Preface

Here are my online notes for my Calculus II course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus II or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and basic integration and integration by substitution.

Calculus II tends to be a very difficult course for many students. There are many reasons for this.

The first reason is that this course does require that you have a very good working knowledge of Calculus I. The Calculus I portion of many of the problems tends to be skipped and left to the student to verify or fill in the details. If you don’t have good Calculus I skills, and you are constantly getting stuck on the Calculus I portion of the problem, you will find this course very difficult to complete.

The second, and probably larger, reason many students have difficulty with Calculus II is that you will be asked to truly think in this class. That is not meant to insult anyone; it is simply an acknowledgment that you can’t just memorize a bunch of formulas and expect to pass the course as you can do in many math classes. There are formulas in this class that you will need to know, but they tend to be fairly general. You will need to understand them, how they work, and more importantly whether they can be used or not. As an example, the first topic we will look at is Integration by Parts. The integration by parts formula is very easy to remember. However, just because you’ve got it memorized doesn’t mean that you can use it. You’ll need to be able to look at an integral and realize that integration by parts can be used (which isn’t always obvious) and then decide which portions of the integral correspond to the parts in the formula (again, not always obvious).

Finally, many of the problems in this course will have multiple solution techniques and so you’ll need to be able to identify all the possible techniques and then decide which will be the easiest technique to use.

So, with all that out of the way let me also get a couple of warnings out of the way to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus II many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often
Calculus II

don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Spherical Coordinates

In this section we will introduce spherical coordinates. Spherical coordinates can take a little getting used to. It’s probably easiest to start things off with a sketch.

Spherical coordinates consist of the following three quantities.

First there is \( r \). This is the distance from the origin to the point and we will require \( 0 \leq r \leq 3 \).

Next there is \( \theta \). This is the same angle that we saw in polar/cylindrical coordinates. It is the angle between the positive \( x \)-axis and the line above denoted by \( r \) (which is also the same \( r \) as in polar/cylindrical coordinates). There are no restrictions on \( \theta \).

Finally there is \( \varphi \). This is the angle between the positive \( z \)-axis and the line from the origin to the point. We will require \( 0 \leq \varphi \leq \pi \).

In summary, \( r \) is the distance from the origin to the point, \( \varphi \) is the angle that we need to rotate down from the positive \( z \)-axis to get to the point and \( \theta \) is how much we need to rotate around the \( z \)-axis to get to the point.

We should first derive some conversion formulas. Let’s first start with a point in spherical coordinates and ask what the cylindrical coordinates of the point are. So, we know \( (\rho, \theta, \varphi) \) and want to find \( (r, \theta, z) \). Of course we really only need to find \( r \) and \( z \) since \( \theta \) is the same in both coordinate systems.

We will be able to do all of our work by looking at the right triangle shown above in our sketch. With a little geometry we see that the angle between \( z \) and \( \rho \) is \( \varphi \) and so we can see that,
and these are exactly the formulas that we were looking for. So, given a point in spherical coordinates the cylindrical coordinates of the point will be,

\[
\begin{align*}
    r &= \rho \sin \varphi \\
    \theta &= \theta \\
    z &= \rho \cos \varphi
\end{align*}
\]

Note as well that,

\[
r^2 + z^2 = \rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi = \rho^2 \left( \cos^2 \varphi + \sin^2 \varphi \right) = \rho^2
\]

Or,

\[
\rho^2 = r^2 + z^2
\]

Next, let’s find the Cartesian coordinates of the same point. To do this we’ll start with the cylindrical conversion formulas from the previous section.

\[
\begin{align*}
    x &= r \cos \theta \\
    y &= r \sin \theta \\
    z &= z
\end{align*}
\]

Now all that we need to do is use the formulas from above for \( r \) and \( z \) to get,

\[
\begin{align*}
    x &= \rho \sin \varphi \cos \theta \\
    y &= \rho \sin \varphi \sin \theta \\
    z &= \rho \cos \varphi
\end{align*}
\]

Also note that since we know that \( r^2 = x^2 + y^2 \) we get,

\[
\rho^2 = x^2 + y^2 + z^2
\]

Converting points from Cartesian or cylindrical coordinates into spherical coordinates is usually done with the same conversion formulas. To see how this is done let’s work an example of each.

