Here are the solutions to the practice problems for my Calculus I notes. Some solutions will have more or less detail than other solutions. The level of detail in each solution will depend up on several issues. If the section is a review section, this mostly applies to problems in the first chapter, there will probably not be as much detail to the solutions given that the problems really should be review. As the difficulty level of the problems increases less detail will go into the basics of the solution under the assumption that if you’ve reached the level of working the harder problems then you will probably already understand the basics fairly well and won’t need all the explanation.

This document was written with presentation on the web in mind. On the web most solutions are broken down into steps and many of the steps have hints. Each hint on the web is given as a popup however in this document they are listed prior to each step. Also, on the web each step can be viewed individually by clicking on links while in this document they are all showing. Also, there are liable to be some formatting parts in this document intended for help in generating the web pages that haven’t been removed here. These issues may make the solutions a little difficult to follow at times, but they should still be readable.
Parabolas

1. Sketch the graph of the following parabola. The graph should contain the vertex, the $y$-intercept, $x$-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = (x + 4)^2 - 3$$

Step 1
Let’s find the vertex first. Because the parabola is in the form $f(x) = a(x - h)^2 + k$ we know that the vertex is just the point $(h, k)$. Therefore, we can compare our equation to this form and see that the vertex is : $(4, -3)$.

Be careful with minus signs here! For the $h$ the term the general form is $(x - h)^2$ and so we need to write ours as $(x + 4)^2 = (x - (-4))^2$. This in turn means we must have $h = -4$. Likewise, for the $k$ in the general form it is “$+k$” and so to match our equation we need $k = -3$.

Also note that $a = 1 > 0$ for this parabola and so the parabola will open upwards.

Step 2
The $y$-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = 13$ and so for our equation the $y$-intercept is $(0, 13)$.

Step 3
For the $x$-intercepts we just need to solve the equation $f(x) = 0$. So, let’s solve that for our equation.

$$(x + 4)^2 - 3 = 0$$

$$(x + 4)^2 = 3$$

$$x + 4 = \pm\sqrt{3}$$

$$x = -4 \pm \sqrt{3}$$

The two $x$-intercepts for this parabola are then : $(4 - \sqrt{3}, 0) = (-5.7321, 0)$ and $(4 + \sqrt{3}, 0) = (-2.2679, 0)$. Do not get excited about the “messy” values for the intercept coordinates. There is nothing wrong with them, they are just decimals rather than the integers we are used to dealing with.

Step 4
Because we had two $x$-intercepts for this parabola we already have at least one point on either side of the vertex and so we don’t really need to find any more points for our graph.

However, just for the practice let’s find the point corresponding to the $y$-intercept on the other side the vertex.

The $y$-intercept is a distance of 4 to the right of the vertex and so there will be a corresponding point at the same $y$ value to the left and it will be a distance of 4 to the left of the vertex. Therefore the point to the left of the vertex corresponding to the $y$-intercept is $(-8, 13)$.

Step 5
Here is a sketch of the parabola including all the points we found above.

---

2. Sketch the graph of the following parabola. The graph should contain the vertex, the $y$-intercept, $x$-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = 5(x-1)^2 - 20$$

Step 1
Let’s find the vertex first. Because the parabola is in the form $f(x) = a(x-h)^2 + k$ we know that the vertex is just the point $(h, k)$. Therefore, we can compare our equation to this form and see that the vertex is $(1, -20)$.

Be careful with minus signs here! For the $h$ the term the general form is $(x-h)^2$ and so to match our equation we must have $h = 1$. Likewise, for the $k$ in the general form it is “$+k$” and so to match our equation we need $k = -20$. 

Also note that $a = 5 > 0$ for this parabola and so the parabola will open upwards.

Step 2
The $y$-intercept is just the point $(0, f(0))$. A quick function evaluation gives us that $f(0) = -15$ and so for our equation the $y$-intercept is $(0, -15)$.

Step 3
For the $x$-intercepts we just need to solve the equation $f(x) = 0$. So, let’s solve that for our equation.

$$5(x - 1)^2 - 20 = 0$$
$$5(x - 1)^2 = 20$$
$$(x - 1)^2 = 4$$
$$x - 1 = \pm 2$$
$$x = 1 \pm 2 = -1, 3$$

The two $x$-intercepts for this parabola are then: $(-1, 0)$ and $(3, 0)$.

Step 4
Because we had two $x$-intercepts for this parabola we already have at least one point on either side of the vertex and so we don’t really need to find any more points for our graph.

However, just for the practice let’s find the point corresponding to the $y$-intercept on the other side the vertex.

The $y$-intercept is a distance of 1 to the left of the vertex and so there will be a corresponding point at the same $y$ value to the right and it will be a distance of 1 to the right of the vertex. Therefore the point to the right of the vertex corresponding to the $y$-intercept is $(2, -15)$.

