Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Higher Order Derivatives

Let’s start this section with the following function.

\[ f(x) = 5x^3 - 3x^2 + 10x - 5 \]

By this point we should be able to differentiate this function without any problems. Doing this we get,

\[ f''(x) = 15x^2 - 6x + 10 \]

Now, this is a function and so it can be differentiated. Here is the notation that we’ll use for that, as well as the derivative.

\[ f'''(x) = (f''(x))' = 30x - 6 \]

This is called the second derivative and \( f''(x) \) is now called the first derivative.

Again, this is a function so we can differentiate it again. This will be called the third derivative. Here is that derivative as well as the notation for the third derivative.

\[ f'''(x) = (f''(x))' = 30 \]

Continuing, we can differentiate again. This is called, oddly enough, the fourth derivative. We’re also going to be changing notation at this point. We can keep adding on primes, but that will get cumbersome after awhile.

\[ f^{(4)}(x) = (f'''(x))' = 0 \]

This process can continue but notice that we will get zero for all derivatives after this point. This set of derivatives leads us to the following fact about the differentiation of polynomials.

**Fact**

If \( p(x) \) is a polynomial of degree \( n \) (i.e. the largest exponent in the polynomial) then,

\[ p^{(k)}(x) = 0 \quad \text{for} \ k \ge n + 1 \]

We will need to be careful with the “non-prime” notation for derivatives. Consider each of the following.

\[ f^{(2)}(x) = f''(x) \]

\[ f^2(x) = [f(x)]^2 \]

The presence of parenthesis in the exponent denotes differentiation while the absence of parenthesis denotes exponentiation.

Collectively the second, third, fourth, etc. derivatives are called higher order derivatives.
Let’s take a look at some examples of higher order derivatives.

### Example 1
Find the first four derivatives for each of the following.

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>(a)</td>
<td>( R(t) = 3t^2 + 8t^2 + e^t )</td>
</tr>
<tr>
<td>(b)</td>
<td>( y = \cos x )</td>
</tr>
<tr>
<td>(c)</td>
<td>( f(y) = \sin(3y) + e^{-2y} + \ln(7y) )</td>
</tr>
</tbody>
</table>

#### Solution

**(a) \( R(t) = 3t^2 + 8t^2 + e^t \)**

There really isn’t a lot to do here other than do the derivatives.

\[
R'(t) = 6t + 4t^{-\frac{1}{2}} + e^t
\]

\[
R''(t) = 6 - 2t^{-\frac{3}{2}} + e^t
\]

\[
R'''(t) = 3t^{-\frac{5}{2}} + e^t
\]

\[
R^{(4)}(t) = -\frac{15}{2}t^{-\frac{7}{2}} + e^t
\]

Notice that differentiating an exponential function is very simple. It doesn’t change with each differentiation.

**(b) \( y = \cos x \)**

Again, let’s just do some derivatives.

\[
y = \cos x
\]

\[
y' = -\sin x
\]

\[
y'' = -\cos x
\]

\[
y''' = \sin x
\]

\[
y^{(4)} = \cos x
\]

Note that cosine (and sine) will repeat every four derivatives. The other four trig functions will not exhibit this behavior. You might want to take a few derivatives to convince yourself of this.

**(c) \( f(y) = \sin(3y) + e^{-2y} + \ln(7y) \)**

In the previous two examples we saw some patterns in the differentiation of exponential functions, cosines and sines. We need to be careful however since they only work if there is just a \( t \) or an \( x \) in the argument. This is the point of this example. In this example we will need to use the chain rule on each derivative.
\[
f'(y) = 3\cos(3y) - 2e^{-2y} + \frac{1}{y} = 3\cos(3y) - 2e^{-2y} + y^{-1}
\]
\[
f''(y) = -9\sin(3y) + 4e^{-2y} - y^{-2}
\]
\[
f'''(y) = -27\cos(3y) - 8e^{-2y} + 2y^{-3}
\]
\[
f^{(4)}(y) = 81\sin(3y) + 16e^{-2y} - 6y^{-4}
\]

So, we can see with slightly more complicated arguments the patterns that we saw for exponential functions, sines and cosines no longer completely hold.

