Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.

**Review : Solving Trig Equations**

1. Without using a calculator find all the solutions to \( 4 \sin(3t) = 2 \).

   **Hint 1** : Isolate the sine (with a coefficient of one) on one side of the equation.

   **Step 1**
   Isolating the sine (with a coefficient of one) on one side of the equation gives,
   \[
   \sin(3t) = \frac{1}{2}
   \]

   **Hint 2** : Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which sine will have this value.

   **Step 2**
   Because we’re dealing with sine in this problem and we know that the \(y\)-axis represents sine on a unit circle we’re looking for angles that will have a \(y\) coordinate of \(\frac{1}{2}\). This means we’ll have an angle in the first quadrant and an angle in the second quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.
Clearly the angle in the first quadrant is $\frac{\pi}{6}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{6}$ with the negative $x$-axis as shown above and so it will be: $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

Hint 3: Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “+ $2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.

This then means that we must have,
Finally, to get all the solutions to the equation all we need to do is divide both sides by 3.

\[
t = \frac{\pi}{18} + \frac{2\pi n}{3} \quad \text{OR} \quad t = \frac{5\pi}{18} + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots
\]

2. Without using a calculator find the solution(s) to \(4\sin(3t) = 2\) that are in \(\left[0, \frac{4\pi}{3}\right]\).

Hint 1: First, find all the solutions to the equation without regard to the given interval.

Step 1
Because we found all the solutions to this equation in Problem 1 of this section we’ll just list the result here. For full details on how these solutions were obtained please see the solution to Problem 1.

All solutions to the equation are,

\[
t = \frac{\pi}{18} + \frac{2\pi n}{3} \quad \text{OR} \quad t = \frac{5\pi}{18} + \frac{2\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 2: Now all we need to do is plug in values of \(n\) to determine which solutions will actually fall in this interval.

Step 2
Note that because at least some of the solutions will have a denominator of 18 it will probably be convenient to also have the interval written in terms of fractions with denominators of 18. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[
\left[0, \frac{4\pi}{3}\right] = \left[0, \frac{24\pi}{18}\right]
\]

With the interval written in this form, if our potential solutions have a denominator of 18, all we need to do is compare numerators. As long as the numerators are positive and less than 24\(\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \(n\) it will be much easier to have both fractions in the solutions have denominators of 18. So the solutions, written in this form, are.
Calculus I

\[ t = \frac{\pi}{18} + \frac{12\pi n}{18} \quad \text{OR} \quad t = \frac{5\pi}{18} + \frac{12\pi n}{18} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \( n = 0 \) and see what we get.

\[
\begin{align*}
n = 0: & \quad t = \frac{\pi}{18} \quad \text{OR} \quad t = \frac{5\pi}{18} \\
n = 1: & \quad t = \frac{13\pi}{18} \quad \text{OR} \quad t = \frac{17\pi}{18} \\
n = 2: & \quad t = \frac{25\pi}{18} > \frac{24\pi}{18} \quad \text{OR} \quad t = \frac{29\pi}{18} > \frac{24\pi}{18}
\end{align*}
\]

Note that we didn’t really need to plug in \( n = 2 \) above to see that they would not work. With each increase in \( n \) we were really just adding another \( \frac{12\pi}{18} \) onto the previous results and by a quick inspection we could see that adding \( 12\pi \) to the numerator of either solution from the \( n = 1 \) step would result in a numerator that is larger than \( 24\pi \) and so would result in a solution that is outside of the interval. This is not something that must be noticed in order to work the problem, but noticing this would definitely help reduce the amount of actual work.

So, it looks like we have the four solutions to this equation in the given interval.

\[
\begin{align*}
t & = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{13\pi}{18}, \frac{17\pi}{18}
\end{align*}
\]

3. Without using a calculator find all the solutions to \( 2\cos\left(\frac{x}{3}\right) + \sqrt{2} = 0. \)

Hint 1: Isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos\left(\frac{x}{3}\right) = -\frac{\sqrt{2}}{2}
\]
Hint 2 : Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the \(x\)-axis represents cosine on a unit circle we’re looking for angles that will have a \(x\) coordinate of \(-\frac{\sqrt{2}}{2}\). This means that we’ll have angles in the second and third quadrant.

Because of the negative value we can’t just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.

