Preface

Here are my online notes for my differential equations course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn how to solve differential equations or needing a refresher on differential equations.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from a Calculus or Algebra class or contained in other sections of the notes.

A couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn differential equations I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Differential Equation many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Complex Eigenvalues

In this section we will look at solutions to
\[ \mathbf{x}' = A\mathbf{x} \]
where the eigenvalues of the matrix \( A \) are complex. With complex eigenvalues we are going to have the same problem that we had back when we were looking at second order differential equations. We want our solutions to only have real numbers in them, however since our solutions to systems are of the form,
\[ \mathbf{x} = \eta e^{\lambda t} \]
we are going to have complex numbers come into our solution from both the eigenvalue and the eigenvector. Getting rid of the complex numbers here will be similar to how we did it back in the second order differential equation case, but will involve a little more work this time around. It’s easiest to see how to do this in an example.

**Example 1** Solve the following IVP.
\[ \mathbf{x}' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 2 \\ -4 \end{pmatrix} \]

**Solution**
We first need the eigenvalues and eigenvectors for the matrix.
\[
\det (A - \lambda I) = \begin{vmatrix} 3 - \lambda & -9 \\ 4 & -3 - \lambda \end{vmatrix} = \lambda^2 + 27
\]
\[ \lambda_{1,2} = \pm 3\sqrt{3}i \]

So, now that we have the eigenvalues recall that we only need to get the eigenvector for one of the eigenvalues since we can get the second eigenvector for free from the first eigenvector.

\( \lambda_2 = 3\sqrt{3}i \):
We need to solve the following system.
\[
\begin{pmatrix} 3 - 3\sqrt{3}i & -9 \\ 4 & -3 - 3\sqrt{3}i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Using the first equation we get,
\[
(3 - 3\sqrt{3}i)\eta_1 - 9\eta_2 = 0
\]
\[
\eta_2 = \frac{1}{3}(1 - \sqrt{3}i)\eta_1
\]

So, the first eigenvector is,
\[
\eta = \begin{pmatrix} \eta_1 \\ \frac{1}{3}(1 - \sqrt{3}i)\eta_1 \end{pmatrix} \quad \eta_1 = 3
\]
When finding the eigenvectors in these cases make sure that the complex number appears in the numerator of any fractions since we’ll need it in the numerator later on. Also try to clear out any fractions by appropriately picking the constant. This will make our life easier down the road.

Now, the second eigenvector is,

\[ \mathbf{\tilde{\eta}}^{(2)} = \begin{pmatrix} 3 \\ 1 + \sqrt{3} i \end{pmatrix} \]

However, as we will see we won’t need this eigenvector.

The solution that we get from the first eigenvalue and eigenvector is,

\[ \mathbf{x}_1(t) = e^{3\sqrt{3}it} \begin{pmatrix} 3 \\ 1 - \sqrt{3} i \end{pmatrix} \]

So, as we can see there are complex numbers in both the exponential and vector that we will need to get rid of in order to use this as a solution. Recall from the complex roots section of the second order differential equation chapter that we can use Euler’s formula to get the complex number out of the exponential. Doing this gives us,

\[ \mathbf{x}_1(t) = \left( \cos(3\sqrt{3}t) + i \sin(3\sqrt{3}t) \right) \begin{pmatrix} 3 \\ 1 - \sqrt{3} i \end{pmatrix} \]

The next step is to multiply the cosines and sines into the vector.

\[ \mathbf{x}_1(t) = \begin{pmatrix} 3 \cos(3\sqrt{3}t) + 3i \sin(3\sqrt{3}t) \\ \cos(3\sqrt{3}t) + i \sin(3\sqrt{3}t) - \sqrt{3} i \cos(3\sqrt{3}t) + \sqrt{3} \sin(3\sqrt{3}t) \end{pmatrix} \]

Now combine the terms with an “i” in them and split these terms off from those terms that don’t contain an “i”. Also factor the “i” out of this vector.

