Here are my online notes for my Algebra course that I teach here at Lamar University, although I have to admit that it’s been years since I last taught this course. At this point in my career I mostly teach Calculus and Differential Equations.

Despite the fact that these are my “class notes” they should be accessible to anyone wanting to learn Algebra or needing a refresher in for algebra. I’ve tried to make the notes as self contained as possible and do not reference any book. However, they do assume that you’ve has some exposure to the basics of algebra at some point prior to this. While there is some review of exponents, factoring and graphing it is assumed that not a lot of review will be needed to remind you how these topics work.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Combining Functions

The topic with functions that we need to deal with is combining functions. For the most part this means performing basic arithmetic (addition, subtraction, multiplication, and division) with functions. There is one new way of combing functions that we’ll need to look at as well.

Let’s start with basic arithmetic of functions. Given two functions \( f(x) \) and \( g(x) \) we have the following notation and operations.

\[
(f + g)(x) = f(x) + g(x) \quad \quad (f - g)(x) = f(x) - g(x)
\]

\[
(fg)(x) = f(x) g(x) \quad \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}
\]

Sometimes we will drop the \((x)\) part and just write the following,

\[
 f + g = f(x) + g(x) \quad \quad f - g = f(x) - g(x)
\]

\[
 fg = f(x) g(x) \quad \quad \frac{f}{g} = \frac{f(x)}{g(x)}
\]

Note as well that we put \(x\)’s in the parenthesis, but we will often put in numbers as well. Let’s take a quick look at an example.

**Example 1** Given \( f(x) = 2 + 3x - x^2 \) and \( g(x) = 2x - 1 \) evaluate each of the following.

(a) \((f + g)(4)\)  \quad \quad [Solution]

(b) \(g - f\)  \quad \quad [Solution]

(c) \((fg)(x)\)  \quad \quad [Solution]

(d) \(\left(\frac{f}{g}\right)(0)\)  \quad \quad [Solution]

**Solution**

By evaluate we mean one of two things depending on what is in the parenthesis. If there is a number in the parenthesis then we want a number. If there is an \(x\) (or no parenthesis, since that implies and \(x\)) then we will perform the operation and simplify as much as possible.

(a) \((f + g)(4)\)

In this case we’ve got a number so we need to do some function evaluation.

\[
(f + g)(4) = f(4) + g(4)
\]

\[
= (2 + 3(4) - 4^2) + (2(4) - 1)
\]

\[
= -2 + 7
\]

\[
= 5
\]

[Return to Problems]
(b) $g - f$
Here we don’t have an $x$ or a number so this implies the same thing as if there were an $x$ in parenthesis. Therefore, we’ll subtract the two functions and simplify. Note as well that this is written in the opposite order from the definitions above, but it works the same way.
\[
g - f = g(x) - f(x)
= 2x - 1 - \left(2 + 3x - x^2\right)
= 2x - 1 - 2 - 3x + x^2
= x^2 - x - 3
\]

(e) $(fg)(x)$
As with the last part this has an $x$ in the parenthesis so we’ll multiply and then simplify.
\[
(fg)(x) = f(x)g(x)
= \left(2 + 3x - x^2\right)(2x - 1)
= 4x + 6x^2 - 2x^3 - 2 - 3x + x^2
= -2x^3 + 7x^2 + x - 2
\]

(d) $\left(\frac{f}{g}\right)(0)$
In this final part we’ve got a number so we’ll once again be doing function evaluation.
\[
\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)}
= \frac{2 + 3(0) - (0)^2}{2(0) - 1}
= \frac{2}{-1}
= -2
\]

Now we need to discuss the new method of combining functions. The new method of combining functions is called **function composition**. Here is the definition.

Given two functions $f(x)$ and $g(x)$ we have the following two definitions.

1. The **composition** of $f(x)$ and $g(x)$ (note the order here) is,
   \[
   (f \circ g)(x) = f\left[g(x)\right]
   \]
2. The **composition** of $g(x)$ and $f(x)$ (again, note the order) is,
   \[
   (g \circ f)(x) = g\left[f(x)\right]
   \]
We need to note a couple of things here about function composition. First this is **NOT** multiplication. Regardless of what the notation may suggest to you this is simply not multiplication.

Second, the order we’ve listed the two functions is very important since more often than not we will get different answers depending on the order we’ve listed them.

Finally, function composition is really nothing more than function evaluation. All we’re really doing is plugging the second function listed into the first function listed. In the definitions we used \[\text{for the function evaluation instead of the standard } (\ ) \text{ to avoid confusion with too many sets of parenthesis, but the evaluation will work the same.}\]

Let’s take a look at a couple of examples.

**Example 2** Given \( f(x) = 2 + 3x - x^2 \) and \( g(x) = 2x - 1 \) evaluate each of the following.

(a) \((fg)(x)\)  [Solution]

(b) \((f \circ g)(x)\)  [Solution]

(c) \((g \circ f)(x)\)  [Solution]

**Solution**

(a) These are the same functions that we used in the first set of examples and we’ve already done this part there so we won’t redo all the work here. It is here only here to prove the point that function composition is **NOT** function multiplication.

Here is the multiplication of these two functions.
\[
(fg)(x) = -2x^3 + 7x^2 + x - 2
\]

(b) Now, for function composition all you need to remember is that we are going to plug the second function listed into the first function listed. If you can remember that you should always be able to write down the basic formula for the composition.

