Preface

Here are my online notes for my Calculus I course that I teach here at Lamar University. Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn Calculus I or needing a refresher in some of the early topics in calculus.

I’ve tried to make these notes as self contained as possible and so all the information needed to read through them is either from an Algebra or Trig class or contained in other sections of the notes.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible when writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Review: Solving Trig Equations

In this section we will take a look at solving trig equations. This is something that you will be asked to do on a fairly regular basis in my class.

Let’s just jump into the examples and see how to solve trig equations.

Example 1 Solve \( 2 \cos(t) = \sqrt{3} \).

Solution
There’s really not a whole lot to do in solving this kind of trig equation. All we need to do is divide both sides by 2 and go to the unit circle.

\[
2 \cos(t) = \sqrt{3} \\
\cos(t) = \frac{\sqrt{3}}{2}
\]

So, we are looking for all the values of \( t \) for which cosine will have the value of \( \frac{\sqrt{3}}{2} \). So, let’s take a look at the following unit circle.

From quick inspection we can see that \( t = \frac{\pi}{6} \) is a solution. However, as I have shown on the unit
circle there is another angle which will also be a solution. We need to determine what this angle is. When we look for these angles we typically want positive angles that lie between 0 and $2\pi$. This angle will not be the only possibility of course, but by convention we typically look for angles that meet these conditions.

To find this angle for this problem all we need to do is use a little geometry. The angle in the first quadrant makes an angle of $\frac{\pi}{6}$ with the positive $x$-axis, then so must the angle in the fourth quadrant. So we could use $-\frac{\pi}{6}$, but again, it's more common to use positive angles so, we'll use $t = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

We aren't done with this problem. As the discussion about finding the second angle has shown there are many ways to write any given angle on the unit circle. Sometimes it will be $-\frac{\pi}{6}$ that we want for the solution and sometimes we will want both (or neither) of the listed angles. Therefore, since there isn't anything in this problem (contrast this with the next problem) to tell us which is the correct solution we will need to list ALL possible solutions.

This is very easy to do. Recall from the previous section and you’ll see there that I used $\frac{\pi}{6} + 2\pi n , \ n = 0, \pm 1, \pm 2, \pm 3, \ldots$ to represent all the possible angles that can end at the same location on the unit circle, i.e. angles that end at $\frac{\pi}{6}$. Remember that all this says is that we start at $\frac{\pi}{6}$ then rotate around in the counter-clockwise direction ($n$ is positive) or clockwise direction ($n$ is negative) for $n$ complete rotations. The same thing can be done for the second solution.

So, all together the complete solution to this problem is

\[
\frac{\pi}{6} + 2\pi n , \ n = 0, \pm 1, \pm 2, \pm 3, \ldots \\
\frac{11\pi}{6} + 2\pi n , \ n = 0, \pm 1, \pm 2, \pm 3, \ldots
\]

As a final thought, notice that we can get $-\frac{\pi}{6}$ by using $n = -1$ in the second solution.

Now, in a calculus class this is not a typical trig equation that we’ll be asked to solve. A more typical example is the next one.
**Example 2** Solve \(2 \cos(t) = \sqrt{3}\) on \([-2\pi, 2\pi]\).

**Solution**
In a calculus class we are often more interested in only the solutions to a trig equation that fall in a certain interval. The first step in this kind of problem is to first find all possible solutions. We did this in the first example.

\[
\begin{align*}
\frac{\pi}{6} + 2\pi n, & \quad n = 0, \pm 1, \pm 2, \pm 3, \\
\frac{11\pi}{6} + 2\pi n, & \quad n = 0, \pm 1, \pm 2, \pm 3, \\
\end{align*}
\]

Now, to find the solutions in the interval all we need to do is start picking values of \(n\), plugging them in and getting the solutions that will fall into the interval that we’ve been given.

\(n=0\).

\[
\begin{align*}
\frac{\pi}{6} + 2\pi(0) &= \frac{\pi}{6} < 2\pi \\
\frac{11\pi}{6} + 2\pi(0) &= \frac{11\pi}{6} < 2\pi \\
\end{align*}
\]

Now, notice that if we take any positive value of \(n\) we will be adding on positive multiples of \(2\pi\) onto a positive quantity and this will take us past the upper bound of our interval and so we don’t need to take any positive value of \(n\).