Step 5
Here is a sketch of the parabola including all the points we found above.
3. Sketch the graph of the following parabola. The graph should contain the vertex, the \( y \)-intercept, \( x \)-intercepts (if any) and at least one point on either side of the vertex.

\[ f(x) = 3x^2 + 7 \]

**Step 1**
Let’s find the vertex first. In this case we can consider the equation to be in the form

\[ f(x) = a(x - h)^2 + k \]

or we can use the form \( f(x) = ax^2 + bx + c \). Either form will work and which you find to be the easiest will probably depend on you. We’ll use the first form so we can get another example of that form.

To make our equation up to the first form let’s do a little rewrite on our equation. Let’s write it as,

\[ f(x) = 3x^2 + 7 = 3(x - 0)^2 + 7 \]

Note that we haven’t changed the equation! All we’ve done is use \( x = x - 0 \) to get the forms to match up.

After doing this we can see that the vertex is: \( (0, 7) \).

Also note that \( a = 3 > 0 \) for this parabola and so the parabola will open upwards.

**Step 2**
In this case we can see that the vertex above is on the \( y \)-axis (the \( x \) coordinate is zero!) and so it is also the \( y \)-intercept for the parabola!

**Step 3**
For the \( x \)-intercepts we just need to solve the equation \( f(x) = 0 \). So, let’s solve that for our equation.
\[3x^2 + 7 = 0\]
\[3x^2 = -7\]
\[x^2 = \frac{-7}{3}\]
\[x = \pm \sqrt{-\frac{7}{3}} \pm \sqrt{\frac{7}{3}}i\]

So, in this case the solutions to this equation are complex numbers and so we know that this parabola will have no \(x\)-intercepts.

Note that we did not really need to solve the equation above to see that there would be no \(x\)-intercepts for this problem. An alternate method would be to do the following analysis.

From the first step we found that the vertex was \((0, 7)\), which is above the \(x\)-axis, and we also noted that the parabola opened upwards. So, the parabola starts above the \(x\)-axis and opens upwards and we know that once a parabola starts opening in a given direction it won’t turn around and start going in the opposite direction. Therefore, because there is no way for the parabola to go *down* to the \(x\)-axis, there is no way for there to be \(x\)-intercepts for this problem.

**Step 4**
In this case we didn’t have \(x\)-intercepts and the \(y\)-intercept also happens to be the vertex. So, at this point we have only have one point on the graph. To get points on either side of the vertex all we need to do is do a couple of quick function evaluations to find points on either side of the vertex.

We’ll use the following two points.

\[(-2, f(-2)) = (-2, 19)\]
\[(2, f(2)) = (2, 19)\]

**Step 5**
Here is a sketch of the parabola including all the points we found above.
4. Sketch the graph of the following parabola. The graph should contain the vertex, the $y$-intercept, $x$-intercepts (if any) and at least one point on either side of the vertex.

\[ f(x) = x^2 + 12x + 11 \]

**Step 1**
Let’s find the vertex first. In this case the equation is in the form \( f(x) = ax^2 + bx + c \). And so we know the vertex is the point \( \left( \frac{-b}{2a}, f\left(\frac{-b}{2a}\right) \right) \). The vertex is then,

\[
\left( \frac{-12}{2(1)}, f\left(\frac{-12}{2(1)}\right) \right) = (-6, f(-6)) = (-6, -25)
\]

Also note that \( a = 1 > 0 \) for this parabola and so the parabola will open upwards.

**Step 2**
The $y$-intercept is just the point \( (0, f(0)) \). A quick function evaluation gives us that \( f(0) = 11 \) and so for our equation the $y$-intercept is \( (0, 11) \).

**Step 3**
For the $x$-intercepts we just need to solve the equation \( f(x) = 0 \). So, let’s solve that for our equation.

\[
x^2 + 12x + 11 = 0 \\
(x+1)(x+11) = 0 \\
\rightarrow \quad x = -1, \, x = -11
\]

The two $x$-intercepts for this parabola are then: \((-1, 0)\) and \((-11, 0)\).

**Step 4**
Because we had two $x$-intercepts for this parabola we already have at least one point on either side of the vertex and so we don’t really need to find any more points for our graph.

However, just for the practice let’s find the point corresponding to the $y$-intercept on the other side the vertex.

The $y$-intercept is a distance of 6 to the right of the vertex and so there will be a corresponding point at the same $y$ value to the left and it will be a distance of 6 to the left of the vertex. Therefore the point to the left of the vertex corresponding to the $y$-intercept is \((-12, 11)\).

