Preface

Here are my online notes for my Algebra course that I teach here at Lamar University, although I have to admit that it’s been years since I last taught this course. At this point in my career I mostly teach Calculus and Differential Equations.

Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Algebra or needing a refresher for algebra. I’ve tried to make the notes as self contained as possible and do not reference any book. However, they do assume that you’ve had some exposure to the basics of algebra at some point prior to this. While there is some review of exponents, factoring and graphing it is assumed that not a lot of review will be needed to remind you how these topics work.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn algebra I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. Because I want these notes to provide some more examples for you to read through, I don’t always work the same problems in class as those given in the notes. Likewise, even if I do work some of the problems in here I may work fewer problems in class than are presented here.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
**Finding Zeroes of Polynomials**

We’ve been talking about zeroes of polynomial and why we need them for a couple of sections now. We haven’t, however, really talked about how to actually find them for polynomials of degree greater than two. That is the topic of this section. Well, that’s kind of the topic of this section. In general, finding all the zeroes of any polynomial is a fairly difficult process. In this section we will give a process that will find all rational (i.e. integer or fractional) zeroes of a polynomial. We will be able to use the process for finding all the zeroes of a polynomial provided all but at most two of the zeroes are rational. If more than two of the zeroes are not rational then this process will not find all of the zeroes.

We will need the following theorem to get us started on this process.

**Rational Root Theorem**

If the rational number \( \frac{b}{c} \) is a zero of the \( n \)th degree polynomial,

\[
P(x) = s x^n + \cdots + t
\]

where all the coefficients are integers then \( b \) will be a factor of \( t \) and \( c \) will be a factor of \( s \).

Note that in order for this theorem to work then the zero must be reduced to lowest terms. In other words it will work for \( \frac{4}{3} \) but not necessarily for \( \frac{20}{15} \).

Let’s verify the results of this theorem with an example.

**Example 1** Verify that the roots of the following polynomial satisfy the rational root theorem.

\[
P(x) = 12x^3 - 41x^2 - 38x + 40 = (x - 4)(3x - 2)(4x + 5)
\]

**Solution**

From the factored form we can see that the zeroes are,

\[
x = 4, \quad x = \frac{2}{3}, \quad x = -\frac{5}{4}
\]

Notice that we wrote the integer as a fraction to fit it into the theorem. Also, with the negative zero we can put the negative onto the numerator or denominator. It won’t matter.

So, according to the rational root theorem the numerators of these fractions (with or without the minus sign on the third zero) must all be factors of 40 and the denominators must all be factors of 12.

Here are several ways to factor 40 and 12.

\[
40 = (4)(10), \quad 40 = (2)(20), \quad 40 = (5)(8), \quad 40 = (-5)(-8)
\]

\[
12 = (1)(12), \quad 12 = (3)(4), \quad 12 = (-3)(-4)
\]

From these we can see that in fact the numerators are all factors of 40 and the denominators are all factors of 12. Also note that, as shown, we can put the minus sign on the third zero on either the numerator or the denominator and it will still be a factor of the appropriate number.
So, why is this theorem so useful? Well, for starters it will allow us to write down a list of possible rational zeroes for a polynomial and more importantly, any rational zeroes of a polynomial WILL be in this list.

In other words, we can quickly determine all the rational zeroes of a polynomial simply by checking all the numbers in our list.

Before getting into the process of finding the zeroes of a polynomial let’s see how to come up with a list of possible rational zeroes for a polynomial.

**Example 2** Find a list of all possible rational zeroes for each of the following polynomials.

(a) \( P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 \)  
   \([\text{Solution}]\)

(b) \( P(x) = 2x^4 + x^3 + 3x^2 + 3x - 9 \)  
   \([\text{Solution}]\)

**Solution**

(a) \( P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 \)

Now, just what does the rational root theorem say? It says that if \( \frac{b}{c} \) is to be a zero of \( P(x) \) then \( b \) must be a factor of 6 and \( c \) must be a factor of 1. Also, as we saw in the previous example we can’t forget negative factors.

So, the first thing to do is actually to list all possible factors of 1 and 6. Here they are.

6: \( \pm 1, \pm 2, \pm 3, \pm 6 \)

1: \( \pm 1 \)

Now, to get a list of possible rational zeroes of the polynomial all we need to do is write down all possible fractions that we can form from these numbers where the numerators must be factors of 6 and the denominators must be factors of 1. This is actually easier than it might at first appear to be.

