Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Finding Absolute Extrema**

It’s now time to see our first major application of derivatives in this chapter. Given a continuous function, \( f(x) \), on an interval \([a,b]\) we want to determine the absolute extrema of the function. To do this we will need many of the ideas that we looked at in the previous section.

First, since we have an interval and we are assuming that the function is continuous the Extreme Value Theorem tells us that we can in fact do this. This is a good thing of course. We don’t want to be trying to find something that may not exist.

Next, we saw in the previous section that absolute extrema can occur at endpoints or at relative extrema. Also, from Fermat’s Theorem we know that the list of critical points is also a list of all possible relative extrema. So the endpoints along with the list of all critical points will in fact be a list of all possible absolute extrema.

Now we just need to recall that the absolute extrema are nothing more than the largest and smallest values that a function will take so all that we really need to do is get a list of possible absolute extrema, plug these points into our function and then identify the largest and smallest values.

Here is the procedure for finding absolute extrema.

**Finding Absolute Extrema of \( f(x) \) on \([a,b]\).**

1. Verify that the function is continuous on the interval \([a,b]\).
2. Find all critical points of \( f(x) \) that are in the interval \([a,b]\). This makes sense if you think about it. Since we are only interested in what the function is doing in this interval we don’t care about critical points that fall outside the interval.
3. Evaluate the function at the critical points found in step 1 and the end points.
4. Identify the absolute extrema.

There really isn’t a whole lot to this procedure. We called the first step in the process step 0, mostly because all of the functions that we’re going to look at here are going to be continuous, but it is something that we do need to be careful with. This process will only work if we have a function that is continuous on the given interval. The most labor intensive step of this process is the second step (step 1) where we find the critical points. It is also important to note that all we want are the critical points that are in the interval.

Let’s do some examples.

**Example 1** Determine the absolute extrema for the following function and interval.

\[
g(t) = 2t^3 + 3t^2 - 12t + 4 \quad \text{on} \quad [-4, 2]
\]

**Solution**

All we really need to do here is follow the procedure given above. So, first notice that this is a polynomial and so in continuous everywhere and in particular is then continuous on the given
interval.

Now, we need to get the derivative so that we can find the critical points of the function.

\[
g'(t) = 6t^2 + 6t - 12 = 6(t + 2)(t - 1)
\]

It looks like we’ll have two critical points, \( t = -2 \) and \( t = 1 \). Note that we actually want something more than just the critical points. We only want the critical points of the function that lie in the interval in question. Both of these do fall in the interval as so we will use both of them. That may seem like a silly thing to mention at this point, but it is often forgotten, usually when it becomes important, and so we will mention it at every opportunity to make sure it’s not forgotten.

Now we evaluate the function at the critical points and the end points of the interval.

\[
\begin{align*}
g(-2) &= 24 \\
g(0) &= -36 \\
g(2) &= 8
\end{align*}
\]

Absolute extrema are the largest and smallest the function will ever be and these four points represent the only places in the interval where the absolute extrema can occur. So, from this list we see that the absolute maximum of \( g(t) \) is 24 and it occurs at \( t = -2 \) (a critical point) and the absolute minimum of \( g(t) \) is -28 which occurs at \( t = -4 \) (an endpoint).

In this example we saw that absolute extrema can and will occur at both endpoints and critical points. One of the biggest mistakes that students make with these problems is to forget to check the endpoints of the interval.

**Example 2** Determine the absolute extrema for the following function and interval.

\[
g(t) = 2t^3 + 3t^2 - 12t + 4 \quad \text{on} \quad [0, 2]
\]

**Solution**

Note that this problem is almost identical to the first problem. The only difference is the interval that we’re working on. This small change will completely change our answer however. With this change we have excluded both of the answers from the first example.

The first step is to again find the critical points. From the first example we know these are \( t = -2 \) and \( t = 1 \). At this point it’s important to recall that we only want the critical points that actually fall in the interval in question. This means that we only want \( t = 1 \) since \( t = -2 \) falls outside the interval.

Now evaluate the function at the single critical point in the interval and the two endpoints.

\[
\begin{align*}
g(1) &= -3 \\
g(0) &= 4 \\
g(2) &= 8
\end{align*}
\]

From this list of values we see that the absolute maximum is 8 and will occur at \( t = 2 \) and the absolute minimum is -3 which occurs at \( t = 1 \).
As we saw in this example a simple change in the interval can completely change the answer. It also has shown us that we do need to be careful to exclude critical points that aren’t in the interval. Had we forgotten this and included \( t = -2 \) we would have gotten the wrong absolute maximum!

This is the other big mistakes that students make in these problems. All too often they forget to exclude critical points that aren’t in the interval. If your instructor is anything like me this will mean that you will get the wrong answer. It’s not too hard to make sure that a critical point outside of the interval is larger or smaller than any of the points in the interval.

**Example 3** Suppose that the population (in thousands) of a certain kind of insect after \( t \) months is given by the following formula.

\[
P(t) = 3t + \sin(4t) + 100
\]

Determine the minimum and maximum population in the first 4 months.

**Solution**

The question that we’re really asking is to find the absolute extrema of \( P(t) \) on the interval \([0,4]\). Since this function is continuous everywhere we know we can do this.

