Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
More Substitution Rule

In order to allow these pages to be displayed on the web we’ve broken the substitution rule examples into two sections. The previous section contains the introduction to the substitution rule and some fairly basic examples. The examples in this section tend towards the slightly more difficult side. Also, we’ll not be putting quite as much explanation into the solutions here as we did in the previous section.

In the first couple of sets of problems in this section the difficulty is not with the actual integration itself, but with the set up for the integration. Most of the integrals are fairly simple and most of the substitutions are fairly simple. The problems arise in getting the integral set up properly for the substitution(s) to be done. Once you see how these are done it’s easy to see what you have to do, but the first time through these can cause problems if you aren’t on the lookout for potential problems.

Example 1 Evaluate each of the following integrals.

(a) \( \int 2 \sec^2 \tan^2 t \, dt \) \[Solution\]

(b) \( \int \sin(t)(4 \cos^3(t) + 6 \cos^2(t) - 8) \, dt \) \[Solution\]

(c) \( \int x \cos(x^2 + 1) \, dt \) \[Solution\]

Solution

(a) \( \int e^{2t} + \sec(2t) \tan(2t) \, dt \)

This first integral has two terms in it and both will require the same substitution. This means that we won’t have to do anything special to the integral. One of the more common “mistakes” here is to break the integral up and do a separate substitution on each part. This isn’t really mistake but will definitely increase the amount of work we’ll need to do. So, since both terms in the integral use the same substitution we’ll just do everything as a single integral using the following substitution.

\[
\begin{align*}
    u &= 2t \\
    du &= 2 \, dt \\
    dt &= \frac{1}{2} \, du
\end{align*}
\]

The integral is then,

\[
\begin{align*}
    \int e^{2t} + \sec(2t) \tan(2t) \, dt &= \frac{1}{2} \int e^u + \sec(u) \tan(u) \, du \\
    &= \frac{1}{2} \left( e^u + \sec(u) \right) + c \\
    &= \frac{1}{2} \left( e^{2t} + \sec(2t) \right) + c
\end{align*}
\]

Often a substitution can be used multiple times in an integral so don’t get excited about that if it happens. Also note that since there was a \( \frac{1}{2} \) in front of the whole integral there must also be a \( \frac{1}{2} \) in front of the answer from the integral.

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(b) \( \int \sin(t) \left( 4 \cos^3(t) + 6 \cos^2(t) - 8 \right) dt \)

This integral is similar to the previous one, but it might not look like it at first glance. Here is the substitution for this problem,

\[
u = \cos(t) \quad du = -\sin(t) \, dt \quad \Rightarrow \quad \sin(t) \, dt = -du
\]

We’ll plug the substitution into the problem twice (since there are two cosines) and will only work because there is a sine multiplying everything. Without that sine in front we would not be able to use this substitution.

The integral in this case is,

\[
\int \sin(t) \left( 4 \cos^3(t) + 6 \cos^2(t) - 8 \right) dt = -\int 4u^3 + 6u^2 - 8 \, du
\]

\[
= -(u^4 + 2u^3 - 8u) + c
\]

\[
= -(\cos^4(t) + 2 \cos^3(t) - 8 \cos(t)) + c
\]

Again, be careful with the minus sign in front of the whole integral.

(e) \( \int x \cos(x^2 + 1) + \frac{x}{x^2 + 1} \, dx \)

It should be fairly clear that each term in this integral will use the same substitution, but let’s rewrite things a little to make things really clear.

\[
\int x \cos(x^2 + 1) + \frac{x}{x^2 + 1} \, dx = \int x \left( \cos(x^2 + 1) + \frac{1}{x^2 + 1} \right) dx
\]

Since each term had an \( x \) in it and we’ll need that for the differential we factored that out of both terms to get it into the front. This integral is now very similar to the previous one. Here’s the substitution.

\[
u = x^2 + 1 \quad du = 2x \, dx \quad \Rightarrow \quad x \, dx = \frac{1}{2} du
\]

The integral is,

\[
\int x \cos(x^2 + 1) + \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int \cos(u) + \frac{1}{u} \, du
\]

\[
= \frac{1}{2} (\sin(u) + \ln |u|) + c
\]

\[
= \frac{1}{2} (\sin(x^2 + 1) + \ln |x^2 + 1|) + c
\]
Example 2  Evaluate each of the following integrals.