If we didn’t have the negative value then the angle would be \(\frac{\pi}{4}\). Now, based on the symmetry in the unit circle, the terminal line for both of the angles will form an angle of \(\frac{\pi}{4}\) with the negative
x-axis. The angle in the second quadrant will then be \( \pi - \frac{\pi}{4} = \frac{3\pi}{4} \) and the angle in the third quadrant will be \( \pi + \frac{\pi}{4} = \frac{5\pi}{4} \).

Hint 3 : Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “+2\(\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\)” onto each of these.

This then means that we must have,

\[
\frac{x}{3} = \frac{3\pi}{4} + 2\pi n \quad \text{OR} \quad \frac{x}{3} = \frac{5\pi}{4} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 3.

\[
x = \frac{9\pi}{4} + 6\pi n \quad \text{OR} \quad x = \frac{15\pi}{4} + 6\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

4. Without using a calculator find the solution(s) to \(2\cos \left(\frac{x}{3}\right) + \sqrt{2} = 0\) that are in \([-\pi, \pi]\).

Hint 1 : First, find all the solutions to the equation without regard to the given interval.

Step 1
Because we found all the solutions to this equation in Problem 3 of this section we’ll just list the result here. For full details on how these solutions were obtained please see the solution to Problem 3.

All solutions to the equation are,

\[
x = \frac{9\pi}{4} + 6\pi n \quad \text{OR} \quad x = \frac{15\pi}{4} + 6\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 2 : Now all we need to do is plug in values of \(n\) to determine which solutions will actually fall in this interval.

Step 2
Note that because at least some of the solutions will have a denominator of 4 it will probably be convenient to also have the interval written in terms of fractions with denominators of 4. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[-7\pi, 7\pi\] = \left[\frac{-28\pi}{4}, \frac{28\pi}{4}\right]

With the interval written in this form, if our potential solutions have a denominator of 4, all we need to do is compare numerators. As long as the numerators are between \(-28\pi\) and \(28\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \(n\) it will be much easier to have both fractions in the solutions have denominators of 4. So the solutions, written in this form, are.

\[x = \frac{9\pi}{4} + \frac{24\pi n}{4}\quad \text{OR} \quad x = \frac{15\pi}{4} + \frac{24\pi n}{4}\quad n = 0, \pm 1, \pm 2, \ldots\]

Now let’s find all the solutions.

\(n = -2:\quad x = -\frac{39\pi}{4} < -\frac{28\pi}{4}\quad \text{OR} \quad x = -\frac{33\pi}{4} < -\frac{28\pi}{4}\)

\(n = -1:\quad x = -\frac{15\pi}{4}\quad \text{OR} \quad x = -\frac{9\pi}{4}\)

\(n = 0:\quad x = \frac{9\pi}{4}\quad \text{OR} \quad x = \frac{15\pi}{4}\)

\(n = 1:\quad x = \frac{33\pi}{4} > \frac{28\pi}{4}\quad \text{OR} \quad x = \frac{39\pi}{4} > \frac{28\pi}{4}\)

Note that we didn’t really need to plug in \(n = 1\) or \(n = -2\) above to see that they would not work. With each increase in \(n\) we were really just adding (for positive \(n\)) or subtracting (for negative \(n\)) another \(\frac{24\pi}{4}\) from the previous results. By a quick inspection we could see that adding \(24\pi\) to the numerator of either solution from the \(n = 1\) step would result in a numerator that is larger than \(28\pi\) and so would result in a solution that is outside of the interval. Likewise, for the \(n = -2\) case, subtracting \(24\pi\) from each of the numerators will result in numerators that will be less than \(-28\pi\) and so will not be in the interval. This is not something that must be noticed in order to work the problem, but noticing this would definitely help reduce the amount of actual work.

So, it looks like we have the four solutions to this equation in the given interval.

\[x = -\frac{15\pi}{4}, -\frac{9\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}\]
5. Without using a calculator find the solution(s) to \( 4 \cos(6z) = \sqrt{12} \) that are in \( \left[ 0, \frac{\pi}{2} \right] \).

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos(6z) = \frac{\sqrt{12}}{4} = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}
\]

Notice that we needed to do a little simplification of the root to get the value into a more recognizable form. This kind of simplification is usually a good thing to do.

