Preface

Here are my online notes for my Algebra course that I teach here at Lamar University, although I have to admit that it’s been years since I last taught this course. At this point in my career I mostly teach Calculus and Differential Equations.

Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Algebra or needing a refresher for algebra. I’ve tried to make the notes as self contained as possible and do not reference any book. However, they do assume that you’ve had some exposure to the basics of algebra at some point prior to this. While there is some review of exponents, factoring and graphing it is assumed that not a lot of review will be needed to remind you how these topics work.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Rational Inequalities**

In this section we will solve inequalities that involve rational expressions. The process for solving rational inequalities is nearly identical to the process for solving polynomial inequalities with a few minor differences.

Let’s just jump straight into some examples.

**Example 1** Solve \( \frac{x+1}{x-5} \leq 0 \).

**Solution**

Before we get into solving these we need to point out that these DON’T solve in the same way that we’ve solve equations that contained rational expressions. With equations the first thing that we always did was clear out the denominators by multiplying by the least common denominator. That won’t work with these however.

Since we don’t know the value of \( x \) we can’t multiply both sides by anything that contains an \( x \). Recall that if we multiply both sides of an inequality by a negative number we will need to switch the direction of the inequality. However, since we don’t know the value of \( x \) we don’t know if the denominator is positive or negative and so we won’t know if we need to switch the direction of the inequality or not. In fact, to make matters worse, the denominator will be both positive and negative for values of \( x \) in the solution and so that will create real problems.

So, we need to leave the rational expression in the inequality.

Now, the basic process here is the same as with polynomial inequalities. The first step is to get a zero on one side and write the other side as a single rational inequality. This has already been done for us here.

The next step is to factor the numerator and denominator as much as possible. Again, this has already been done for us in this case.

The next step is to determine where both the numerator and the denominator are zero. In this case these values are.

\[
\text{numerator} : x = -1 \quad \quad \text{denominator} : x = 5
\]

Now, we need these numbers for a couple of reasons. First, just like with polynomial inequalities these are the only numbers where the rational expression may change sign. So, we’ll build a number line using these points to define ranges out of which to pick test points just like we did with polynomial inequalities.

There is another reason for needing the value of \( x \) that make the denominator zero however. No matter what else is going on here we do have a rational expression and that means we need to avoid division by zero and so knowing where the denominator is zero will give us the values of \( x \) to avoid for this.

Here is the number line for this inequality.
So, we need regions that make the rational expression negative. That means the middle region. Also, since we’ve got an “or equal to” part in the inequality we also need to include where the inequality is zero, so this means we include $x = -1$. Notice that we will also need to avoid $x = 5$ since that gives division by zero.

The solution for this inequality is, $-1 \leq x < 5$ $[-1, 5)$

**Example 2** Solve $\frac{x^2 + 4x + 3}{x - 1} > 0$.

**Solution**

We’ve got zero on one side so let’s first factor the numerator and determine where the numerator and denominator are both zero.

$$\frac{(x+1)(x+3)}{x-1} > 0$$

numerator: $x = -1, x = -3$ denominator: $x = 1$

Here is the number line for this one.

In the problem we are after values of $x$ that make the inequality strictly positive and so that looks like the second and fourth region and we won’t include any of the endpoints here. The solution is then,

$$-3 < x < -1 \quad \text{and} \quad 1 < x < \infty$$

$$(-3, -1) \quad \text{and} \quad (1, \infty)$$
Example 3  Solve \( \frac{x^2-16}{(x-1)^2} < 0 \).

Solution
There really isn’t too much to this example. We’ll first need to factor the numerator and then determine where the numerator and denominator are zero.

\[
\frac{(x-4)(x+4)}{(x-1)^2} < 0
\]

numerator : \( x = -4, \ x = 4 \)
denominator : \( x = 1 \)

The number line for this problem is,

- \( x = -5 \) \( -9 \cdot (-1) > 0 \)
- \( x = 0 \) \( -4 \cdot 4 < 0 \)
- \( x = 2 \) \( -2 \cdot 6 < 0 \)
- \( x = 5 \) \( 1 \cdot 9 > 0 \)

So, as with the polynomial inequalities we can not just assume that the regions will always alternate in sign. Also, note that while the middle two regions do give negative values in the rational expression we need to avoid \( x = 1 \) to make sure we don’t get division by zero. This means that we will have to write the answer as two inequalities and/or intervals.

\(-4 < x < 1 \) and \( 1 < x < 4 \)

\((-4,1) \) and \( (1,4) \)

Once again, it’s important to note that we really do need to test each region and not just assume that the regions will alternate in sign.

Next we need to take a look at some examples that don’t already have a zero on one side of the inequality.

Example 4  Solve \( \frac{3x+1}{x+4} \geq 1 \).

Solution
The first thing that we need to do here is subtract 1 from both sides and then get everything into a single rational expression.
In this case there is no factoring to do so we can go straight to identifying where the numerator and denominator are zero.

\[
\frac{3x + 1}{x + 4} - 1 \geq 0 \\
\frac{3x + 1}{x + 4} - \frac{x + 4}{x + 4} \geq 0 \\
\frac{3x + 1 - (x + 4)}{x + 4} \geq 0 \\
\frac{2x - 3}{x + 4} \geq 0
\]

In this case there is no factoring to do so we can go straight to identifying where the numerator and denominator are zero.

**numerator :** \( x = \frac{3}{2} \) \[**denominator :** x = -4 \]

Here is the number line for this problem.

Okay, we want values of \( x \) that give positive and/or zero in the rational expression. This looks like the outer two regions as well as \( x = \frac{3}{2} \). As with the first example we will need to avoid \( x = -4 \) since that will give a division by zero error.

The solution for this problem is then,

\[-\infty < x < -4 \quad \text{and} \quad \frac{3}{2} \leq x < \infty \]

\((-\infty, -4) \quad \text{and} \quad \left[ \frac{3}{2}, \infty \right)\]

**Example 5** Solve \( \frac{x - 8}{x} \leq 3 - x \).

**Solution**

So, again, the first thing to do is to get a zero on one side and then get everything into a single rational expression.
\[
\frac{x-8}{x} + x - 3 \leq 0 \\
\frac{x-8}{x} + \frac{x(x-3)}{x} \leq 0 \\
\frac{x-8+x^2-3x}{x} \leq 0 \\
\frac{x^2-2x-8}{x} \leq 0 \\
\frac{(x-4)(x+2)}{x} \leq 0
\]

We also factored the numerator above so we can now determine where the numerator and denominator are zero.

numerator : \(x = -2, \ x = 4\)  
denominator : \(x = 0\)

Here is the number line for this problem.

\[
\begin{align*}
x = -3 & \quad \frac{(1)(-5)}{-1} > 0 \\
(-1)(-7) < 0 & \quad \frac{(-3)(3)}{1} < 0 \\
\frac{7(1)}{5} > 0 & \quad \frac{1}{5} > 0
\end{align*}
\]

The solution for this inequality is then,

\(-\infty < x \leq -2\) and \(0 < x \leq 4\)

\((-\infty, -2]\) and \((0, 4]\)