(a) \( \int x^2 + e^{1-x} \, dx \)  \[Solution\]

(b) \( \int x \cos(x^2 + 1) + \frac{1}{x^2 + 1} \, dx \)  \[Solution\]

Solution

(a) \( \int x^2 + e^{1-x} \, dx \)

In this integral the first term does not need any substitution while the second term does need a substitution. So, to deal with that we’ll need to split the integral up as follows,

\[
\int x^2 + e^{1-x} \, dx = \int x^2 \, dx + \int e^{1-x} \, dx
\]

The substitution for the second integral is then,

\[
u = 1 - x \quad du = -dx \quad \Rightarrow \quad dx = -du
\]

The integral is,

\[
\int x^2 + e^{1-x} \, dx = \int x^2 \, dx - \int e^u \, du
\]

\[
= \frac{1}{3} x^3 - e^u + c
\]

\[
= \frac{1}{3} x^3 - e^{1-x} + c
\]

Be careful with this kind of integral. One of the more common mistakes here is do the following “shortcut”.

\[
\int x^2 + e^{1-x} \, dx = -\int x^2 + e^u \, du
\]

In other words, some students will try do the substitution just the second term without breaking up the integral. There are two issues with this. First, there is a “-” in front of the whole integral that shouldn’t be there. It should only be on the second term because that is the term getting the substitution. Secondly, and probably more importantly, there are \(x\)’s in the integral and we have a \(du\) for the differential. We can’t mix variables like this. When we do integrals all the variables in the integrand must match the variable in the differential.

(b) \( \int x \cos(x^2 + 1) + \frac{1}{x^2 + 1} \, dx \)

This integral looks very similar to Example 1c above, but it is different. In this integral we no longer have the \(x\) in the numerator of the second term and that means that the substitution we’ll use for the first term will no longer work for the second term. In fact,
the second term doesn’t need a substitution at all since it is just an inverse tangent.

The substitution for the first term is then,
\[ u = x^2 + 1 \quad \Rightarrow \quad x \, dx = \frac{1}{2} \, du \]

Now let’s do the integral. Remember to first break it up into two terms before using the substitution.
\[
\int x \cos(x^2 + 1) + \frac{1}{x^2 + 1} \, dx = \int x \cos(x^2 + 1) \, dx + \int \frac{1}{x^2 + 1} \, dx \\
= \frac{1}{2} \int \cos(u) \, du + \int \frac{1}{x^2 + 1} \, dx \\
= \frac{1}{2} \sin(u) + \tan^{-1}(x) + c \\
= \frac{1}{2} \sin(x^2 + 1) + \tan^{-1}(x) + c
\]

In this set of examples we saw that sometimes one (or potentially more than one) term in the integrand will not require a substitution. In these cases we’ll need to break up the integral into two integrals, one involving the terms that don’t need a substitution and another with the term(s) that do need a substitution.

**Example 3** Evaluate each of the following integrals.

(a) \( \int e^{-z} + \sec^2\left(\frac{z}{10}\right) \, dz \)  \[\text{[Solution]}\]

(b) \( \int \sin w \sqrt{1 - 2 \cos w + \frac{1}{7w + 2}} \, dw \)  \[\text{[Solution]}\]

(c) \( \int \frac{10x + 3}{x^2 + 16} \, dx \)  \[\text{[Solution]}\]

**Solution**

(a) \( \int e^{-z} + \sec^2\left(\frac{z}{10}\right) \, dz \)

In this integral, unlike any integrals that we’ve yet done, there are two terms and each will require a different substitution. So, to do this integral we’ll first need to split up the integral as follows,
\[
\int e^{-z} + \sec^2\left(\frac{z}{10}\right) \, dz = \int e^{-z} \, dz + \int \sec^2\left(\frac{z}{10}\right) \, dz
\]

Here are the substitutions for each integral.
\[ u = -z \quad du = -dz \quad \Rightarrow \quad dz = -du \]
\[ v = \frac{z}{10} \quad dv = \frac{1}{10} dz \quad \Rightarrow \quad dz = 10dv \]

Notice that we used different letters for each substitution to avoid confusion when we go to plug back in for \( u \) and \( v \).