Here is this function composition.
\[
(f \circ g)(x) = f\left[g(x)\right] = f\left[2x-1\right]
\]

Now, notice that since we’ve got a formula for \( g(x) \) we went ahead and plugged that in first. Also, we did this kind of function evaluation in the first section we looked at for functions. At the time it probably didn’t seem all that useful to be looking at that kind of function evaluation, yet here it is.

Let’s finish this problem out.
\[
\begin{align*}
(f \circ g)(x) &= f\left(g(x)\right) \\
&= f(2x - 1) \\
&= 2 + 3(2x - 1) - (2x - 1)^2 \\
&= 2 + 6x - 3 - (4x^2 - 4x + 1) \\
&= -1 + 6x - 4x^2 + 4x - 1 \\
&= -4x^2 + 10x - 2
\end{align*}
\]

Notice that this is very different from the multiplication! Remember that function composition is NOT function multiplication.

(c) We’ll not put in the detail in this part as it works essentially the same as the previous part.

\[
\begin{align*}
(g \circ f)(x) &= g\left(f(x)\right) \\
&= g\left(2 + 3x - x^2\right) \\
&= 2\left(2 + 3x - x^2\right) - 1 \\
&= 4 + 6x - 2x^2 - 1 \\
&= -2x^2 + 6x + 3
\end{align*}
\]

Notice that this is NOT the same answer as that from the second part. In most cases the order in which we do the function composition will give different answers.

The ideas from the previous example are important enough to make again. First, function composition is NOT function multiplication. Second, the order in which we do function composition is important. In most case we will get different answers with a different order. Note however, that there are times when we will get the same answer regardless of the order.

Let’s work a couple more examples.

**Example 3** Given \( f(x) = x^2 - 3 \) and \( h(x) = \sqrt{x+1} \) evaluate each of the following.

(a) \( (f \circ h)(x) \) \hspace{1cm} [Solution]

(b) \( (h \circ f)(x) \) \hspace{1cm} [Solution]

(c) \( (f \circ f)(x) \) \hspace{1cm} [Solution]

(d) \( (h \circ h)(8) \) \hspace{1cm} [Solution]

(e) \( (f \circ h)(4) \) \hspace{1cm} [Solution]

**Solution**

(a) \( (f \circ h)(x) \)

Not much to do here other than run through the formula.
\[(f \circ h)(x) = f[h(x)]\]
\[= f[\sqrt{x+1}]\]
\[= (\sqrt{x+1})^2 - 3\]
\[= x + 1 - 3\]
\[= x - 2\]

(b) \((h \circ f)(x)\)
Again, not much to do here.
\[(h \circ f)(x) = h[f(x)]\]
\[= h[x^2 - 3]\]
\[= \sqrt{x^2 - 3 + 1}\]
\[= \sqrt{x^2 - 2}\]

(c) \((f \circ f)(x)\)
Now in this case do not get excited about the fact that the two functions here are the same. Composition works the same way.
\[(f \circ f)(x) = f[f(x)]\]
\[= f[x^2 - 3]\]
\[= (x^2 - 3)^2 - 3\]
\[= x^4 - 6x^2 + 9 - 3\]
\[= x^4 - 6x^2 + 6\]

(d) \((h \circ h)(8)\)
In this case, unlike all the previous examples, we’ve got a number in the parenthesis instead of an \(x\), but it works in exactly the same manner.
\[(h \circ h)(8) = h[h(8)]\]
\[= h[\sqrt{8+1}]\]
\[= h[\sqrt{9}]\]
\[= h[3]\]
\[= \sqrt{3+1}\]
\[= 2\]
Again, we’ve got a number here. This time there are actually two ways to do this evaluation. The first is to simply use the results from the first part since that is a formula for the general function composition.

If we do the problem that way we get,

\[(f \circ h)(4) = 4 - 2 = 2\]

We could also do the evaluation directly as we did in the previous part. The answers should be the same regardless of how we get them. So, to get another example down of this kind of evaluation let’s also do the evaluation directly.

\[(f \circ h)(4) = f[h(4)]\]
\[= f[\sqrt{4} + 1]\]
\[= f[\sqrt{5}]\]
\[= (\sqrt{5})^2 - 3\]
\[= 5 - 3\]
\[= 2\]

So, sure enough we got the same answer, although it did take more work to get it.

Example 4  Given \(f(x) = 3x - 2\) and \(g(x) = \frac{x}{3} + \frac{2}{3}\) evaluate each of the following.

(a) \((f \circ g)(x)\)  [Solution]

(b) \((g \circ f)(x)\)  [Solution]

Solution
(a) Hopefully, by this point these aren’t too bad.

\[(f \circ g)(x) = f[g(x)]\]
\[= f\left[\frac{x}{3} + \frac{2}{3}\right]\]
\[= 3\left(\frac{x}{3} + \frac{2}{3}\right) - 2\]
\[= x + 2 - 2\]
\[= x\]

Looks like things simplified down considerable here.  [Return to Problems]
(b) All we need to do here is use the formula so let’s do that.

\[
(g \circ f)(x) = g[f(x)] \\
= g[3x - 2] \\
= \frac{1}{3}(3x - 2) + \frac{2}{3} \\
= x - \frac{2}{3} + \frac{2}{3} \\
= x
\]

So, in this case we get the same answer regardless of the order we did the composition in.

So, as we’ve seen from this last example it is possible to get the same answer from both compositions on occasion. In fact when the answer from both composition is \(x\), as it is in this case, we know that these two functions are very special functions. In fact, they are so special that we’re going to devote the whole next section to these kinds of functions. So, let’s move onto the next section.