**Step 5**
Here is a sketch of the parabola including all the points we found above.
5. Sketch the graph of the following parabola. The graph should contain the vertex, the \( y \)-intercept, \( x \)-intercepts (if any) and at least one point on either side of the vertex.

\[ f(x) = 2x^2 - 12x + 26 \]

**Step 1**

Let's find the vertex first. In this case the equation is in the form \( f(x) = ax^2 + bx + c \). And so we know the vertex is the point \( \left( -\frac{b}{2a}, f\left(-\frac{b}{2a}\right) \right) \). The vertex is then,

\[
\left( -\frac{-12}{2(2)}, f\left(\frac{-12}{2(2)}\right) \right) = \left( 3, f(3) \right) = (3, 8)
\]

Also note that \( a = 2 > 0 \) for this parabola and so the parabola will open upwards.

**Step 2**

The \( y \)-intercept is just the point \( (0, f(0)) \). A quick function evaluation gives us that \( f(0) = 26 \) and so for our equation the \( y \)-intercept is \( (0, 26) \).

**Step 3**

For the \( x \)-intercepts we just need to solve the equation \( f(x) = 0 \). So, let's solve that for our equation.

\[
2x^2 - 12x + 26 = 0 \quad \rightarrow \quad x = \frac{12 \pm \sqrt{(-12)^2 - 4(2)(26)}}{2(2)} = \frac{12 \pm \sqrt{-64}}{4} = 3 \pm 2i
\]
So, in this case the solutions to this equation are complex numbers and so we know that this parabola will have no $x$-intercepts.

Note that we did not really need to solve the equation above to see that there would be no $x$-intercepts for this problem. An alternate method would be to do the following analysis.

From the first step we found that the vertex was $(3, 8)$, which is above the $x$-axis, and we also noted that the parabola opened upwards. So, the parabola starts above the $x$-axis and opens upwards and we know that once a parabola starts opening in a given direction it won’t turn around and start going in the opposite direction. Therefore, because there is no way for the parabola to go down to the $x$-axis, there is no way for there to be $x$-intercepts for this problem.

Step 4
In this case all we have are the vertex and the $y$-intercept (which is on the right side of the vertex). So, we’ll need a point that is on the left side of the vertex and we can find the point on the left side of the vertex that corresponds to the $y$-intercept for this point.

The $y$-intercept is a distance of 3 to the left of the vertex and so there will be a corresponding point at the same $y$ value to the right and it will be a distance of 3 to the right of the vertex. Therefore the point to the right of the vertex corresponding to the $y$-intercept is $(6, 26)$.

Step 5
Here is a sketch of the parabola including all the points we found above.

6. Sketch the graph of the following parabola. The graph should contain the vertex, the $y$-intercept, $x$-intercepts (if any) and at least one point on either side of the vertex.

$$f(x) = 4x^2 - 4x + 1$$

Step 1
Let’s find the vertex first. In this case the equation is in the form \( f(x) = ax^2 + bx + c \). And so we know the vertex is the point \( \left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right) \). The vertex is then,

\[
\left( -\frac{-4}{2(4)}, f\left( -\frac{-4}{2(4)} \right) \right) = \left( \frac{1}{2}, f\left( \frac{1}{2} \right) \right) = \left( \frac{1}{2}, 0 \right)
\]

Also note that \( a = 4 > 0 \) for this parabola and so the parabola will open upwards.

Step 2
The \( y \)-intercept is just the point \( (0, f(0)) \). A quick function evaluation gives us that \( f(0) = 1 \) and so for our equation the \( y \)-intercept is \( (0,1) \).

Step 3
For the \( x \)-intercepts we would normally solve the equation \( f(x) = 0 \). However, in this case we don’t need to do that. From the first step we see that the vertex has a \( y \)-coordinate of zero and hence is also an \( x \)-intercept. Also, because it is the vertex this can be the only \( x \)-intercept for this function.

Note that if we’d solved the equation we would have also arrived at this single \( x \)-intercept.

Step 4
In this case all we have are the vertex (which also happens to be the single \( x \)-intercept) and the \( y \)-intercept (which is on the right side of the vertex). So, we’ll need a point that is on the left side of the vertex and we can find the point on the left side of the vertex that corresponds to the \( y \)-intercept for this point.

The \( y \)-intercept is a distance of 1 to the left of the vertex and so there will be a corresponding point at the same \( y \) value to the right and it will be a distance of 1 to the right of the vertex. Therefore the point to the right of the vertex corresponding to the \( y \)-intercept is \( (1,1) \).

Step 5
Here is a sketch of the parabola including all the points we found above.
7. Sketch the graph of the following parabola. The graph should contain the vertex, the $y$-intercept, $x$-intercepts (if any) and at least one point on either side of the vertex.

\[ f(x) = -3x^2 + 6x + 3 \]

**Step 1**
Let’s find the vertex first. In this case the equation is in the form \( f(x) = ax^2 + bx + c \). And so we know the vertex is the point \( \left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right) \). The vertex is then,

\[ \left( -\frac{6}{2(-3)}, f\left( -\frac{6}{2(-3)} \right) \right) = \left( 1, f(1) \right) = (1, 6) \]

Also note that \( a = -3 < 0 \) for this parabola and so the parabola will open downwards.