Here is the integral.
\[
\int e^{-z} + \sec^2 \left( \frac{z}{10} \right) dz = -\int e^u du + 10 \int \sec^2 (v) dv
\]
\[
= -e^u + 10 \tan (v) + c
\]
\[
= -e^{-z} + 10 \tan \left( \frac{z}{10} \right) + c
\]

(b) \[ \int \sin w \sqrt{1 - 2 \cos w} + \frac{1}{7w + 2} \, dw \]

As with the last problem this integral will require two separate substitutions. Let’s first break up the integral.
\[
\int \sin w \sqrt{1 - 2 \cos w} + \frac{1}{7w + 2} \, dw = \int \sin w \left(1 - 2 \cos w \right)^{\frac{1}{2}} \, dw + \int \frac{1}{7w + 2} \, dw
\]

Here are the substitutions for this integral.
\[ u = 1 - 2 \cos (w) \quad du = 2 \sin (w) \, dw \quad \Rightarrow \quad \sin (w) \, dw = \frac{1}{2} du \]
\[ v = 7w + 2 \quad dv = 7 \, dw \quad \Rightarrow \quad dw = \frac{1}{7} dv \]

The integral is then,
\[
\int \sin w \sqrt{1 - 2 \cos w} + \frac{1}{7w + 2} \, dw = \frac{1}{2} \int u^{\frac{1}{2}} du + \frac{1}{7} \int \frac{1}{v} \, dv
\]
\[
= \frac{1}{2} \left( \frac{2}{3} \right) u^{\frac{3}{2}} + \frac{1}{7} \ln |v| + c
\]
\[
= \frac{1}{3} (1 - 2 \cos w)^{\frac{3}{2}} + \frac{1}{7} \ln |7w + 2| + c
\]

(c) \[ \int \frac{10x + 3}{x^2 + 16} \, dx \]

The last problem in this set can be tricky. If there was just an \( x \) in the numerator we could do a quick substitution to get a natural logarithm. Likewise if there wasn’t an \( x \) in the numerator we
would get an inverse tangent after a quick substitution.

To get this integral into a form that we can work with we will first need to break it up as follows.

\[
\int \frac{10x + 3}{x^2 + 16} \, dx = \int \frac{10x}{x^2 + 16} \, dx + \int \frac{3}{x^2 + 16} \, dx
\]

\[
= \int \frac{10x}{x^2 + 16} \, dx + \frac{1}{16} \int \frac{3}{\frac{x^2}{16} + 1} \, dx
\]

We now have two integrals each requiring a different substitution. The substitutions for each of the integrals above are,

\[
u = x^2 + 16 \quad \Rightarrow \quad xdx = \frac{1}{2} \, du
\]
\[
v = \frac{x}{4} \quad \Rightarrow \quad dx = 4 \, dv
\]

The integral is then,

\[
\int \frac{10x + 3}{x^2 + 16} \, dx = 5 \int \frac{1}{u} \, du + \frac{3}{4} \int \frac{1}{v^2 + 1} \, dv
\]

\[
= 5 \ln |u| + \frac{3}{4} \tan^{-1} (v) + c
\]

\[
= 5 \ln |x^2 + 16| + \frac{3}{4} \tan^{-1} \left( \frac{x}{4} \right) + c
\]

We’ve now seen a set of integrals in which we need to do more than one substitution. In these cases we will need to break up the integral into separate integrals and do separate substitutions for each.

We now need to move onto a different set of examples that can be a little tricky. Once you’ve seen how to do these they aren’t too bad, but doing them for the first time can be difficult if you aren’t ready for them.
Example 4  Evaluate each of the following integrals.

(a) \( \int \tan x \, dx \)  [Solution]

(b) \( \int \sec y \, dy \)  [Solution]

(c) \( \int \frac{\cos(\sqrt{x})}{\sqrt{x}} \, dx \)  [Solution]

(d) \( \int e^{x^2} e^x \, dx \)  [Solution]

(e) \( \int 2x^3\sqrt{x^2 + 1} \, dx \)  [Solution]

\[ \text{Solution} \]

(a) \( \int \tan x \, dx \)

The first question about this problem is probably why is it here? Substitution rule problems generally require more than a single function. The key to this problem is to realize that there really are two functions here. All we need to do is remember the definition of tangent and we can write the integral as,

\[ \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx \]

Written in this way we can see that the following substitution will work for us,

\( u = \cos x \)  \quad \Rightarrow  \quad \sin x \, dx = -du \)

The integral is then,

\[ \int \tan x \, dx = -\int \frac{1}{u} \, du \]

\[ = -\ln |u| + c \]

\[ = -\ln |\cos x| + c \]

Now, while this is a perfectly serviceable answer that minus sign in front is liable to cause problems if we aren’t careful. So, let’s rewrite things a little. Recalling a property of logarithms we can move the minus sign in front to an exponent on the cosine and then do a little simplification.

\[ \int \tan x \, dx = -\ln |\cos x| + c \]

\[ = \ln |\cos x|^{-1} + c \]

\[ = \ln \frac{1}{|\cos x|} + c \]

\[ = \ln |\sec x| + c \]
This is the formula that is typically given for the integral of tangent.