However, just because we aren’t going to take any positive value of \(n\) doesn’t mean that we shouldn’t also look at negative values of \(n\).

\(n=-1\).

\[
\begin{align*}
\frac{\pi}{6} + 2\pi(-1) &= -\frac{11\pi}{6} > -2\pi \\
\frac{11\pi}{6} + 2\pi(-1) &= -\frac{\pi}{6} > -2\pi \\
\end{align*}
\]

These are both greater than \(-2\pi\) and so are solutions, but if we subtract another \(2\pi\) off (i.e. use \(n = -2\)) we will once again be outside of the interval so we’ve found all the possible solutions that lie inside the interval \([-2\pi, 2\pi]\).

So, the solutions are : \(\frac{\pi}{6}, \frac{11\pi}{6}, -\frac{\pi}{6}, -\frac{11\pi}{6}\).

So, let’s see if you’ve got all this down.
Example 3  Solve \(2 \sin(5x) = -\sqrt{3}\) on \([-\pi, 2\pi]\)

Solution
This problem is very similar to the other problems in this section with a very important difference. We’ll start this problem in exactly the same way. We first need to find all possible solutions.

\[
2 \sin(5x) = -\sqrt{3} \\
\sin(5x) = -\frac{\sqrt{3}}{2}
\]

So, we are looking for angles that will give \(-\frac{\sqrt{3}}{2}\) out of the sine function. Let’s again go to our trusty unit circle.

Now, there are no angles in the first quadrant for which sine has a value of \(-\frac{\sqrt{3}}{2}\). However, there are two angles in the lower half of the unit circle for which sine will have a value of \(-\frac{\sqrt{3}}{2}\).

So, what are these angles? We’ll notice \(\sin\left(\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\), so the angle in the third quadrant will be
below the negative x-axis or \( \frac{\pi}{3} + \frac{\pi}{3} = \frac{4\pi}{3} \). Likewise, the angle in the fourth quadrant will \( \frac{\pi}{3} \) below the positive x-axis or \( 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \). Remember that we’re typically looking for positive angles between 0 and \( 2\pi \).

Now we come to the very important difference between this problem and the previous problems in this section. The solution is NOT

\[
x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots
\]

This is not the set of solutions because we are NOT looking for values of \( x \) for which

\[
\sin(x) = -\frac{\sqrt{3}}{2},
\]

but instead we are looking for values of \( x \) for which \( \sin(5x) = -\frac{\sqrt{3}}{2} \). Note the difference in the arguments of the sine function! One is \( x \) and the other is \( 5x \). This makes all the difference in the world in finding the solution! Therefore, the set of solutions is

\[
5x = \frac{4\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
5x = \frac{5\pi}{3} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots
\]

Well, actually, that’s not quite the solution. We are looking for values of \( x \) so divide everything by 5 to get.

\[
x = \frac{4\pi}{15} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots
\]

\[
x = \frac{\pi}{3} + \frac{2\pi n}{5}, \quad n = 0, \pm 1, \pm 2, \ldots
\]

Notice that we also divided the \( 2\pi n \) by 5 as well! This is important! If we don’t do that you WILL miss solutions. For instance, take \( n = 1 \).

\[
x = \frac{4\pi}{15} + \frac{2\pi}{5} = \frac{10\pi}{15} = \frac{2\pi}{3} \quad \Rightarrow \quad \sin \left( 5 \left( \frac{2\pi}{3} \right) \right) = \sin \left( \frac{10\pi}{3} \right) = -\frac{\sqrt{3}}{2}
\]

\[
x = \frac{\pi}{3} + \frac{2\pi}{5} = \frac{11\pi}{15} \quad \Rightarrow \quad \sin \left( 5 \left( \frac{11\pi}{15} \right) \right) = \sin \left( \frac{11\pi}{3} \right) = -\frac{\sqrt{3}}{2}
\]

I’ll leave it to you to verify my work showing they are solutions. However it makes the point. If you didn’t divided the \( 2\pi n \) by 5 you would have missed these solutions!