Hint 2: Use your knowledge of the unit circle to determine all the angles in the range \( \left[ 0, 2\pi \right] \) for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the x-axis represents cosine on a unit circle we’re looking for angles that will have a x coordinate of \( \frac{\sqrt{3}}{2} \). This means we’ll have an angle in the first quadrant and an angle in the fourth quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.
Clearly the angle in the first quadrant is $\frac{\pi}{6}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{6}$ with the positive $x$-axis as shown above and so it will be: $2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$.

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{6}$ for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.
Hint 3 : Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “ + 2πn” for \( n = 0, \pm 1, \pm 2, \ldots \)” onto each of these.

This then means that we must have,

\[
6z = \frac{\pi}{6} + 2\pi n \quad \text{OR} \quad 6z = \frac{11\pi}{6} + 2\pi n \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 6.

\[
z = \frac{\pi}{36} + \frac{\pi n}{3} \quad \text{OR} \quad z = \frac{11\pi}{36} + \frac{\pi n}{3} \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4 : Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 36 it will probably be convenient to also have the interval written in terms of fractions with denominators of 36. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[
\left[ 0, \frac{\pi}{2} \right] = \left[ 0, \frac{18\pi}{36} \right]
\]

With the interval written in this form, if our potential solutions have a denominator of 36, all we need to do is compare numerators. As long as the numerators are positive and less than \( \frac{18\pi}{36} \) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 36. So the solutions, written in this form, are.

\[
z = \frac{\pi}{36} + \frac{12\pi n}{36} \quad \text{OR} \quad z = \frac{11\pi}{36} + \frac{12\pi n}{36} \quad \text{for} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \( n = 0 \) and see what we get.
There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of $n$ will not necessarily be in the given interval. It is completely possible, as this problem shows, that we will only get one or the other solution from a given value of $n$ to fall in the given interval.

Next notice that with each increase in $n$ we were really just adding another $\frac{12\pi}{36}$ onto the previous results and by a quick inspection we could see that adding $12\pi$ to the numerator of the first solution from the $n = 1$ step would result in a numerator that is larger than $\frac{18\pi}{36}$ and so would result in a solution that is outside of the interval. Therefore, there was no reason to plug in $\frac{2}{3}$ into the first set of solutions. Of course, we also didn’t plug $n = 2$ into the second set because once we’ve gotten out of the interval adding anything else on will remain out of the interval.

So, it looks like we have the three solutions to this equation in the given interval.

\[
\begin{align*}
\frac{\pi}{36}, \frac{11\pi}{36}, \frac{13\pi}{36}
\end{align*}
\]

6. Without using a calculator find the solution(s) to $2 \sin \left(\frac{3y}{2}\right) + \sqrt{3} = 0$ that are in $\left[ -\frac{7\pi}{3}, 0 \right]$. 

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the sine (with a coefficient of one) on one side of the equation gives,

\[
\sin \left(\frac{3y}{2}\right) = -\frac{\sqrt{3}}{2}
\]

Hint 2: Use your knowledge of the unit circle to determine all the angles in the range $[0, 2\pi]$ for which cosine will have this value.

Step 2
Because we’re dealing with sine in this problem and we know that the \( y \)-axis represents sine on a unit circle we’re looking for angles that will have a \( y \) coordinate of \(-\frac{\sqrt{3}}{2}\). This means that we’ll have angles in the third and fourth quadrant.

Because of the negative value we can’t just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.

If we didn’t have the negative value then the angle would be \( \frac{\pi}{3} \). Now, based on the symmetry in the unit circle, the terminal line for the first angle will form an angle of \( \frac{\pi}{3} \) with the negative \( x \)-axis and the terminal line for the second angle will form an angle of \( \frac{\pi}{3} \) with the positive \( x \)-axis.
The angle in the third quadrant will then be \( \pi + \frac{\pi}{3} = \frac{4\pi}{3} \) and the angle in the fourth quadrant will be \( 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \).

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used \( -\frac{\pi}{3} \) for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.

Hint 3 : Using the two angles above write down all the angles for which sine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “\( +2\pi n \) for \( n = 0, \pm 1, \pm 2, \ldots \)” onto each of these.