Note that we could integrate cotangent in a similar manner.

(b) \( \int \sec y \, dy \)

This problem also at first appears to not belong in the substitution rule problems. This is even more of a problem upon noticing that we can’t just use the definition of the secant function to write this in a form that will allow the use of the substitution rule.

This problem is going to require a technique that isn’t used terribly often at this level, but is a useful technique to be aware of. Sometimes we can make an integral doable by multiplying the top and bottom by a common term. This will not always work and even when it does it is not always clear what we should multiply by but when it works it is very useful.

Here is how we’ll use this idea for this problem.

\[
\int \sec y \, dy = \int \frac{\sec y \left( \sec y + \tan y \right)}{1 \left( \sec y + \tan y \right)} \, dy
\]

First, we will think of the secant as a fraction and then multiply the top and bottom of the fraction by the same term. It is probably not clear why one would want to do this here but doing this will actually allow us to use the substitution rule. To see how this will work let’s simplify the integrand somewhat.

\[
\int \sec y \, dy = \int \frac{\sec^2 y + \tan y \sec y}{\sec y + \tan y} \, dy
\]

We can now use the following substitution.

\[
u = \sec y + \tan y \quad \quad du = \left( \sec y \tan y + \sec^2 y \right) dy
\]

The integral is then,

\[
\int \sec y \, dy = \int \frac{1}{u} \, du
\]

\[
= \ln |u| + c
\]

\[
= \ln |\sec y + \tan y| + c
\]

Sometimes multiplying the top and bottom of a fraction by a carefully chosen term will allow us to work a problem. It does however take some thought sometimes to determine just what the term should be.

We can use a similar process for integrating cosecant.
This next problem has a subtlety to it that can get us in trouble if we aren’t paying attention. Because of the root in the cosine it makes some sense to use the following substitution.

\[ u = x^{\frac{1}{2}} \quad \Rightarrow \quad \frac{1}{2} x^{-\frac{1}{2}} dx \]

This is where we need to be careful. Upon rewriting the differential we get,

\[ 2du = \frac{1}{\sqrt{x}} dx \]

The root that is in the denominator will not become a \( u \) as we might have been tempted to do. Instead it will get taken care of in the differential.

The integral is,

\[ \int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx = 2 \int \cos(u) du \]

\[ = 2\sin(u) + c \]

\[ = 2\sin(\sqrt{x}) + c \]

(d) \[ \int e^{t^2} dt \]

With this problem we need to very carefully pick our substitution. As the problem is written we might be tempted to use the following substitution,

\[ u = t + e^t \quad \Rightarrow \quad du = (1 + e^t) dt \]

However, this won’t work as you can probably see. The differential doesn’t show up anywhere in the integrand and we just wouldn’t be able to eliminate all the \( t \)'s with this substitution.

In order to work this problem we will need to rewrite the integrand as follows,

\[ \int e^{t^2} dt = \int e^t e^t dt \]

We will now use the substitution,

\[ u = e^t \quad \Rightarrow \quad du = e^t dt \]

The integral is,
Some substitutions can be really tricky to see and it’s not unusual that you’ll need to do some simplification and/or rewriting to get a substitution to work.

\[
\int e^{e^t} \, dt = \int e^u \, du = e^u + c = e^{e^t} + c
\]

(e) \( \int 2x^3 \sqrt{x^2 + 1} \, dx \)

This last problem in this set is different from all the other substitution problems that we’ve worked to this point. Given the fact that we’ve got more than an \( x \) under the root it makes sense that the substitution pretty much has to be,

\[
u = x^2 + 1 \quad \Rightarrow \quad du = 2x \, dx
\]

At first glance it looks like this might not work for the substitution because we have an \( x^3 \) in front of the root. However, if we first rewrite \( 2x^3 = x^2 \cdot (2x) \) we could then move the \( 2x \) to the end of the integral so at least the \( du \) will show up explicitly in the integral. Doing this gives the following,

\[
\int 2x^3 \sqrt{x^2 + 1} \, dx = \int x^2 \sqrt{x^2 + 1} \cdot (2x) \, dx = \int x^2 u^\frac{1}{2} \, du
\]

This is a real problem. Our integrals can’t have two variables in them. Normally this would mean that we chose our substitution incorrectly. However, in this case we can rewrite the substitution as follows,

\[
x^2 = u - 1
\]

and now, we can eliminate the remaining \( x \)’s from our integral. Doing this gives,

\[
\int 2x^3 \sqrt{x^2 + 1} \, dx = \int (u - 1)u^\frac{1}{2} \, du = \int u^\frac{3}{2} - u^\frac{1}{2} \, du = \frac{2}{5}u^\frac{5}{2} - \frac{2}{3}u^\frac{3}{2} + c
\]

\[
= \frac{2}{5}(x^2 + 1)^{\frac{5}{2}} - \frac{2}{3}(x^2 + 1)^{\frac{3}{2}} + c
\]

Sometimes, we will need to use a substitution more than once.