There is a very simple shorthanded way of doing this. Let’s go through the first one in detail then we’ll do the rest quicker. First, take the first factor from the numerator list, including the \( \pm \), and divide this by the first factor (okay, only factor in this case) from the denominator list, again including the \( \pm \). Doing this gives,

\[
\pm 1
\]

\[
\pm 1
\]

This looks like a mess, but it isn’t too bad. There are four fractions here. They are,

\[
\frac{+1}{+1} = 1 \quad \frac{+1}{-1} = -1 \quad \frac{-1}{+1} = -1 \quad \frac{-1}{-1} = 1
\]

Notice however, that the four fractions all reduce down to two possible numbers. This will always happen with these kinds of fractions. What we’ll do from now on is form the fraction, do any simplification of the numbers, ignoring the \( \pm \), and then drop one of the \( \pm \).

So, the list possible rational zeroes for this polynomial is,
So, it looks there are only 8 possible rational zeroes and in this case they are all integers. Note as well that any rational zeroes of this polynomial WILL be somewhere in this list, although we haven’t found them yet.

(b) \( P(x) = 2x^4 + x^3 + 3x^2 + 3x - 9 \)

We’ll not put quite as much detail into this one. First get a list of all factors of -9 and 2. Note that the minus sign on the 9 isn’t really all that important since we will still get a ± on each of the factors.

\(-9:\quad ±1, ±3, ±9\)
\(2:\quad ±1, ±2\)

Now, the factors of -9 are all the possible numerators and the factors of 2 are all the possible denominators.

Here then is a list of all possible rational zeroes of this polynomial.

\[ \frac{±1}{±1} = ±1 \quad \frac{±3}{±1} = ±3 \quad \frac{±9}{±1} = ±9 \]
\[ \frac{±1}{±2} = ±\frac{1}{2} \quad \frac{±3}{±2} = ±\frac{3}{2} \quad \frac{±9}{±2} = ±\frac{9}{2} \]

So, we’ve got a total of 12 possible rational zeroes, half are integers and half are fractions.

The following fact will also be useful on occasion in finding the zeroes of a polynomial.

**Fact**

If \( P(x) \) is a polynomial and we know that \( P(a) > 0 \) and \( P(b) < 0 \) then somewhere between \( a \) and \( b \) is a zero of \( P(x) \).

What this fact is telling us is that if we evaluate the polynomial at two points and one of the evaluations gives a positive value (i.e. the point is above the \( x \)-axis) and the other evaluation gives a negative value (i.e. the point is below the \( x \)-axis), then the only way to get from one point to the other is to go through the \( x \)-axis. Or, in other words, the polynomial must have a zero, since we know that zeroes are where a graph touches or crosses the \( x \)-axis.

Note that this fact doesn’t tell us what the zero is, it only tells us that one will exist. Also, note that if both evaluations are positive or both evaluations are negative there may or may not be a zero between them.

Here is the process for determining all the rational zeroes of a polynomial.
Process for Finding Rational Zeroes

1. Use the rational root theorem to list all possible rational zeroes of the polynomial $P(x)$.

2. Evaluate the polynomial at the numbers from the first step until we find a zero. Let’s suppose the zero is $x = r$, then we will know that it’s a zero because $P(r) = 0$. Once this has been determined that it is in fact a zero write the original polynomial as

$$P(x) = (x - r)Q(x)$$

3. Repeat the process using $Q(x)$ this time instead of $P(x)$. This repeating will continue until we reach a second degree polynomial. At this point we can solve this directly for the remaining zeroes.

To simplify the second step we will use synthetic division. This will greatly simplify our life in several ways. First, recall that the last number in the final row is the polynomial evaluated at $r$ and if we do get a zero the remaining numbers in the final row are the coefficients for $Q(x)$ and so we won’t have to go back and find that.

Also, in the evaluation step it is usually easiest to evaluate at the possible integer zeroes first and then go back and deal with any fractions if we have to.

Let’s see how this works.

**Example 3** Determine all the zeroes of $P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6$.

**Solution**
We found the list of all possible rational zeroes in the previous example. Here they are.

$$\pm 1, \pm 2, \pm 3, \pm 6$$

We now need to start evaluating the polynomial at these numbers. We can start anywhere in the list and will continue until we find zero.

