Here are my online notes for my Algebra course that I teach here at Lamar University, although I have to admit that it’s been years since I last taught this course. At this point in my career I mostly teach Calculus and Differential Equations.

Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Algebra or needing a refresher for algebra. I’ve tried to make the notes as self contained as possible and do not reference any book. However, they do assume that you’ve had some exposure to the basics of algebra at some point prior to this. While there is some review of exponents, factoring and graphing it is assumed that not a lot of review will be needed to remind you how these topics work.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Equations with Radicals

The title of this section is maybe a little misleading. The title seems to imply that we’re going to look at equations that involve any radicals. However, we are going to restrict ourselves to equations involving square roots. The techniques we are going to apply here can be used to solve equations with other radicals, however the work is usually significantly messier than when dealing with square roots. Therefore, we will work only with square roots in this section.

Before proceeding it should be mentioned as well that in some Algebra textbooks you will find this section in with the equations reducible to quadratic form material. The reason is that we will in fact end up solving a quadratic equation in most cases. However, the approach is significantly different and so we’re going to separate the two topics into different sections in this course.

It is usually best to see how these work with an example.

**Example 1** Solve $x = \sqrt{x+6}$.

**Solution**
In this equation the basic problem is the square root. If that weren’t there we could do the problem. The whole process that we’re going to go through here is set up to eliminate the square root. However, as we will see, the steps that we’re going to take can actually cause problems for us. So, let’s see how this all works.

Let’s notice that if we just square both sides we can make the square root go away. Let’s do that and see what happens.

\[
(x)^2 = (\sqrt{x+6})^2
\]

\[
x^2 = x + 6
\]

\[
x^2 - x - 6 = 0
\]

\[
(x-3)(x+2) = 0 \quad \Rightarrow \quad x = 3, \ x = -2
\]

Upon squaring both sides we see that we get a factorable quadratic equation that gives us two solutions $x = 3$ and $x = -2$.

Now, for no apparent reason, let’s do something that we haven’t actually done since the section on solving linear equations. Let’s check our answers. Remember as well that we need to check the answers in the original equation! That is very important.

Let’s first check $x = 3$

\[
3 = \sqrt{3+6}
\]

\[
3 = \sqrt{9} \quad \text{OK}
\]

So $x = 3$ is a solution. Now let’s check $x = -2$. 

We have a problem. Recall that square roots are ALWAYS positive and so $x = -2$ does not work in the original equation. One possibility here is that we made a mistake somewhere. We can go back and look however and we’ll quickly see that we haven’t made a mistake.

So, what is the deal? Remember that our first step in the solution process was to square both sides. Notice that if we plug $x = -2$ into the quadratic we solved it would in fact be a solution to that. When we squared both sides of the equation we actually changed the equation and in the process introduced a solution that is not a solution to the original equation.

With these problems it is vitally important that you check your solutions as this will often happen. When this does we only take the values that are actual solutions to the original equation.

So, the original equation had a single solution $x = 3$.

Now, as this example has shown us, we have to be very careful in solving these equations. When we solve the quadratic we will get two solutions and it is possible both of these, one of these, or none of these values to be solutions to the original equation. The only way to know is to check your solutions!

Let’s work a couple more examples that are a little more difficult.

**Example 2** Solve each of the following equations.

(a) $y + \sqrt{y - 4} = 4$  \[Solution\]

(b) $1 = t + \sqrt{2t - 3}$  \[Solution\]

(c) $\sqrt{5z + 6} - 2 = z$  \[Solution\]

**Solution**

(a) $y + \sqrt{y - 4} = 4$

In this case let’s notice that if we just square both sides we’re going to have problems.

$\left(y + \sqrt{y - 4}\right)^2 = (4)^2$

$y^2 + 2y\sqrt{y - 4} + y - 4 = 16$

Before discussing the problem we’ve got here let’s make sure you can do the squaring that we did above since it will show up on occasion. All that we did here was use the formula

$\left(a + b\right)^2 = a^2 + 2ab + b^2$

with $a = y$ and $b = \sqrt{y - 4}$. You will need to be able to do these because while this may not have worked here we will need to this kind of work in the next set of problems.