\[ \mathbf{x}_1(t) = \begin{pmatrix} 3 \cos(3\sqrt{3}t) \\ \cos(3\sqrt{3}t) + \sqrt{3} \sin(3\sqrt{3}t) \end{pmatrix} + i \begin{pmatrix} 3 \sin(3\sqrt{3}t) \\ \sin(3\sqrt{3}t) - \sqrt{3} \cos(3\sqrt{3}t) \end{pmatrix} \]

\[ = \mathbf{u}(t) + i \mathbf{v}(t) \]

Now, it can be shown (we’ll leave the details to you) that \( \mathbf{u}(t) \) and \( \mathbf{v}(t) \) are two linearly independent solutions to the system of differential equations. This means that we can use them to form a general solution and they are both real solutions.

So, the general solution to a system with complex roots is

\[ \mathbf{x}(t) = c_1 \mathbf{u}(t) + c_2 \mathbf{v}(t) \]

where \( \mathbf{u}(t) \) and \( \mathbf{v}(t) \) are found by writing the first solution as

\[ \mathbf{x}(t) = \mathbf{u}(t) + i \mathbf{v}(t) \]
For our system then, the general solution is,

$$\vec{x}(t) = c_1 \begin{pmatrix} 3 \cos(3\sqrt{3}t) \\ \cos(3\sqrt{3}t) + \sqrt{3} \sin(3\sqrt{3}t) \end{pmatrix} + c_2 \begin{pmatrix} 3 \sin(3\sqrt{3}t) \\ \sin(3\sqrt{3}t) - \sqrt{3} \cos(3\sqrt{3}t) \end{pmatrix}$$

We now need to apply the initial condition to this to find the constants.

$$\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -\sqrt{3} \end{pmatrix}$$

This leads to the following system of equations to be solved,

$$\begin{aligned} 3c_1 &= 2 \\ c_1 - \sqrt{3}c_2 &= -4 \end{aligned} \quad \Rightarrow \quad \begin{cases} c_1 = \frac{2}{3} \\ c_2 = \frac{14}{3\sqrt{3}} \end{cases}$$

The actual solution is then,

$$\vec{x}(t) = \frac{2}{3} \begin{pmatrix} 3 \cos(3\sqrt{3}t) \\ \cos(3\sqrt{3}t) + \sqrt{3} \sin(3\sqrt{3}t) \end{pmatrix} + \frac{14}{3\sqrt{3}} \begin{pmatrix} 3 \sin(3\sqrt{3}t) \\ \sin(3\sqrt{3}t) - \sqrt{3} \cos(3\sqrt{3}t) \end{pmatrix}$$

As we did in the last section we’ll do the phase portraits separately from the solution of the system in case phase portraits haven’t been taught in your class.

**Example 2** Sketch the phase portrait for the system.

$$\vec{x}' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \vec{x}$$

**Solution**

When the eigenvalues of a matrix $A$ are purely complex, as they are in this case, the trajectories of the solutions will be circles or ellipses that are centered at the origin. The only thing that we really need to concern ourselves with here are whether they are rotating in a clockwise or counterclockwise direction.

This is easy enough to do. Recall when we first looked at these phase portraits a couple of sections ago that if we pick a value of $\vec{x}(t)$ and plug it into our system we will get a vector that will be tangent to the trajectory at that point and pointing in the direction that the trajectory is traveling. So, let’s pick the following point and see what we get.

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \vec{x}' = \begin{pmatrix} 3 & -9 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

Therefore at the point $(1,0)$ in the phase plane the trajectory will be pointing in a upwards direction. The only way that this can be is if the trajectories are traveling in a counterclockwise direction.

Here is the sketch of some of the trajectories for this problem.
The equilibrium solution in the case is called a **center** and is stable.

Note in this last example that the equilibrium solution is stable and not asymptotically stable. Asymptotically stable refers to the fact that the trajectories are moving in toward the equilibrium solution as \( t \) increases. In this example the trajectories are simply revolving around the equilibrium solution and not moving in towards it. The trajectories are also not moving away from the equilibrium solution and so they aren’t unstable. Therefore we call the equilibrium solution stable.

Not all complex eigenvalues will result in centers so let’s take a look at an example where we get something different.