**Example 1** Perform each of the following conversions.

(a) Convert the point \( \left( \sqrt{6}, \frac{\pi}{4}, \sqrt{2} \right) \) from cylindrical to spherical coordinates.

[b] [Solution]

(b) Convert the point \( \left( -1, 1, -\sqrt{2} \right) \) from Cartesian to spherical coordinates.

[b] [Solution]
Solution

(a) Convert the point \( \left( \sqrt{6}, \frac{\pi}{4}, \sqrt{2} \right) \) from cylindrical to spherical coordinates.

We’ll start by acknowledging that \( \theta \) is the same in both coordinate systems and so we don’t need to do anything with that.

Next, let’s find \( \rho \).

\[
\rho = \sqrt{r^2 + z^2} = \sqrt{6 + 2} = \sqrt{8} = 2\sqrt{2}
\]

Finally, let’s get \( \varphi \). To do this we can use either the conversion for \( r \) or \( z \). We’ll use the conversion for \( z \).

\[
z = \rho \cos \varphi \quad \Rightarrow \quad \cos \varphi = \frac{z}{\rho} = \frac{\sqrt{2}}{2\sqrt{2}} \quad \Rightarrow \quad \varphi = \cos^{-1}\left( \frac{1}{2} \right) = \frac{\pi}{3}
\]

Notice that there are many possible values of \( \varphi \) that will give \( \cos \varphi = \frac{1}{2} \), however, we have restricted \( \varphi \) to the range \( 0 \leq \varphi \leq \pi \) and so this is the only possible value in that range.

So, the spherical coordinates of this point will are \( \left( 2\sqrt{2}, \frac{\pi}{4}, \frac{\pi}{3} \right) \).

(b) Convert the point \( (-1,1,-\sqrt{2}) \) from Cartesian to spherical coordinates.

The first thing that we’ll do here is find \( \rho \).

\[
\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 2} = 2
\]

Now we’ll need to find \( \varphi \). We can do this using the conversion for \( z \).

\[
z = \rho \cos \varphi \quad \Rightarrow \quad \cos \varphi = \frac{z}{\rho} = \frac{-\sqrt{2}}{2} \quad \Rightarrow \quad \varphi = \cos^{-1}\left( -\frac{\sqrt{2}}{2} \right) = \frac{3\pi}{4}
\]

As with the last parts this will be the only possible \( \varphi \) in the range allowed.

Finally, let’s find \( \theta \). To do this we can use the conversion for \( x \) or \( y \). We will use the conversion for \( y \) in this case.

\[
\sin \theta = \frac{y}{\rho \sin \varphi} = \frac{1}{2 \left( \frac{\sqrt{2}}{2} \right)} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}
\]

Now, we actually have more possible choices for \( \theta \) but all of them will reduce down to one of the two angles above since they will just be one of these two angles with one or more complete rotations around the unit circle added on.

We will however, need to decide which one is the correct angle since only one will be. To do
this let’s notice that, in two dimensions, the point with coordinates \( x = -1 \) and \( y = 1 \) lies in the second quadrant. This means that \( \theta \) must be angle that will put the point into the second quadrant. Therefore, the second angle, \( \theta = \frac{3\pi}{4} \), must be the correct one.

The spherical coordinates of this point are then \( \left( 2, \frac{3\pi}{4}, \frac{3\pi}{4} \right) \).

Now, let’s take a look at some equations and identify the surfaces that they represent.

**Example 2** Identify the surface for each of the following equations.

(a) \( \rho = 5 \) [Solution]

(b) \( \varphi = \frac{\pi}{3} \) [Solution]

(c) \( \theta = \frac{2\pi}{3} \) [Solution]

(d) \( \rho \sin \varphi = 2 \) [Solution]

**Solution**

(a) \( \rho = 5 \)

There are a couple of ways to think about this one.