Okay, now that we’ve gotten all possible solutions it’s time to find the solutions on the given interval. We’ll do this as we did in the previous problem. Pick values of \( n \) and get the solutions.
Calculus I

\( n = 0 \).

\[
\begin{align*}
\frac{4\pi}{15} + \frac{2\pi(0)}{5} &= \frac{4\pi}{15} < 2\pi \\
\frac{\pi}{3} + \frac{2\pi(0)}{5} &= \frac{\pi}{3} < 2\pi 
\end{align*}
\]

\( n = 1 \).

\[
\begin{align*}
\frac{4\pi}{15} + \frac{2\pi(1)}{5} &= \frac{2\pi}{3} < 2\pi \\
\frac{\pi}{3} + \frac{2\pi(1)}{5} &= \frac{11\pi}{15} < 2\pi 
\end{align*}
\]

\( n = 2 \).

\[
\begin{align*}
\frac{4\pi}{15} + \frac{2\pi(2)}{5} &= \frac{16\pi}{15} < 2\pi \\
\frac{\pi}{3} + \frac{2\pi(2)}{5} &= \frac{17\pi}{15} < 2\pi 
\end{align*}
\]

\( n = 3 \).

\[
\begin{align*}
\frac{4\pi}{15} + \frac{2\pi(3)}{5} &= \frac{22\pi}{15} < 2\pi \\
\frac{\pi}{3} + \frac{2\pi(3)}{5} &= \frac{23\pi}{15} < 2\pi 
\end{align*}
\]

\( n = 4 \).

\[
\begin{align*}
\frac{4\pi}{15} + \frac{2\pi(4)}{5} &= \frac{28\pi}{15} < 2\pi \\
\frac{\pi}{3} + \frac{2\pi(4)}{5} &= \frac{29\pi}{15} < 2\pi 
\end{align*}
\]

\( n = 5 \).

\[
\begin{align*}
\frac{4\pi}{15} + \frac{2\pi(5)}{5} &= \frac{34\pi}{15} > 2\pi \\
\frac{\pi}{3} + \frac{2\pi(5)}{5} &= \frac{35\pi}{15} > 2\pi 
\end{align*}
\]

Okay, so we finally got past the right endpoint of our interval so we don’t need any more positive \( n \). Now let’s take a look at the negative \( n \) and see what we’ve got.

\( n = -1 \).

\[
\begin{align*}
\frac{4\pi}{15} + \frac{2\pi(-1)}{5} &= -\frac{2\pi}{15} > -\pi \\
\frac{\pi}{3} + \frac{2\pi(-1)}{5} &= -\frac{\pi}{15} > -\pi 
\end{align*}
\]
Calculus I

\( n = -2. \)

\[
x = \frac{4\pi}{15} + \frac{2\pi(-2)}{5} = -\frac{8\pi}{15} > -\pi
\]

\[
x = \frac{\pi}{3} + \frac{2\pi(-2)}{5} = -\frac{7\pi}{15} > -\pi
\]

\( n = -3. \)

\[
x = \frac{4\pi}{15} + \frac{2\pi(-3)}{5} = -\frac{14\pi}{15} > -\pi
\]

\[
x = \frac{\pi}{3} + \frac{2\pi(-3)}{5} = -\frac{13\pi}{15} > -\pi
\]

\( n = -4. \)

\[
x = \frac{4\pi}{15} + \frac{2\pi(-4)}{5} = -\frac{4\pi}{3} < -\pi
\]

\[
x = \frac{\pi}{3} + \frac{2\pi(-4)}{5} = -\frac{19\pi}{15} < -\pi
\]

And we’re now past the left endpoint of the interval. Sometimes, there will be many solutions as there were in this example. Putting all of this together gives the following set of solutions that lie in the given interval.

\[
\frac{4\pi}{15}, \frac{\pi}{3}, \frac{2\pi}{15}, \frac{11\pi}{15}, \frac{16\pi}{15}, \frac{17\pi}{15}, \frac{22\pi}{15}, \frac{23\pi}{15}, \frac{28\pi}{15}, \frac{29\pi}{15}, -\frac{\pi}{15}, -\frac{2\pi}{15}, -\frac{7\pi}{15}, -\frac{8\pi}{15}, -\frac{13\pi}{15}, -\frac{14\pi}{15}
\]

Let’s work another example.