Now, just what is the problem with this? Well recall that the point behind squaring both sides in the first problem was to eliminate the square root. We haven’t done that. There is still a square root in the problem and we’ve made the remainder of the problem messier as well.
So, what we’re going to need to do here is make sure that we’ve got a square root all by itself on one side of the equation before squaring. Once that is done we can square both sides and the square root really will disappear.

Here is the correct way to do this problem.

\[
\sqrt{y-4} = 4 - y \\
(\sqrt{y-4})^2 = (4 - y)^2 \\
y - 4 = 16 - 8y + y^2 \\
0 = y^2 - 9y + 20 \\
0 = (y-5)(y-4) \quad \Rightarrow \quad y = 4, \quad y = 5
\]

As with the first example we will need to make sure and check both of these solutions. Again, make sure that you check in the original equation. Once we’ve square both sides we’ve changed the problem and so checking there won’t do us any good. In fact checking there could well lead us into trouble.

First \( y = 4 \).

\[
4 + \sqrt{4-4} = 4 \\
4 = 4 \quad \text{OK}
\]

So, that is a solution. Now \( y = 5 \).

\[
5 + \sqrt{5-4} = 4 \\
5 + \sqrt{1} = 4 \\
6 \neq 4 \quad \text{NOT OK}
\]

So, as with the first example we worked there is in fact a single solution to the original equation, \( y = 4 \).

(b) \( 1 = t + \sqrt{2t - 3} \)

Okay, so we will again need to get the square root on one side by itself before squaring both sides.

\[
1 - t = \sqrt{2t - 3} \\
(1 - t)^2 = (\sqrt{2t - 3})^2 \\
1 - 2t + t^2 = 2t - 3 \\
t^2 - 4t + 4 = 0 \\
(t - 2)^2 = 0 \quad \Rightarrow \quad t = 2
\]

So, we have a double root this time. Let’s check it to see if it really is a solution to the original equation.
So, \( t = 2 \) isn’t a solution to the original equation. Since this was the only possible solution, this means that there are no solutions to the original equation. This doesn’t happen too often, but it does happen so don’t be surprised by it when it does.

\[
\begin{align*}
1^2 &\neq 2 + \sqrt{2(2) - 3} \\
1^2 &\neq 2 + \sqrt{1} \\
1 &\neq 3
\end{align*}
\]

(c) \( \sqrt{5z + 6} - 2 = z \)

This one will work the same as the previous two.

\[
\begin{align*}
\sqrt{5z + 6} &= z + 2 \\
(\sqrt{5z + 6})^2 &= (z + 2)^2 \\
5z + 6 &= z^2 + 4z + 4 \\
0 &= z^2 - z - 2 \\
0 &= (z - 2)(z + 1) \Rightarrow z &= -1, \ z = 2
\end{align*}
\]

Let’s check these possible solutions start with \( z = -1 \).

\[
\begin{align*}
\sqrt{5(-1) + 6} - 2 &= -1 \\
\sqrt{1} - 2 &= -1 \\
-1 &= -1 \quad \text{OK}
\end{align*}
\]

So, that’s was a solution. Now let’s check \( z = 2 \).

\[
\begin{align*}
\sqrt{5(2) + 6} - 2 &= 2 \\
\sqrt{16} - 2 &= 2 \\
4 - 2 &= 2 \quad \text{OK}
\end{align*}
\]

This was also a solution.

So, in this case we’ve now seen an example where both possible solutions are in fact solutions to the original equation as well.

So, as we’ve seen in the previous set of examples once we get our list of possible solutions anywhere from none to all of them can be solutions to the original equation. Always remember to check your answers!

Okay, let’s work one more set of examples that have an added complexity to them. To this point all the equations that we’ve looked at have had a single square root in them. However, there can be more than one square root in these equations. The next set of examples is designed to show us how to deal with these kinds of problems.
Example 3  Solve each of the following equations.