This then means that we must have,

\[ \frac{3y}{2} = \frac{4\pi}{3} + 2\pi n \quad \text{OR} \quad \frac{3y}{2} = \frac{5\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by \( \frac{2}{3} \).

\[ y = \frac{8\pi}{9} + \frac{4\pi n}{3} \quad \text{OR} \quad y = \frac{10\pi}{9} + \frac{4\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4 : Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 9 it will probably be convenient to also have the interval written in terms of fractions with denominators of 9. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[ \left[ -\frac{7\pi}{3}, 0 \right] = \left[ -\frac{21\pi}{9}, 0 \right] \]

With the interval written in this form, if our potential solutions have a denominator of 9, all we need to do is compare numerators. As long as the numerators are negative and greater than \(-21\pi\) we’ll know that the solution is in the interval.
Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 9. So the solutions, written in this form, are:

\[
y = \frac{8\pi}{9} + \frac{12\pi n}{9} \quad \text{OR} \quad y = \frac{10\pi}{9} + \frac{12\pi n}{9} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Now let’s find all the solutions. First notice that, in this case, if we plug in positive values of \( n \) or zero we will get positive solutions and these will not be in the interval and so there is no reason to even try these. So, let’s start at \( n = -1 \) and see what we get.

\[
n = -1: \quad y = -\frac{4\pi}{9} \quad \text{OR} \quad y = -\frac{2\pi}{9}
\]

\[
n = -2: \quad y = -\frac{16\pi}{9} \quad \text{OR} \quad y = -\frac{14\pi}{9}
\]

Notice that with each increase (in the negative sense anyway) in \( n \) we were really just subtracting another \( \frac{12\pi}{9} \) from the previous results and by a quick inspection we could see that subtracting \( 12\pi \) from either of the numerators from the \( n = -2 \) solutions the numerators will be less than \( -21\pi \) and so will be out of the interval. There is no reason to write down the \( n = -3 \) solutions since we know that they will not be in the given interval.

So, it looks like we have the four solutions to this equation in the given interval.

\[
y = -\frac{16\pi}{9}, -\frac{14\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}
\]

7. Without using a calculator find the solution(s) to \( 8 \tan(2x) - 5 = 3 \) that are in \( [-\frac{\pi}{2}, \frac{3\pi}{2}] \).

Hint 1 : Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.

Step 1
Isolating the tangent (with a coefficient of one) on one side of the equation gives,

\[
\tan(2x) = 1
\]

Hint 2 : Determine all the angles in the range \([0, 2\pi]\) for which tangent will have this value.
Step 2
If tangent has a value of 1 then we know that sine and cosine must be the same. This means that, in the first quadrant, the solution is $\frac{\pi}{4}$. We also know that sine and cosine will be the same in the third quadrant and we can use the basic symmetry on our unit circle to determine this value. Here is a unit circle for this situation.

By basic symmetry we can see that the line terminal line for the second angle must form an angle of $\frac{\pi}{4}$ with the negative $x$-axis as shown above and so it will be: $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$.

Hint 3: Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.
This then means that we must have,
\[ 2x = \frac{\pi}{4} + 2\pi n \quad \text{OR} \quad 2x = \frac{5\pi}{4} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 2.
\[ x = \frac{\pi}{8} + \pi n \quad \text{OR} \quad x = \frac{5\pi}{8} + \pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4 : Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 8 it will probably be convenient to also have the interval written in terms of fractions with denominators of 8. Doing this will make it much easier to identify solutions that fall inside the interval so,
\[ \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] = \left[ -\frac{4\pi}{8}, \frac{12\pi}{8} \right] \]

With the interval written in this form, if our potential solutions have a denominator of 8, all we need to do is compare numerators. As long as the numerators are between \(-4\pi\) and \(12\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 8. So the solutions, written in this form, are.
\[ x = \frac{\pi}{8} + \frac{8\pi n}{8} \quad \text{OR} \quad x = \frac{5\pi}{8} + \frac{8\pi n}{8} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions.
\[ n = -1 : \quad x = \frac{-7\pi}{8} \quad < \frac{4\pi}{8} \quad \text{OR} \quad x = -\frac{3\pi}{8} \]
\[ n = 0 : \quad x = \frac{\pi}{8} \quad \text{OR} \quad x = \frac{5\pi}{8} \]
\[ n = 1 : \quad x = \frac{9\pi}{8} \quad \text{OR} \quad x = \frac{13\pi}{8} > \frac{12\pi}{8} \]

There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of \( n \) will not necessarily be in the given interval. It
Calculus I

is completely possible, as this problem shows, that we will only get one or the other solution from a given value of \( n \) to fall in the given interval.