**Example 4** Solve \( \sin(2x) = -\cos(2x) \) on \( \left[ -\frac{3\pi}{2}, \frac{3\pi}{2} \right] \)

**Solution**

This problem is a little different from the previous ones. First, we need to do some rearranging and simplification.

\[
\sin(2x) = -\cos(2x)
\]

\[
\tan(2x) = -1
\]

So, solving \( \sin(2x) = -\cos(2x) \) is the same as solving \( \tan(2x) = -1 \). At some level we didn’t need to do this for this problem as all we’re looking for is angles in which sine and cosine have the same value, but opposite signs. However, for other problems this won’t be the case and we’ll want to convert to tangent.
Looking at our trusty unit circle it appears that the solutions will be,

\[ 2x = \frac{3\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ 2x = \frac{7\pi}{4} + 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots \]

Or, upon dividing by the 2 we get all possible solutions.

\[ x = \frac{3\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots \]

\[ x = \frac{7\pi}{8} + \pi n, \quad n = 0, \pm 1, \pm 2, \ldots \]

Now, let’s determine the solutions that lie in the given interval.

\( n = 0. \)

\[ x = \frac{3\pi}{8} + \pi (0) = \frac{3\pi}{8} < \frac{3\pi}{2} \]

\[ x = \frac{7\pi}{8} + \pi (0) = \frac{7\pi}{8} < \frac{3\pi}{2} \]

\( n = 1. \)

\[ x = \frac{3\pi}{8} + \pi (1) = \frac{11\pi}{8} < \frac{3\pi}{2} \]

\[ x = \frac{7\pi}{8} + \pi (1) = \frac{15\pi}{8} > \frac{3\pi}{2} \]

Unlike the previous example only one of these will be in the interval. This will happen occasionally so don’t always expect both answers from a particular \( n \) to work. Also, we should now check \( n = 2 \) for the first to see if it will be in or out of the interval. I’ll leave it to you to check that it’s out of the interval.

Now, let’s check the negative \( n. \)

\( n = -1. \)

\[ x = \frac{3\pi}{8} + \pi (-1) = -\frac{5\pi}{8} > -\frac{3\pi}{2} \]

\[ x = \frac{7\pi}{8} + \pi (-1) = -\frac{\pi}{8} > -\frac{3\pi}{2} \]

\( n = -2. \)

\[ x = \frac{3\pi}{8} + \pi (-2) = -\frac{13\pi}{8} < -\frac{3\pi}{2} \]

\[ x = \frac{7\pi}{8} + \pi (-2) = -\frac{9\pi}{8} > -\frac{3\pi}{2} \]

Again, only one will work here. I’ll leave it to you to verify that \( n = -3 \) will give two answers.
that are both out of the interval.

The complete list of solutions is then,

\[-\frac{9\pi}{8}, -\frac{5\pi}{8}, -\frac{\pi}{8}, \frac{3\pi}{8}, \frac{7\pi}{8}, \frac{11\pi}{8}\]

Let’s work one more example so that I can make a point that needs to be understood when solving some trig equations.

**Example 5** Solve \( \cos(3x) = 2 \).

**Solution**

This example is designed to remind you of certain properties about sine and cosine. Recall that \(-1 \leq \cos(\theta) \leq 1\) and \(-1 \leq \sin(\theta) \leq 1\). Therefore, since cosine will never be greater that 1 it definitely can’t be 2. So **THERE ARE NO SOLUTIONS** to this equation!

It is important to remember that not all trig equations will have solutions.

In this section we solved some simple trig equations. There are more complicated trig equations that we can solve so don’t leave this section with the feeling that there is nothing harder out there in the world to solve. In fact, we’ll see at least one of the more complicated problems in the next section. Also, every one of these problems came down to solutions involving one of the “common” or “standard” angles. Most trig equations won’t come down to one of those and will in fact need a calculator to solve. The next section is devoted to this kind of problem.