To do the evaluations we will build a **synthetic division table**. In a synthetic division table do the multiplications in our head and drop the middle row just writing down the third row and since we will be going through the process multiple times we put all the rows into a table.

Here is the first synthetic division table for this problem.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-7</th>
<th>17</th>
<th>-17</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>1</td>
<td>-8</td>
<td>25</td>
<td>-42</td>
<td>48</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-6</td>
<td>11</td>
<td>-6</td>
<td>0</td>
</tr>
</tbody>
</table>

So, we found a zero. Before getting into that let’s recap the computations here to make sure you can do them.
The top row is the coefficients from the polynomial and the first column is the numbers that we’re evaluating the polynomial at.

Each row (after the first) is the third row from the synthetic division process. Let’s quickly look at the first couple of numbers in the second row. The number in the second column is the first coefficient dropped down. The number in the third column is then found by multiplying the -1 by 1 and adding to the -7. This gives the -8. For the fourth number is then -1 times -8 added onto 17. This is 25, etc.

You can do regular synthetic division if you need to, but it’s a good idea to be able to do these tables as it can help with the process.

Okay, back to the problem. We now know that $x = 1$ is a zero and so we can write the polynomial as,

$$P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 = (x - 1)(x^3 - 6x^2 + 11x - 6)$$

Now we need to repeat this process with the polynomial $Q(x) = x^3 - 6x^2 + 11x - 6$. So, the first thing to do is to write down all possible rational roots of this polynomial and in this case we’re lucky enough to have the first and last numbers in this polynomial be the same as the original polynomial, that usually won’t happen so don’t always expect it. Here is the list of all possible rational zeroes of this polynomial.

$$\pm 1, \pm 2, \pm 3, \pm 6$$

Now, before doing a new synthetic division table let’s recall that we are looking for zeroes to $P(x)$ and from our first division table we determined that $x = -1$ is NOT a zero of $P(x)$ and so there is no reason to bother with that number again.

This is something that we should always do at this step. Take a look at the list of new possible rational zeros and ask are there any that can’t be rational zeroes of the original polynomial. If there are some, throw them out as we will already know that they won’t work. So, a reduced list of numbers to try here is,

$$1, \pm 2, \pm 3, \pm 6$$

Note that we do need to include $x = 1$ in the list since it is possible for a zero to occur more than once (i.e. multiplicity greater than one).

Here is the synthetic division table for this polynomial.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>-6</th>
<th>11</th>
<th>-6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>-5</td>
<td>6</td>
<td>0</td>
</tr>
</tbody>
</table>

$P(1) = 0!!$

So, $x = 1$ is also a zero of $Q(x)$ and we can now write $Q(x)$ as,

$$Q(x) = x^3 - 6x^2 + 11x - 6 = (x - 1)(x^2 - 5x + 6)$$
Now, technically we could continue the process with \( x^2 - 5x + 6 \), but this is a quadratic equation and we know how to find zeroes of these without a complicated process like this so let’s just solve this like we normally would.

\[
x^2 - 5x + 6 = (x - 2)(x - 3) = 0 \quad \Rightarrow \quad x = 2, x = 3
\]

Note that these two numbers are in the list of possible rational zeroes.

Finishing up this problem then gives the following list of zeroes for \( P(x) \).

\[
\begin{align*}
x &= 1 \quad \text{(multiplicity 2)} \\
x &= 2 \quad \text{(multiplicity 1)} \\
x &= 3 \quad \text{(multiplicity 1)}
\end{align*}
\]

Note that \( x = 1 \) has a multiplicity of 2 since it showed up twice in our work above.

Before moving onto the next example let’s also note that we can now completely factor the polynomial \( P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 \). We know that each zero will give a factor in the factored form and that the exponent on the factor will be the multiplicity of that zero. So, the factored form is,

\[
P(x) = x^4 - 7x^3 + 17x^2 - 17x + 6 = (x - 1)^2(x - 2)(x - 3)
\]

Let’s take a look at another example.

