Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Indefinite Integrals

In the past two chapters we’ve been given a function, \( f(x) \), and asking what the derivative of this function was. Starting with this section we are now going to turn things around. We now want to ask what function we differentiated to get the function \( f(x) \).

Let’s take a quick look at an example to get us started.

Example 1  What function did we differentiate to get the following function.

\[
f(x) = x^4 + 3x - 9
\]

Solution

Let’s actually start by getting the derivative of this function to help us see how we’re going to have to approach this problem. The derivative of this function is,

\[
f'(x) = 4x^3 + 3
\]

The point of this was to remind us of how differentiation works. When differentiating powers of \( x \) we multiply the term by the original exponent and then drop the exponent by one.

Now, let’s go back and work the problem. In fact let’s just start with the first term. We got \( x^4 \) by differentiating a function and since we drop the exponent by one it looks like we must have differentiated \( x^5 \). However, if we had differentiated \( x^5 \) we would have \( 5x^4 \) and we don’t have a 5 in front our first term, so the 5 needs to cancel out after we’ve differentiated. It looks then like we would have to differentiate \( \frac{1}{5}x^5 \) in order to get \( x^4 \).

Likewise for the second term, in order to get \( 3x \) after differentiating we would have to differentiate \( \frac{3}{2}x^2 \). Again, the fraction is there to cancel out the 2 we pick up in the differentiation.

The third term is just a constant and we know that if we differentiate \( x \) we get 1. So, it looks like we had to differentiate \( -9x \) to get the last term.

Putting all of this together gives the following function,

\[
F(x) = \frac{1}{5}x^5 + \frac{3}{2}x^2 - 9x
\]

Our answer is easy enough to check. Simply differentiate \( F(x) \).

\[
F'(x) = x^4 + 3x - 9 = f(x)
\]
So, it looks like we got the correct function. Or did we? We know that the derivative of a constant is zero and so any of the following will also give \( f(x) \) upon differentiating.

\[
F(x) = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + 10 \\
F(x) = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x - 1954 \\
F(x) = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + \frac{3469}{123} \\
etc.
\]

In fact, any function of the form,

\[
F(x) = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + c, \quad c \text{ is a constant}
\]

will give \( f(x) \) upon differentiating.

There were two points to this last example. The first point was to get you thinking about how to do these problems. It is important initially to remember that we are really just asking what we differentiated to get the given function.

The other point is to recognize that there are actually an infinite number of functions that we could use and they will all differ by a constant.

Now that we’ve worked an example let’s get some of the definitions and terminology out of the way.

**Definitions**

Given a function, \( f(x) \), an **anti-derivative** of \( f(x) \) is any function \( F(x) \) such that

\[
F'(x) = f(x)
\]

If \( F(x) \) is any anti-derivative of \( f(x) \) then the most general anti-derivative of \( f(x) \) is called an **indefinite integral** and denoted,

\[
\int f(x) \, dx = F(x) + c, \quad c \text{ is any constant}
\]

In this definition the \( \int \) is called the **integral symbol**, \( f(x) \) is called the **integrand**, \( x \) is called the **integration variable** and the “\( c \)” is called the **constant of integration**.
Note that often we will just say integral instead of indefinite integral (or definite integral for that matter when we get to those). It will be clear from the context of the problem that we are talking about an indefinite integral (or definite integral).

The process of finding the indefinite integral is called integration or integrating \( f(x) \). If we need to be specific about the integration variable we will say that we are integrating \( f(x) \) with respect to \( x \).

Let’s rework the first problem in light of the new terminology.

**Example 2** Evaluate the following indefinite integral.

\[
\int x^4 + 3x - 9 \, dx
\]

**Solution**

Since this is really asking for the most general anti-derivative we just need to reuse the final answer from the first example.

The indefinite integral is,

\[
\int x^4 + 3x - 9 \, dx = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + c
\]

A couple of warnings are now in order. One of the more common mistakes that students make with integrals (both indefinite and definite) is to drop the \( dx \) at the end of the integral. This is required! Think of the integral sign and the \( dx \) as a set of parentheses. You already know and are probably quite comfortable with the idea that every time you open a parenthesis you must close it. With integrals, think of the integral sign as an “open parenthesis” and the \( dx \) as a “close parenthesis”.

If you drop the \( dx \) it won’t be clear where the integrand ends. Consider the following variations of the above example.

\[
\int x^4 + 3x - 9 \, dx = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + c
\]

\[
\int x^4 + 3x \, dx - 9 = \frac{1}{5} x^5 + \frac{3}{2} x^2 + c - 9
\]

\[
\int x^4 \, dx + 3x - 9 = \frac{1}{5} x^5 + c + 3x - 9
\]

You only integrate what is between the integral sign and the \( dx \). Each of the above integrals end in a different place and so we get different answers because we integrate a different number of terms each time. In the second integral the “-9” is outside the integral and so is left alone and not integrated. Likewise, in the third integral the “3x – 9” is outside the integral and so is left alone.
Knowing which terms to integrate is not the only reason for writing the \( dx \) down. In the Substitution Rule section we will actually be working with the \( dx \) in the problem and if we aren’t in the habit of writing it down it will be easy to forget about it and then we will get the wrong answer at that stage.