Let’s do a couple more examples to make a couple of points.

**Example 2** Find the second derivative for each of the following functions.

(a) \( Q(t) = \sec(5t) \) [Solution]

(b) \( g(w) = e^{1-2w^3} \) [Solution]

(c) \( f(t) = \ln(1 + t^2) \) [Solution]

**Solution**

(a) \( Q(t) = \sec(5t) \)

Here’s the first derivative.

\[ Q'(t) = 5\sec(5t)\tan(5t) \]

Notice that the second derivative will now require the product rule.

\[ Q''(t) = 25\sec(5t)\tan(5t)\tan(5t) + 25\sec(5t)\sec^2(5t) \]
\[ = 25\sec(5t)\tan^2(5t) + 25\sec^3(5t) \]

Notice that each successive derivative will require a product and/or chain rule and that as noted above this will not end up returning back to just a secant after four (or another other number for that matter) derivatives as sine and cosine will.

(b) \( g(w) = e^{1-2w^3} \)

Again, let’s start with the first derivative.

\[ g'(w) = -6w^2e^{1-2w^3} \]

As with the first example we will need the product rule for the second derivative.

\[ g''(w) = -12we^{1-2w^3} - 6w^2(-6w^2)e^{1-2w^3} \]
\[ = -12we^{1-2w^3} + 36w^4e^{1-2w^3} \]
(e) \( f(t) = \ln(1 + t^2) \)

Same thing here.

\[
f''(t) = \frac{2t}{1 + t^2}
\]

The second derivative this time will require the quotient rule.

\[
f''(t) = \frac{2(1 + t^2) - (2t)(2t)}{(1 + t^2)^2}
\]

\[
= \frac{2 - 2t^2}{(1 + t^2)^2}
\]

As we saw in this last set of examples we will often need to use the product or quotient rule for the higher order derivatives, even when the first derivative didn’t require these rules.

Let’s work one more example that will illustrate how to use implicit differentiation to find higher order derivatives.

**Example 3** Find \( y'' \) for

\[
x^2 + y^4 = 10
\]

**Solution**

Okay, we know that in order to get the second derivative we need the first derivative and in order to get that we’ll need to do implicit differentiation. Here is the work for that.

\[
2x + 4y^3y' = 0
\]

\[
y' = -\frac{x}{2y^3}
\]

Now, this is the first derivative. We get the second derivative by differentiating this, which will require implicit differentiation again.

\[
y'' = \left(-\frac{x}{2y^3}\right)' = \frac{-2y^3 - x(6y^2y')}{(2y^3)^2} = \frac{-2y^3 - 6xy^2y'}{4y^6} = \frac{y - 3xy'}{2y^4}
\]
This is fine as far as it goes. However, we would like there to be no derivatives in the answer. We don’t, generally, mind having $x$’s and/or $y$’s in the answer when doing implicit differentiation, but we really don’t like derivatives in the answer. We can get rid of the derivative however by acknowledging that we know what the first derivative is and substituting this into the second derivative equation. Doing this gives,

\[
y'' = -\frac{y - 3xy'}{2y^4}
\]

\[
= -\frac{y - 3x\left(-\frac{x}{2y^3}\right)}{2y^4}
\]

\[
= -\frac{y + \frac{3}{2}x^2y^{-3}}{2y^4}
\]

Now that we’ve found some higher order derivatives we should probably talk about an interpretation of the second derivative.

If the position of an object is given by $s(t)$ we know that the velocity is the first derivative of the position.

\[
v(t) = s'(t)
\]

The acceleration of the object is the first derivative of the velocity, but since this is the first derivative of the position function we can also think of the acceleration as the second derivative of the position function.

\[
a(t) = v'(t) = s''(t)
\]

**Alternate Notation**

There is some alternate notation for higher order derivatives as well. Recall that there was a fractional notation for the first derivative.

\[
f'(x) = \frac{df}{dx}
\]

We can extend this to higher order derivatives.

\[
f''(x) = \frac{d^2y}{dx^2}
\]

\[
f'''(x) = \frac{d^3y}{dx^3}
\]

etc.