**Example 4** Find all the zeroes of \( P(x) = 2x^4 + x^3 + 3x^2 + 3x - 9 \).

**Solution**

From the second example we know that the list of all possible rational zeroes is,

\[
\begin{align*}
\pm \frac{1}{1} &= \pm 1 \\
\pm \frac{3}{1} &= \pm 3 \\
\pm \frac{9}{1} &= \pm 9 \\
\pm \frac{1}{2} &= \pm 0.5 \\
\pm \frac{3}{2} &= \pm 1.5 \\
\pm \frac{9}{2} &= \pm 4.5
\end{align*}
\]

The next step is to build up the synthetic division table. When we’ve got fractions it’s usually best to start with the integers and do those first. Also, this time we’ll start with doing all the negative integers first. We are doing this to make a point on how we can use the fact given above to help us identify zeroes.
\[
\begin{array}{rrrrr}
2 & 1 & 3 & 3 & -9 \\
-9 & 2 & -17 & 156 & -1401 & 12600 = P(-9) \neq 0 \\
-3 & 2 & -5 & 18 & -51 & 144 = P(-3) \neq 0 \\
-1 & 2 & -1 & 4 & -1 & -8 = P(-1) \neq 0 \\
\end{array}
\]

Now, we haven’t found a zero yet, however let’s notice that \( P(-3) = 144 > 0 \) and \( P(-1) = -8 < 0 \) and so by the fact above we know that there must be a zero somewhere between \( x = -3 \) and \( x = -1 \). Now, we can also notice that \( x = -\frac{3}{2} \approx -1.5 \) is in this range and is the only number in our list that is in this range and so there is a chance that this is a zero. Let’s run through synthetic division real quick to check and see if it’s a zero and to get the coefficients for \( Q(x) \) if it is a zero.

\[
\begin{array}{rrrrr}
2 & 1 & 3 & 3 & -9 \\
-\frac{3}{2} & 2 & -2 & 6 & -6 & 0 \\
\end{array}
\]

So, we got a zero in the final spot which tells us that this was a zero and \( Q(x) \) is,

\[Q(x) = 2x^3 - 2x^2 + 6x - 6\]

We now need to repeat the whole process with this polynomial. Also, unlike the previous example we can’t just reuse the original list since the last number is different this time. So, here are the factors of -6 and 2.

\[-6: \quad \pm 1, \pm 2, \pm 3, \pm 6 \\
2: \quad \pm 1, \pm 2\]

Here is a list of all possible rational zeroes for \( Q(x) \).

\[
\begin{aligned}
\frac{\pm 1}{\pm 1} &= \pm 1 & \frac{\pm 2}{\pm 1} &= \pm 2 & \frac{\pm 3}{\pm 1} &= \pm 3 & \frac{\pm 6}{\pm 1} &= \pm 6 \\
\frac{\pm 1}{\pm 2} &= \pm \frac{1}{2} & \frac{\pm 2}{\pm 2} &= \pm 1 & \frac{\pm 3}{\pm 2} &= \pm \frac{3}{2} & \frac{\pm 6}{\pm 2} &= \pm 3
\end{aligned}
\]

Notice that some of the numbers appear in both rows and so we can shorten the list by only writing them down once. Also, remember that we are looking for zeroes of \( P(x) \) and so we can exclude any number in this list that isn’t also in the original list we gave for \( P(x) \). So, excluding previously checked numbers that were not zeros of \( P(x) \) as well as those that aren’t in the original list gives the following list of possible number that we’ll need to check.
Again, we’ve already checked \( x = -3 \) and \( x = -1 \) and know that they aren’t zeroes so there is no reason to recheck them. Let’s again start with the integers and see what we get.

\[
\begin{array}{c|cccc}
2 & -2 & 6 & -6 \\
1 & 2 & 0 & 6 & 0 = P(1) = 0
\end{array}
\]

So, \( x = 1 \) is a zero of \( Q(x) \) and we can now write \( Q(x) \) as,

\[
Q(x) = 2x^3 - 2x^2 + 6x - 6 = (x - 1)(2x^2 + 6)
\]

and as with the previous example we can solve the quadratic by other means.

\[
\begin{align*}
2x^2 + 6 &= 0 \\
x^2 &= -3 \\
x &= \pm \sqrt{3}i
\end{align*}
\]

So, in this case we get a couple of complex zeroes. That can happen.

Here is a complete list of all the zeroes for \( P(x) \) and note that they all have multiplicity of one.

\[
x = -\frac{3}{2}, \ x = 1, \ x = -\sqrt{3}i, \ x = \sqrt{3}i
\]

So, as you can see this is a fairly lengthy process and we only did the work for two \( 4^{th} \) degree polynomials. The larger the degree the longer and more complicated the process. With that being said, however, it is sometimes a process that we’ve got to go through to get zeroes of a polynomial.