
To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.

\[
\begin{array}{ccc}
A'(-1) = -3e^6 & A'(1) = e^4 & A'(3) = -3e^3 \\
A'(t) < 0 & A'(t) > 0 & A'(t) < 0 \\
\end{array}
\]
From this we get the following increasing/decreasing information.

<table>
<thead>
<tr>
<th>Increasing</th>
<th>Decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; t &lt; 2$</td>
<td>$-\infty &lt; t &lt; 0, \ 2 &lt; t &lt; \infty$</td>
</tr>
</tbody>
</table>

32. Determine where in the interval $[-1, 20]$ the function $f(x) = \ln(x^4 + 20x^3 + 100)$ is increasing and decreasing.

**Solution**

**Step 1**
We’ll first need the derivative because we know that the derivative will give us the rate of change of the function. Here is the derivative.

$$f'(x) = \frac{4x^3 + 60x^2}{x^4 + 20x^3 + 100} = \frac{4x^2(x + 15)}{x^4 + 20x^3 + 100}$$

Note that we factored the numerator to help with the next step. In general we don’t need to do this.

**Step 2**
Next, we need to know where the function is not changing and so all we need to do is set the derivative equal to zero and solve.

$$\frac{4x^2(x + 15)}{x^4 + 20x^3 + 100} = 0 \quad \Rightarrow \quad 4x^2(x + 15) = 0 \quad \Rightarrow \quad x = 0, \ x = -15$$

Note, that in general, we also need to be concerned with where the derivative is not defined as well. Functions can (and often do) change sign where they are not defined. In this case however we’ve restricted the interval down to a range where the function exists and is continuous on the given interval and so this is something we need to worry about for this problem.

In the next Chapter we will start also looking at what happens if the derivative is also not defined at particular points.

Note as well that we really should at this point step back and recall that we are working on the interval $[-1, 20]$. We are only interested in what is happening on this interval and so we should make sure that the points found above are inside the interval.
In this case only \( x = 0 \) is in the interval and so we’ll need to exclude \( x = -15 \) from our work for the rest of this problem.

Step 3
To get the answer to this problem all we need to know is where the derivative is positive (and hence the function is increasing) or negative (and hence the function is decreasing). Because the derivative is continuous we know that the only place it can change sign is where the derivative is zero. So, as we did in this section a quick number line will give us the sign of the derivative for the various intervals.

Here is the number line for this problem.

\[
\begin{align*}
\text{f''(x) > 0} & \quad f'(1) = 0.53 \\
\text{f'(x) > 0} & \quad f'(x) > 0
\end{align*}
\]

So, we can see that, in this case function is increasing everywhere in the interval \([-1, 20]\) except \( x = 0 \). Recall that at this point the derivative was zero and hence the function is not changing (and therefore can’t be increasing).

So, the formal answer for this problem is,

\[
\text{Increasing : } -1 \leq x < 0, \quad 0 < x \leq 20
\]

Note that because we’ve only looked at what is happening in the interval \([-1, 20]\) we can’t say anything about the increasing/decreasing nature of the function outside of this interval.