(a)  $\sqrt{2x-1} - \sqrt{x-4} = 2$  [Solution]

(b)  $\sqrt{t + 7} + 2 = \sqrt{3 - t}$  [Solution]

Solution

In both of these there are two square roots in the problem. We will work these in basically the same manner however. The first step is to get one of the square roots by itself on one side of the equation then square both sides. At this point the process is different so we’ll see how to proceed from this point once we reach it in the first example.

(a)  $\sqrt{2x-1} - \sqrt{x-4} = 2$

So, the first thing to do is get one of the square roots by itself. It doesn’t matter which one we get by itself. We’ll end up the same solution(s) in the end.

\[
\sqrt{2x-1} = 2 + \sqrt{x-4}
\]

\[
\left(\sqrt{2x-1}\right)^2 = \left(2 + \sqrt{x-4}\right)^2
\]

\[
x - 1 = 4 + 4\sqrt{x-4} + x - 4
\]

\[
x - 1 = 4\sqrt{x-4} + x
\]

Now, we still have a square root in the problem, but we have managed to eliminate one of them. Not only that, but what we’ve got left here is identical to the examples we worked in the first part of this section. Therefore, we will continue now work this problem as we did in the previous sets of examples.

\[
(x-1)^2 = \left(4\sqrt{x-4}\right)^2
\]

\[
x^2 - 2x + 1 = 16(x-4)
\]

\[
x^2 - 2x + 1 = 16x - 64
\]

\[
x^2 = 18x + 65 = 0
\]

\[
(x-13)(x-5) = 0 \quad \Rightarrow \quad x = 13, \ x = 5
\]

Now, let’s check both possible solutions in the original equation. We’ll start with $x = 13$

\[
\sqrt{2(13)} - 1 - \sqrt{13-4} \neq 2
\]

\[
\sqrt{25} - \sqrt{9} \neq 2
\]

\[
5 - 3 = 2 \quad \text{OK}
\]

So, the one is a solution. Now let’s check $x = 5$.

\[
\sqrt{2(5)} - 1 - \sqrt{5-4} \neq 2
\]

\[
\sqrt{9} - \sqrt{1} \neq 2
\]

\[
3 - 1 = 2 \quad \text{OK}
\]

So, they are both solutions to the original equation.  

[Return to Problems]
In this case we’ve already got a square root on one side by itself so we can go straight to squaring both sides.

\[
\left(\sqrt{t+7} + 2\right)^2 = \left(\sqrt{3-t}\right)^2
\]

\[
t + 7 + 4\sqrt{t+7} + 4 = 3 - t
\]

\[
t + 11 + 4\sqrt{t+7} = 3 - t
\]

Next, get the remaining square root back on one side by itself and square both sides again.

\[
4\sqrt{t+7} = -8 - 2t
\]

\[
\left(4\sqrt{t+7}\right)^2 = (-8 - 2t)^2
\]

\[
16(t + 7) = 64 + 32t + 4t^2
\]

\[
16t + 112 = 64 + 32t + 4t^2
\]

\[
0 = 4t^2 + 16t - 48
\]

\[
0 = 4(t^2 + 4t - 12)
\]

\[
0 = 4(t + 6)(t - 2) \quad \Rightarrow \quad t = -6, \quad t = 2
\]

Now check both possible solutions starting with \( t = 2 \).

\[
\sqrt{2 + 7} + 2 = \sqrt{3 - 2}
\]

\[
\sqrt{9} + 2 = \sqrt{1}
\]

\[
3 + 2 \neq 1 \quad \text{NOT OK}
\]

So, that wasn’t a solution. Now let’s check \( t = -6 \).

\[
\sqrt{-6 + 7} + 2 = \sqrt{3 - (-6)}
\]

\[
\sqrt{1} + 2 = \sqrt{9}
\]

\[
1 + 2 = 3 \quad \text{OK}
\]

It looks like in this case we’ve got a single solution, \( t = -6 \).

So, when there is more than one square root in the problem we are again faced with the task of checking our possible solutions. It is possible that anywhere from none to all of the possible solutions will in fact be solutions and the only way to know for sure is to check them in the original equation.