First, think about what this equation is saying. This equation says that, no matter what \( \theta \) and \( \varphi \) are, the distance from the origin must be 5. So, we can rotate as much as we want away from the \( z \)-axis and around the \( z \)-axis, but we must always remain at a fixed distance from the origin. This is exactly what a sphere is. So, this is a sphere of radius 5 centered at the origin.

The other way to think about it is to just convert to Cartesian coordinates.

\[
\rho = 5 \\
\rho^2 = 25 \\
x^2 + y^2 + z^2 = 25
\]

Sure enough a sphere of radius 5 centered at the origin. [Return to Problems]

(b) \( \varphi = \frac{\pi}{3} \)

In this case there isn’t an easy way to convert to Cartesian coordinates so we’ll just need to think about this one a little. This equation says that no matter how far away from the origin that we move and no matter how much we rotate around the \( z \)-axis the point must always be at an angle of \( \frac{\pi}{3} \) from the \( z \)-axis.

This is exactly what happens in a cone. All of the points on a cone are a fixed angle from the \( z \)-axis along with a fixed distance from the origin. [Return to Problems]
axis. So, we have a cone whose points are all at an angle of $\frac{2\pi}{3}$ from the $z$-axis.

(e) $\theta = \frac{2\pi}{3}$

As with the last part we won’t be able to easily convert to Cartesian coordinates here. In this case no matter how far from the origin we get or how much we rotate down from the positive $z$-axis the points must always form an angle of $\frac{2\pi}{3}$ with the $x$-axis.

Points in a vertical plane will do this. So, we have a vertical plane that forms an angle of $\frac{2\pi}{3}$ with the positive $x$-axis.

(d) $\rho \sin \varphi = 2$

In this case we can convert to Cartesian coordinates so let’s do that. There are actually two ways to do this conversion. We will look at both since both will be used on occasion.

**Solution 1**

In this solution method we will convert directly to Cartesian coordinates. To do this we will first need to square both sides of the equation.

$$\rho^2 \sin^2 \varphi = 4$$

Now, for no apparent reason add $\rho^2 \cos^2 \varphi$ to both sides.

$$\rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi = 4 + \rho^2 \cos^2 \varphi$$

$$\rho^2 \left( \sin^2 \varphi + \cos^2 \varphi \right) = 4 + \rho^2 \cos^2 \varphi$$

$$\rho^2 = 4 + (\rho \cos \varphi)^2$$

Now we can convert to Cartesian coordinates.

$$x^2 + y^2 + z^2 = 4 + z^2$$

$$x^2 + y^2 = 4$$

So, we have a cylinder of radius 2 centered on the $z$-axis.

This solution method wasn’t too bad, but it did require some not so obvious steps to complete.

**Solution 2**

This method is much shorter, but also involves something that you may not see the first time around. In this case instead of going straight to Cartesian coordinates we’ll first convert to cylindrical coordinates.

This won’t always work, but in this case all we need to do is recognize that $r = \rho \sin \varphi$ and we will get something we can recognize. Using this we get,
\[
\rho \sin \varphi = 2 \\
r = 2
\]

At this point we know this is a cylinder (remember that we’re in three dimensions and so this isn’t a circle!). However, let’s go ahead and finish the conversion process out.

\[
r^2 = 4 \\
x^2 + y^2 = 4
\]

So, as we saw in the last part of the previous example it will sometimes be easier to convert equations in spherical coordinates into cylindrical coordinates before converting into Cartesian coordinates. This won’t always be easier, but it can make some of the conversions quicker and easier.

The last thing that we want to do in this section is generalize the first three parts of the previous example.

- \(\rho = a\) sphere of radius \(a\) centered at the origin
- \(\varphi = \alpha\) cone that makes an angle of \(\alpha\) with the positive \(z\)–axis
- \(\theta = \beta\) vertical plane that makes an angle of \(\beta\) with the positive \(x\)–axis