**Example 3** Solve the following IVP.

\[
\frac{d\mathbf{x}}{dt} = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 3 \\ -10 \end{pmatrix}
\]

**Solution**

Let’s get the eigenvalues and eigenvectors for the matrix.

\[
\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & -13 \\ 5 & 1 - \lambda \end{vmatrix} = \lambda^2 - 4\lambda + 68
\]

\[\lambda_{1,2} = 2 \pm 8i\]

Now get the eigenvector for the first eigenvalue.

\[\lambda_1 = 2 + 8i: \]

We need to solve the following system.

\[
\begin{pmatrix} 1 - 8i & -13 \\ 5 & -1 - 8i \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Using the second equation we get,
\[ 5\eta_1 + (-1 - 8i)\eta_2 = 0 \]
\[ \eta_1 = \frac{1}{5}(1 + 8i)\eta_2 \]

So, the first eigenvector is,
\[ \vec{\eta} = \begin{pmatrix} \frac{1}{5}(1 + 8i)\eta_2 \\ \eta_2 \end{pmatrix} \]
\[ \vec{\eta}^{(i)} = \begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix} \quad \eta_2 = 5 \]

The solution corresponding the this eigenvalue and eigenvector is
\[ \bar{x}_1(t) = e^{(2 + 8i)t} \begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix} \]
\[ = e^{2t}e^{8it} \begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix} \]
\[ = e^{2t}(\cos(8t) + i\sin(8t))\begin{pmatrix} 1 + 8i \\ 5 \end{pmatrix} \]

As with the first example multiply cosines and sines into the vector and split it up. Don’t forget about the exponential that is in the solution this time.
\[ \bar{x}_1(t) = e^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + ie^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix} \]
\[ = \bar{u}(t) + i\bar{v}(t) \]

The general solution to this system then,
\[ \bar{x}(t) = c_1e^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + c_2e^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix} \]

Now apply the initial condition and find the constants.
\[ \begin{pmatrix} 3 \\ -10 \end{pmatrix} = \bar{x}(0) = c_1\begin{pmatrix} 1 \\ 5 \end{pmatrix} + c_2\begin{pmatrix} 8 \\ 0 \end{pmatrix} \]
\[ c_1 + 8c_2 = 3 \]
\[ 5c_1 = -10 \]
\[ \Rightarrow \quad c_1 = -2, \quad c_2 = \frac{5}{8} \]

The actual solution is then,
\[ \bar{x}(t) = -2e^{2t}\begin{pmatrix} \cos(8t) - 8\sin(8t) \\ 5\cos(8t) \end{pmatrix} + \frac{5}{8}e^{2t}\begin{pmatrix} 8\cos(8t) + \sin(8t) \\ 5\sin(8t) \end{pmatrix} \]

Let’s take a look at the phase portrait for this problem.
Example 4  Sketch the phase portrait for the system.

\[
\mathbf{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \mathbf{x}
\]

Solution

When the eigenvalues of a system are complex with a real part the trajectories will spiral into or out of the origin. We can determine which one it will be by looking at the real portion. Since the real portion will end up being the exponent of an exponential function (as we saw in the solution to this system) if the real part is positive the solution will grow very large as \( t \) increases. Likewise, if the real part is negative the solution will die out as \( t \) increases.

So, if the real part is positive the trajectories will spiral out from the origin and if the real part is negative they will spiral into the origin. We determine the direction of rotation (clockwise vs. counterclockwise) in the same way that we did for the center.

In our case the trajectories will spiral out from the origin since the real part is positive and

\[
\mathbf{x}' = \begin{pmatrix} 3 & -13 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}
\]

will rotate in the counterclockwise direction as the last example did.

Here is a sketch of some of the trajectories for this system.

![Image of trajectories](image)

Here we call the equilibrium solution a **spiral** (oddly enough…) and in this case it’s unstable since the trajectories move away from the origin.

If the real part of the eigenvalue is negative the trajectories will spiral into the origin and in this case the equilibrium solution will be asymptotically stable.