The moral of this is to make sure and put in the \( dx \)! At this stage it may seem like a silly thing to do, but it just needs to be there, if for no other reason than knowing where the integral stops.

On a side note, the \( dx \) notation should seem a little familiar to you. We saw things like this a couple of sections ago. We called the \( dx \) a differential in that section and yes that is exactly what it is. The \( dx \) that ends the integral is nothing more than a differential.

The next topic that we should discuss here is the integration variable used in the integral. Actually there isn’t really a lot to discuss here other than to note that the integration variable doesn’t really matter. For instance,

\[
\int x^4 + 3x - 9 \, dx = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + c
\]

\[
\int t^4 + 3t - 9 \, dt = \frac{1}{5} t^5 + \frac{3}{2} t^2 - 9t + c
\]

\[
\int w^4 + 3w - 9 \, dw = \frac{1}{5} w^5 + \frac{3}{2} w^2 - 9w + c
\]

Changing the integration variable in the integral simply changes the variable in the answer. It is important to notice however that when we change the integration variable in the integral we also changed the differential \( (dx, dt, \text{ or } dw) \) to match the new variable. This is more important that we might realize at this point.

Another use of the differential at the end of integral is to tell us what variable we are integrating with respect to. At this stage that may seem unimportant since most of the integrals that we’re going to be working with here will only involve a single variable. However, if you are on a degree track that will take you into multi-variable calculus this will be very important at that stage since there will be more than one variable in the problem. You need to get into the habit of writing the correct differential at the end of the integral so when it becomes important in those classes you will already be in the habit of writing it down.

To see why this is important take a look at the following two integrals.

\[
\int 2x \, dx \quad \quad \quad \quad \int 2t \, dx
\]

The first integral is simple enough.

\[
\int 2x \, dx = x^2 + c
\]
The second integral is also fairly simple, but we need to be careful. The \( dx \) tells us that we are integrating \( x \)'s. That means that we only integrate \( x \)'s that are in the integrand and all other variables in the integrand are considered to be constants. The second integral is then,

\[
\int 2t \, dx = 2tx + c
\]

So, it may seem silly to always put in the \( dx \), but it is a vital bit of notation that can cause us to get the incorrect answer if we neglect to put it in.

Now, there are some important properties of integrals that we should take a look at.

**Properties of the Indefinite Integral**

1. \( \int k f(x) \, dx = k \int f(x) \, dx \) where \( k \) is any number. So, we can factor multiplicative constants out of indefinite integrals.
   
   See the [Proof of Various Integral Formulas](#) section of the Extras chapter to see the proof of this property.

2. \( \int -f(x) \, dx = -\int f(x) \, dx \). This is really the first property with \( k = -1 \) and so no proof of this property will be given.

3. \( \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx \). In other words, the integral of a sum or difference of functions is the sum or difference of the individual integrals. This rule can be extended to as many functions as we need.
   
   See the [Proof of Various Integral Formulas](#) section of the Extras chapter to see the proof of this property.

Notice that when we worked the first example above we used the first and third property in the discussion. We integrated each term individually, put any constants back in and then put everything back together with the appropriate sign.

Not listed in the properties above were integrals of products and quotients. The reason for this is simple. Just like with derivatives each of the following will NOT work.

\[
\int f(x) g(x) \, dx \neq \int f(x) \, dx \int g(x) \, dx \quad \int \frac{f(x)}{g(x)} \, dx \neq \frac{\int f(x) \, dx}{\int g(x) \, dx}
\]

With derivatives we had a product rule and a quotient rule to deal with these cases. However, with integrals there are no such rules. When faced with a product and quotient in an integral we will have a variety of ways of dealing with it depending on just what the integrand is.
There is one final topic to be discussed briefly in this section. On occasion we will be given $f'(x)$ and will ask what $f(x)$ was. We can now answer this question easily with an indefinite integral.

\[ f(x) = \int f'(x) \, dx \]

**Example 3** If $f'(x) = x^4 + 3x - 9$ what was $f(x)$?

**Solution**

By this point in this section this is a simple question to answer.

\[ f(x) = \int f'(x) \, dx = \int x^4 + 3x - 9 \, dx = \frac{1}{5} x^5 + \frac{3}{2} x^2 - 9x + c \]

In this section we kept evaluating the same indefinite integral in all of our examples. The point of this section was not to do indefinite integrals, but instead to get us familiar with the notation and some of the basic ideas and properties of indefinite integrals. The next couple of sections are devoted to actually evaluating indefinite integrals.