Next notice that with each increase in \( n \) we were really just adding/subtracting (depending upon the sign of \( n \)) another \( \frac{8\pi}{8} \) from the previous results and by a quick inspection we could see that adding \( 8\pi \) to the numerator of the \( n = 1 \) solutions would result in numerators that are larger than \( 12\pi \) and so would result in solutions that are outside of the interval. Likewise, subtracting \( 8\pi \) from the \( n = -1 \) solutions would result in numerators that are smaller than \( -4\pi \) and so would result in solutions that are outside the interval. Therefore, there is no reason to even go past the values of \( n \) listed here.

So, it looks like we have the four solutions to this equation in the given interval.

\[
x = -\frac{3\pi}{8}, -\frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}
\]

8. Without using a calculator find the solution(s) to \( 16 = -9\sin\left(7x\right) - 4 \) that are in \( \left[-2\pi, \frac{9\pi}{4}\right] \).

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the sine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the sine (with a coefficient of one) on one side of the equation gives,

\[
\sin\left(7x\right) = -\frac{20}{9} < -1
\]

Okay, at this point we can stop all work. We know that \( -1 \leq \sin \theta \leq 1 \) for any argument and so in this case there is no solution. This will happen on occasion and we shouldn’t get to excited about it when it happens.

9. Without using a calculator find the solution(s) to \( \sqrt{3} \tan\left(\frac{t}{4}\right) + 5 = 4 \) that are in \( \left[0, 4\pi\right] \).

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the tangent (with a coefficient of one) on one side of the equation.
Step 1
Isolating the tangent (with a coefficient of one) on one side of the equation gives,

$$\tan \left( \frac{t}{4} \right) = -\frac{1}{\sqrt{3}}$$

Hint 2 : Determine all the angles in the range \([0, 2\pi]\) for which tangent will have this value.

Step 2
To get the first angle here let’s recall the definition of tangent in terms of sine and cosine.

$$\tan \left( \frac{t}{4} \right) = \frac{\sin \left( \frac{t}{4} \right)}{\cos \left( \frac{t}{4} \right)} = -\frac{1}{\sqrt{3}}$$

Now, because of the section we’re in, we know that the angle must be one of the “standard” angles and from a quick look at a unit circle (shown below) we know that for \(\frac{\pi}{6}\) we will have,

$$\frac{\sin \left( \frac{\pi}{6} \right)}{\cos \left( \frac{\pi}{6} \right)} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

So, if we had a positive value on the tangent we’d have the first angle. We do have a negative value however, but this work will allow us to get the two angles we’re after. Because the value is negative this simply means that the sine and cosine must have the same values that they have for \(\frac{\pi}{6}\) except that one must be positive and the other must be negative. This means that the angles that we’re after must be in the second and fourth quadrants. Here is a unit circle for this situation.
By basic symmetry we can see that the terminal line for the angle in the second quadrant must form an angle of \( \frac{\pi}{6} \) with the negative \( x \)-axis and the terminal line in the fourth quadrant must form an angle of \( \frac{\pi}{6} \) with the positive \( x \)-axis as shown above. The angle in the second quadrant will then be \( \pi - \frac{\pi}{6} = \frac{5\pi}{6} \) while the angle in the fourth quadrant will be \( 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \).

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used \( -\frac{\pi}{6} \) for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.
Hint 3: Using the two angles above write down all the angles for which tangent will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding \( +2\pi n \) for \( n = 0, \pm 1, \pm 2, \ldots \) onto each of these.

This then means that we must have,

\[
\frac{t}{4} = \frac{5\pi}{6} + 2\pi n \quad \text{OR} \quad \frac{t}{4} = \frac{11\pi}{6} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 4.

\[
t = \frac{10\pi}{3} + 8\pi n \quad \text{OR} \quad t = \frac{22\pi}{3} + 8\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4: Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 3 it will probably be convenient to also have the interval written in terms of fractions with denominators of 3. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[
[0, 4\pi] = \left[0, \frac{12\pi}{3}\right]
\]

With the interval written in this form, if our potential solutions have a denominator of 3, all we need to do is compare numerators. As long as the numerators are positive and less than \( 12\pi \) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 3. So the solutions, written in this form, are.

\[
t = \frac{10\pi}{3} + \frac{24\pi n}{3} \quad \text{OR} \quad t = \frac{22\pi}{3} + \frac{24\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. Next, notice that for any positive \( n \) we will be adding \( \frac{24\pi}{3} \) onto a positive quantity and so are guaranteed to be greater than \( \frac{12\pi}{3} \) and so will out of the given interval. This
leaves $n = 0$ and for this one we can notice that the only solution that will fall in the given interval is then,

$$\frac{10\pi}{3}$$

Before leaving this problem let’s note that on occasion we will only get a single solution out of all the possible solutions that will fall in the given interval. So, don’t get excited about it if this should happen.