This kind of problem doesn’t arise all that often and when it does there will sometimes be
alternate methods of doing the integral. However, it will often work out that the easiest method of doing the integral is to do what we just did here.

This final set of examples isn’t too bad once you see the substitutions and that is the point with this set of problems. These all involve substitutions that we’ve not seen prior to this and so we need to see some of these kinds of problems.

**Example 5** Evaluate each of the following integrals.

(a) \( \int \frac{1}{x \ln x} \, dx \)  

(b) \( \int \frac{e^{2t}}{1 + e^{2t}} \, dt \)

(c) \( \int \frac{e^{2t}}{1 + e^{8t}} \, dt \)

(d) \( \int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx \)

**Solution**

(a) \( \int \frac{1}{x \ln x} \, dx \)

In this case we know that we can’t integrate a logarithm by itself and so it makes some sense (hopefully) that the logarithm will need to be in the substitution. Here is the substitution for this problem.

\[
\begin{align*}
  u &= \ln x & du &= \frac{1}{x} \, dx
\end{align*}
\]

So the \( x \) in the denominator of the integrand will get substituted away with the differential. Here is the integral for this problem.

\[
\int \frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du
\]

\[
= \ln |u| + c
\]

\[
= \ln |\ln x| + c
\]

(b) \( \int \frac{e^{2t}}{1 + e^{2t}} \, dt \)

Again, the substitution here may seem a little tricky. In this case the substitution is,
The integral is then,
\[
\int \frac{e^{2t}}{1 + e^{2t}} \, dt = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln |1 + e^{2t}| + c
\]

(e) \[ \int \frac{e^{2t}}{1 + e^{6t}} \, dt \]

In this case we can’t use the same type of substitution that we used in the previous problem. In order to use the substitution in the previous example the exponential in the numerator and the denominator need to be the same and in this case they aren’t.

To see the correct substitution for this problem note that,
\[
e^{4t} = (e^{2t})^2
\]

Using this, the integral can be written as follows,
\[
\int \frac{e^{2t}}{1 + e^{4t}} \, dt = \int \frac{e^{2t}}{1 + (e^{2t})^2} \, dt
\]

We can now use the following substitution.
\[
u = e^{2t} \quad du = 2e^{2t} \, dt \quad \Rightarrow \quad e^{2t} \, dt = \frac{1}{2} \, du
\]

The integral is then,
\[
\int \frac{e^{2t}}{1 + e^{4t}} \, dt = \frac{1}{2} \int \frac{1}{1 + u^2} \, du = \frac{1}{2} \tan^{-1}(u) + c = \frac{1}{2} \tan^{-1}(e^{2t}) + c
\]

(d) \[ \int \frac{\sin^{-1} x}{\sqrt{1 - x^2}} \, dx \]

This integral is similar to the first problem in this set. Since we don’t know how to integrate inverse sine functions it seems likely that this will be our substitution. If we use this as our
substitution we get,

\[ u = \sin^{-1}(x) \quad du = \frac{1}{\sqrt{1-x^2}} \, dx \]

So, the root in the integral will get taken care of in the substitution process and this will eliminate all the \( x \)'s from the integral. Therefore this was the correct substitution.

The integral is,

\[ \int \frac{\sin^{-1}x}{\sqrt{1-x^2}} \, dx = \int u \, du \]

\[ = \frac{1}{2} u^2 + c \]

\[ = \frac{1}{2} \left( \sin^{-1}x \right)^2 + c \]

Over the last couple of sections we’ve seen a lot of substitution rule examples. There are a couple of general rules that we will need to remember when doing these problems. First, when doing a substitution remember that when the substitution is done all the \( x \)'s in the integral (or whatever variable is being used for that particular integral) should all be substituted away. This includes the \( x \) in the \( dx \). After the substitution only \( u \)'s should be left in the integral. Also, sometimes the correct substitution is a little tricky to find and more often than not there will need to be some manipulation of the differential or integrand in order to actually do the substitution.

Also, many integrals will require us to break them up so we can do multiple substitutions so be on the lookout for those kinds of integrals/substitutions.