**Step 2**
The $y$-intercept is just the point \( (0, f(0)) \). A quick function evaluation gives us that \( f(0) = 3 \) and so for our equation the $y$-intercept is \( (0, 3) \).

**Step 3**
For the $x$-intercepts we just need to solve the equation \( f(x) = 0 \). So, let’s solve that for our equation.

\[ -3x^2 + 6x + 3 = 0 \quad \Rightarrow \quad x = \frac{-6 \pm \sqrt{6^2 - 4(-3)(3)}}{2(-3)} = \frac{-6 \pm \sqrt{72}}{-6} = \frac{-6 \pm 6\sqrt{2}}{-6} = 1 \pm \sqrt{2} \]
The two $x$-intercepts for this parabola are then: $(1 - \sqrt{2}, 0) = (-0.4142, 0)$ and $(1 + \sqrt{2}, 0) = (2.4142, 0)$.

Step 4
Because we had two $x$-intercepts for this parabola we already have at least one point on either side of the vertex and so we don’t really need to find any more points for our graph.

However, just for the practice let’s find the point corresponding to the $y$-intercept on the other side the vertex.

The $y$-intercept is a distance of 1 to the left of the vertex and so there will be a corresponding point at the same $y$ value to the right and it will be a distance of 1 to the right of the vertex. Therefore the point to the right of the vertex corresponding to the $y$-intercept is $(2, 3)$.

Step 5
Here is a sketch of the parabola including all the points we found above.

8. Convert the following equations into the form $y = a(x - h)^2 + k$.

$f(x) = x^2 - 24x + 157$

Step 1
We’ll need to do the modified completing the square process described in the notes for this section.

The first step in this process is to make sure that we have a coefficient of one on the $x^2$, which we already have, so there is nothing we need to do in that regards for this problem.
Step 2
Next, we need to take one-half the coefficient of the $x$, square it and then add and subtract it onto the equation.

\[
\left( \frac{-24}{2} \right)^2 = (-12)^2 = 144
\]

\[
f(x) = x^2 - 24x + 144 - 144 + 157
\]

Step 3
Finally, all we need to do is factor the first three terms and combine the last two numbers to get,

\[
f(x) = (x - 12)^2 + 13
\]

9. Convert the following equations into the form $y = a(x - h)^2 + k$.

\[
f(x) = 6x^2 + 12x + 3
\]

Step 1
We’ll need to do the modified completing the square process described in the notes for this section.

The first step in this process is to make sure that we have a coefficient of one on the $x^2$. So, for this problem that means we need to factor a 6 out of the quadratic to get,

\[
f(x) = 6 \left( x^2 + 2x + \frac{1}{2} \right)
\]

Be careful to not just cancel out a 6 from each term! We need to factor it out.

Step 2
Next, we need to take one-half the coefficient of the $x$, square it and then add and subtract it onto the equation.

\[
\left( \frac{2}{2} \right)^2 = (1)^2 = 1
\]

\[
f(x) = 6 \left( x^2 + 2x + 1 - 1 + \frac{1}{2} \right)
\]

Make sure to do the adding/subtracting inside the parenthesis. If we did it outside of the parenthesis we would not be able to do the next step!
Step 3
Next we need to factor the first three terms and combine the last two numbers to get,

\[ f(x) = 6\left((x + 1)^2 - \frac{1}{2}\right) \]

Step 4
Finally, all we need to do is multiply the 6 back through the parenthesis to get,

\[ f(x) = 6(x + 1)^2 - 3 \]

10. Convert the following equations into the form \( y = a(x - h)^2 + k \).

\[ f(x) = -x^2 - 8x - 18 \]

Step 1
We’ll need to do the modified completing the square process described in the notes for this section.

The first step in this process is to make sure that we have a coefficient of one on the \( x^2 \). So, for this problem that means we need to factor a minus sign out of the quadratic to get,

\[ f(x) = -(x^2 + 8x + 18) \]

Step 2
Next, we need to take one-half the coefficient of the \( x \), square it and then add and subtract it onto the equation.

\[ \left(\frac{8}{2}\right)^2 = (4)^2 = 16 \]

\[ f(x) = -(x^2 + 8x + 16 - 16 + 18) \]

Make sure to do the adding/subtracting inside the parenthesis. If we did it outside of the parenthesis we would not be able to do the next step!

Step 3
Next we need to factor the first three terms and combine the last two numbers to get,

\[ f(x) = -(x + 4)^2 + 2 \]
Step 4
Finally, all we need to do is multiply the 6 back through the parenthesis to get,

\[ f(x) = -(x + 4)^2 - 2 \]