10. Without using a calculator find the solution(s) to $3 \csc(9z) - 7 = -5$ that are in $\left[ -\frac{\pi}{3}, \frac{4\pi}{9} \right]$.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosecant (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosecant (with a coefficient of one) on one side of the equation gives,

$$csc(9z) = \frac{2}{\sqrt{3}}$$

Hint 2: We need to determine all the angles in the range $\left[ 0, 2\pi \right]$ for which cosecant will have this value. The best way to do this is to rewrite this equation into one in terms of a different trig function that we can more easily deal with.

Step 2
The best way to do this is to recall the definition of cosecant in terms of sine and rewrite the equation in terms sine instead as that will be easier to deal with. Doing this gives,

$$csc(9z) = \frac{1}{\sin(9z)} = \frac{2}{\sqrt{3}} \quad \Rightarrow \quad \sin(9z) = \frac{\sqrt{3}}{2}$$

The solution(s) to the equation with sine in it are the same as the solution(s) to the equation with cosecant in it and so let’s work with that instead.

At this point we are now dealing with sine and we know that the $y$-axis represents sine on a unit circle. So we’re looking for angles that will have a $y$ coordinate of $\frac{\sqrt{3}}{2}$. This means we’ll have
an angle in the first quadrant and an angle in the second quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation

Clearly the angle in the first quadrant is $\frac{\pi}{3}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{3}$ with the negative $x$-axis as shown above and so it will be: $\pi - \frac{\pi}{3} = \frac{2\pi}{3}$.

Hint 3: Using the two angles above write down all the angles for which sine/cosecant will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “$+ 2\pi n$ for $n = 0, \pm 1, \pm 2, \ldots$” onto each of these.
This then means that we must have,
\[ 9z = \frac{\pi}{3} + 2\pi n \quad \text{OR} \quad 9z = \frac{2\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots \]

Finally, to get all the solutions to the equation all we need to do is divide both sides by 9.
\[ z = \frac{\pi}{27} + \frac{2\pi n}{9} \quad \text{OR} \quad z = \frac{2\pi}{27} + \frac{2\pi n}{9} \quad n = 0, \pm 1, \pm 2, \ldots \]

Hint 4: Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 27 it will probably be convenient to also have the interval written in terms of fractions with denominators of 27. Doing this will make it much easier to identify solutions that fall inside the interval so,
\[ \left[ -\frac{\pi}{3}, \frac{4\pi}{9} \right] = \left[ -\frac{9\pi}{27}, \frac{12\pi}{27} \right] \]

With the interval written in this form, if our potential solutions have a denominator of 27, all we need to do is compare numerators. As long as the numerators are between \(-9\pi\) and \(12\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 27. So the solutions, written in this form, are.
\[ z = \frac{\pi}{27} + \frac{6\pi n}{27} \quad \text{OR} \quad z = \frac{2\pi}{27} + \frac{6\pi n}{27} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions.
\[ n = -1: \quad z = -\frac{5\pi}{27} \quad \text{OR} \quad z = -\frac{4\pi}{27} \]
\[ n = 0: \quad z = \frac{\pi}{27} \quad \text{OR} \quad z = \frac{2\pi}{27} \]
\[ n = 1: \quad z = \frac{7\pi}{27} \quad \text{OR} \quad z = \frac{8\pi}{27} \]

Notice that with each increase in \( n \) we were really just adding/subtracting (depending upon the sign of \( n \)) another \( \frac{6\pi}{27} \) from the previous results and by a quick inspection we could see that adding \( 6\pi \) to the numerator of the \( n = 1 \) solutions would result in numerators that are larger than \( 12\pi \) and so would result in solutions that are outside of the interval. Likewise, subtracting \( 6\pi \) from the \( n = -1 \) solutions would result in numerators that are smaller than \(-9\pi\) and so would
result in solutions that are outside the interval. Therefore, there is no reason to even go past the values of \( n \) listed here.