Let’s start with the derivative.

\[
P'(t) = 3 + 4 \cos(4t)
\]

We need the critical points of the function. The derivative exists everywhere so there are no critical points from that. So, all we need to do is determine where the derivative is zero.

\[3 + 4 \cos(4t) = 0\]

\[\cos(4t) = -\frac{3}{4}\]

The solutions to this are,

\[4t = 2.4189 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots\]

\[4t = 3.8643 + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots\]

\[t = 0.6047 + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots\]

\[t = 0.9661 + \frac{\pi n}{2}, \quad n = 0, \pm 1, \pm 2, \ldots\]

So, these are all the critical points. We need to determine the ones that fall in the interval \([0,4]\). There’s nothing to do except plug some \( n \)'s into the formulas until we get all of them.

\( n = 0 : \)

\[t = 0.6047 \quad t = 0.9661\]

We’ll need both of these critical points.

\( n = 1 : \)
We’ll need these.

\( n = 2 \) :

\[
t = 0.6047 + \pi = 3.7463 \quad t = 0.9661 + \pi = 4.1077
\]

In this case we only need the first one since the second is out of the interval.

There are five critical points that are in the interval. They are,

\[
0.6047, \ 0.9661, \ 2.1755, \ 2.5369, \ 3.7463
\]

Finally, to determine the absolute minimum and maximum population we only need to plug these values into the function as well as the two end points. Here are the function evaluations.

\[
\begin{align*}
P(0) &= 100.0 \\
P(0.6047) &= 102.4756 \\
P(0.9661) &= 102.2368 \\
P(2.1755) &= 107.1880 \\
P(2.5369) &= 106.9492 \\
P(3.7463) &= 111.9004
\end{align*}
\]

From these evaluations it appears that the minimum population is 100,000 (remember that \( P \) is in thousands…) which occurs at \( t = 0 \) and the maximum population is 111,900 which occurs at \( t = 3.7463 \).

Make sure that you can correctly solve trig equations. If we had forgotten the \( 2\pi n \) we would have missed the last three critical points in the interval and hence gotten the wrong answer since the maximum population was at the final critical point.

Also, note that we do really need to be very careful with rounding answers here. If we’d rounded to the nearest integer, for instance, it would appear that the maximum population would have occurred at two different locations instead of only one.

**Example 4** Suppose that the amount of money in a bank account after \( t \) years is given by,

\[
A(t) = 2000 - 10e^{\frac{t^2}{8}}
\]

Determine the minimum and maximum amount of money in the account during the first 10 years that it is open.

**Solution**

Here we are really asking for the absolute extrema of \( A(t) \) on the interval \([0,10]\). As with the previous examples this function is continuous everywhere and so we know that this can be done.

We’ll first need the derivative so we can find the critical points.
The derivative exists everywhere and the exponential is never zero. Therefore the derivative will only be zero where,

\[-1 + \frac{t^2}{4} = 0 \quad \Rightarrow \quad t^2 = 4 \quad \Rightarrow \quad t = \pm 2\]

We’ve got two critical points, however only \( t = 2 \) is actually in the interval so that is only critical point that we’ll use.

Let’s now evaluate the function at the lone critical point and the end points of the interval. Here are those function evaluations.

\[A(0) = 2000 \quad A(2) = 199.66 \quad A(10) = 1999.94\]

So, the maximum amount in the account will be $2000 which occurs at \( t = 0 \) and the minimum amount in the account will be $199.66 which occurs at the 2 year mark.

In this example there are two important things to note. First, if we had included the second critical point we would have gotten an incorrect answer for the maximum amount so it’s important to be careful with which critical points to include and which to exclude.

All of the problems that we’ve worked to this point had derivatives that existed everywhere and so the only critical points that we looked at were those for which the derivative is zero. Do not get too locked into this always happening. Most of the problems that we run into will be like this, but they won’t all be like this.

Let’s work another example to make this point.

**Example 5** Determine the absolute extrema for the following function and interval.

\[Q(y) = 3y(y + 4)^\frac{2}{3} \quad \text{on} \quad [-5, -1]\]

**Solution**

Again, as with all the other examples here, this function is continuous on the given interval and so we know that this can be done.

First we’ll need the derivative and make sure you can do the simplification that we did here to make the work for finding the critical points easier.
\[
Q'(y) = 3(y + 4)^2 + 3y \left(\frac{2}{3}\right) (y + 4)^{-\frac{1}{3}}
\]
\[
= 3(y + 4)^2 + \frac{2y}{(y + 4)^{\frac{1}{3}}}
\]
\[
= \frac{3(y + 4) + 2y}{(y + 4)^{\frac{1}{3}}}
\]
\[
= \frac{5y + 12}{(y + 4)^{\frac{1}{3}}}
\]

So, it looks like we’ve got two critical points.

\[
y = -4 \quad \text{Because the derivative doesn't exist here.}
\]

\[
y = -\frac{12}{5} \quad \text{Because the derivative is zero here.}
\]

Both of these are in the interval so let’s evaluate the function at these points and the end points of the interval.

\[
Q(-4) = 0 \quad Q\left(-\frac{12}{5}\right) = -9.849
\]

\[
Q(-5) = -15 \quad Q(-1) = -6.241
\]

The function has an absolute maximum of zero at \( y = -4 \) and the function will have an absolute minimum of -15 at \( y = -5 \).

So, if we had ignored or forgotten about the critical point where the derivative doesn’t exist (\( y = -4 \)) we would not have gotten the correct answer.

In this section we’ve seen how we can use a derivative to identify the absolute extrema of a function. This is an important application of derivatives that will arise from time to time so don’t forget about it.