So, it looks like we have the six solutions to this equation in the given interval.

\[
x = -\frac{5\pi}{27}, -\frac{4\pi}{27}, -\frac{\pi}{27}, \frac{2\pi}{27}, \frac{7\pi}{27}, \frac{8\pi}{27}
\]

11. Without using a calculator find the solution(s) to \( 1 - 14 \cos\left(\frac{2x}{5}\right) = -6 \) that are in \( \left[\frac{5\pi}{3}, \frac{40\pi}{3}\right] \).

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

\[
\cos\left(\frac{2x}{5}\right) = \frac{1}{2}
\]

Hint 2: Use your knowledge of the unit circle to determine all the angles in the range \( [0, 2\pi] \) for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the \( x \)-axis represents cosine on a unit circle we’re looking for angles that will have a \( x \) coordinate of \( \frac{1}{2} \). This means we’ll have an angle in the first quadrant and an angle in the fourth quadrant (that we can use the angle in the first quadrant to find). Here is a unit circle for this situation.
Clearly the angle in the first quadrant is $\frac{\pi}{3}$ and by some basic symmetry we can see that the terminal line for the second angle must form an angle of $\frac{\pi}{3}$ with the positive $x$-axis as shown above and so it will be: $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$.

Note that you don’t really need a positive angle for the second one. If you wanted to you could just have easily used $-\frac{\pi}{3}$ for the second angle. There is nothing wrong with this and you’ll get the same solutions in the end. The reason we chose to go with the positive angle is simply to avoid inadvertently losing the minus sign on the second solution at some point in the future. That kind of mistake is easy to make on occasion and by using positive angles here we will not need to worry about making it.
Hint 3: Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

**Step 3**
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “\(+ 2\pi n\) for \(n = 0, \pm 1, \pm 2, \ldots\)” onto each of these.

This then means that we must have,

\[
\frac{2x}{5} = \frac{\pi}{3} + 2\pi n \quad \text{OR} \quad \frac{2x}{5} = \frac{5\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by \(\frac{5}{2}\).

\[
x = \frac{5\pi}{6} + 5\pi n \quad \text{OR} \quad x = \frac{25\pi}{6} + 5\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

**Hint 4**: Now all we need to do is plug in values of \(n\) to determine which solutions will actually fall in the given interval.

**Step 4**
Note that because at least some of the solutions will have a denominator of 6 it will probably be convenient to also have the interval written in terms of fractions with denominators of 6. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[
\left[\frac{5\pi}{3}, \frac{40\pi}{3}\right] = \left[\frac{30\pi}{6}, \frac{80\pi}{6}\right]
\]

With the interval written in this form, if our potential solutions have a denominator of 6, all we need to do is compare numerators. As long as the numerators are between \(30\pi\) and \(80\pi\) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \(n\) it will be much easier to have both fractions in the solutions have denominators of 6. So the solutions, written in this form, are.

\[
x = \frac{5\pi}{6} + \frac{30\pi n}{6} \quad \text{OR} \quad x = \frac{25\pi}{6} + \frac{30\pi n}{6} \quad n = 0, \pm 1, \pm 2, \ldots
\]

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \(n\) we will get negative solutions and these will not be in the interval and so there is no reason to even try these. We can also see from a quick inspection that \(n = 0\) will result in solutions that are not in the interval and so let’s start at \(n = 1\) and see what we get.
There are a couple of things we should note before proceeding. First, it is important to understand both solutions from a given value of $n$ will not necessarily be in the given interval. It is completely possible, as this problem shows, that we will only get one or the other solution from a given value of $n$ to fall in the given interval.

Next notice that with each increase in $n$ we were really just adding another $\frac{30\pi}{6}$ onto the previous results and by a quick inspection we could see that adding $30\pi$ to the numerator of the first solution from the $n = 2$ step would result in a numerator that is larger than $80\pi$ and so would result in a solution that is outside of the interval. Therefore, there was no reason to plug in $n = 3$ into the first set of solutions. Of course, we also didn’t plug $n = 3$ into the second set because once we’ve gotten out of the interval adding anything else on will remain out of the interval.

Finally, unlike most of the problems in this section $n = 0$ did not produce any solutions that were in the given interval. This will happen on occasion so don’t get excited about it when it happens.

So, it looks like we have the three solutions to this equation in the given interval.

$$z = \frac{35\pi}{6}, \frac{55\pi}{6}, \frac{65\pi}{6}$$

12. Without using a calculator find the solution(s) to $15 = 17 + 4\cos\left(\frac{y}{7}\right)$ that are in $[10\pi, 15\pi]$.

Hint 1: Find all the solutions to the equation without regard to the given interval. The first step in this process is to isolate the cosine (with a coefficient of one) on one side of the equation.

Step 1
Isolating the cosine (with a coefficient of one) on one side of the equation gives,

$$\cos\left(\frac{y}{7}\right) = -\frac{1}{2}$$
Hint 2: Use your knowledge of the unit circle to determine all the angles in the range \([0, 2\pi]\) for which cosine will have this value.

Step 2
Because we’re dealing with cosine in this problem and we know that the \(x\)-axis represents cosine on a unit circle we’re looking for angles that will have an \(x\) coordinate of \(-\frac{1}{2}\). This means that we’ll have angles in the second and third quadrant.

Because of the negative value we can’t just find the corresponding angle in the first quadrant and use that to find the second angle. However, we can still use the angles in the first quadrant to find the two angles that we need. Here is a unit circle for this situation.
Calculus I

If we didn’t have the negative value then the angle would be \( \frac{\pi}{3} \). Now, based on the symmetry in the unit circle, the terminal line for both of the angles will form an angle of \( \frac{\pi}{3} \) with the negative \( x \)-axis. The angle in the second quadrant will then be \( \pi - \frac{\pi}{3} = \frac{2\pi}{3} \) and the angle in the third quadrant will be \( \pi + \frac{\pi}{3} = \frac{4\pi}{3} \).

Hint 3: Using the two angles above write down all the angles for which cosine will have this value and use these to write down all the solutions to the equation.

Step 3
From the discussion in the notes for this section we know that once we have these two angles we can get all possible angles by simply adding “+ \( 2\pi n \) for \( n = 0, \pm 1, \pm 2, \ldots \)” onto each of these.

This then means that we must have,

\[
\frac{y}{7} = \frac{2\pi}{3} + 2\pi n \quad \text{OR} \quad \frac{y}{7} = \frac{4\pi}{3} + 2\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Finally, to get all the solutions to the equation all we need to do is multiply both sides by 7.

\[
y = \frac{14\pi}{3} + 14\pi n \quad \text{OR} \quad y = \frac{28\pi}{3} + 14\pi n \quad n = 0, \pm 1, \pm 2, \ldots
\]

Hint 4: Now all we need to do is plug in values of \( n \) to determine which solutions will actually fall in the given interval.

Step 4
Note that because at least some of the solutions will have a denominator of 3 it will probably be convenient to also have the interval written in terms of fractions with denominators of 3. Doing this will make it much easier to identify solutions that fall inside the interval so,

\[
\left[ \frac{30\pi}{3}, \frac{45\pi}{3} \right]
\]

With the interval written in this form, if our potential solutions have a denominator of 3, all we need to do is compare numerators. As long as the numerators are between \( 30\pi \) and \( 45\pi \) we’ll know that the solution is in the interval.

Also, in order to quickly determine the solution for particular values of \( n \) it will be much easier to have both fractions in the solutions have denominators of 3. So the solutions, written in this form, are.
Consider the equations:

\[ y = \frac{14\pi}{3} + \frac{42\pi n}{3} \quad \text{OR} \quad y = \frac{28\pi}{3} + \frac{42\pi n}{3} \quad n = 0, \pm 1, \pm 2, \ldots \]

Now let’s find all the solutions. First notice that, in this case, if we plug in negative values of \( n \), we will get negative solutions and these will not be in the interval and so there is no reason to even try these. We can also see from a quick inspection that \( n = 0 \) will result in solutions that are not in the interval and so let’s start at \( n = 1 \) and see what we get.

For \( n = 1 \):

\[ x = \frac{56\pi}{3} > \frac{45\pi}{3} \quad \text{OR} \quad x = \frac{80\pi}{3} > \frac{45\pi}{3} \]

So, by plugging in \( n = 1 \) we get solutions that are already outside of the interval and increasing \( n \) will simply mean adding another \( \frac{42\pi}{3} \) onto these and so will remain outside of the given interval. We also noticed earlier than all other value of \( n \) will result in solutions outside of the given interval.

What all this means is that while there are solutions to the equation none fall inside the given interval and so the official answer would then be **no solutions in the given interval**.