Here are a set of practice problems for my Calculus III notes. If you are viewing the pdf version of this document (as opposed to viewing it on the web) this document contains only the problems themselves and no solutions are included in this document. Solutions can be found in a number of places on the site.

1. If you’d like a pdf document containing the solutions go to the note page for the section you’d like solutions for and select the download solutions link from there. Or,

2. Go to the download page for the site http://tutorial.math.lamar.edu/download.aspx and select the section you’d like solutions for and a link will be provided there.

3. If you’d like to view the solutions on the web or solutions to an individual problem you can go to the problem set web page, select the problem you want the solution for. At this point I do not provide pdf versions of individual solutions, but for a particular problem you can select “Printable View” from the “Solution Pane Options” to get a printable version.

Note that some sections will have more problems than others and some will have more or less of a variety of problems. Most sections should have a range of difficulty levels in the problems although this will vary from section to section.
Partial Derivatives

1. Find all the 1st order partial derivatives of the following function.

\[ f(x, y, z) = 4x^3 y^2 - e^z y^4 + \frac{z^2}{x^2} + 4y - x^{16} \]

Solution
So, this is clearly a function of \( x, y \) and \( z \) and so we’ll have three 1st order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the \( x \) partial derivative all \( y \)'s and \( z \)'s are treated as constants. This in turn means that, for the \( x \) partial derivative, the second and fourth terms are considered to be constants (they don’t contain any \( x \)'s) and so differentiate to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don’t contain the variable we are differentiating with respect to.

Here are the three 1st order partial derivatives for this problem.

\[
\frac{\partial f}{\partial x} = f_x = 12x^2 y^2 - \frac{2z^3}{x^3} - 16x^{15} \\
\frac{\partial f}{\partial y} = f_y = 8x^3 y - 4e^z y^3 + 4 \\
\frac{\partial f}{\partial z} = f_z = -e^z y^4 + \frac{3z^2}{x^2}
\]

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

2. Find all the 1st order partial derivatives of the following function.

\[ w = \cos(x^2 + 2y) - e^{4x-z^4y} + y^3 \]

Solution
This function isn’t written explicitly with the \( (x, y, z) \) part but it is (hopefully) clearly a function of \( x, y \) and \( z \) and so we’ll have three 1st order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the \( x \) partial derivative all \( y \)'s and \( z \)'s are treated as constants. This in turn means that, for the \( x \) partial derivative, the third term is considered to be a constant (it don’t
contain any $x$’s) and so differentiates to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don’t contain the variable we are differentiating with respect to.

Also be careful with chain rule. Again one of the biggest issues with partial derivatives is students forgetting the “rules” of partial derivatives when it comes to differentiating the inside function. Just remember that you’re just doing the partial derivative of a function and remember which variable we are differentiating with respect to.

Here are the three 1st order partial derivatives for this problem.

\[
\begin{align*}
\frac{\partial w}{\partial x} &= w_x = -2x \sin (x^2 + 2y) - 4e^{4x-z^4} y \\
\frac{\partial w}{\partial y} &= w_y = -2 \sin (x^2 + 2y) + z^4 e^{4x-z^4} + 3y^2 \\
\frac{\partial w}{\partial z} &= w_z = 4z^3 y e^{4x-z^4} 
\end{align*}
\]

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

3. Find all the 1st order partial derivatives of the following function.

\[f(u, v, p, t) = 8u^2 t^3 p - \sqrt{v} p^2 t^{-5} + 2u^2 t + 3p^4 - v\]

Solution

So, this is clearly a function of $u$, $v$, $p$, and $t$ and so we’ll have four 1st order partial derivatives and each of them should be pretty easy to compute.

Just remember that when computing each individual derivative that the other variables are to be treated as constants. So, for instance, when computing the $u$ partial derivative all $v$’s, $p$’s and $t$’s are treated as constants. This in turn means that, for the $u$ partial derivative, the second, fourth and fifth terms are considered to be constants (they don’t contain any $u$’s) and so differentiate to zero. Dealing with these types of terms properly tends to be one of the biggest mistakes students make initially when taking partial derivatives. Too often students just leave them alone since they don’t contain the variable we are differentiating with respect to.

Here are the four 1st order partial derivatives for this problem.
Calculus II

\[
\frac{\partial f}{\partial u} = f_u = 16u^3 + 4ut
\]
\[
\frac{\partial f}{\partial v} = f_v = -\frac{1}{2}v^{-\frac{1}{2}} p^2 t^{-3} - 1
\]
\[
\frac{\partial f}{\partial p} = f_p = 8u^2 t^3 - 2\sqrt{v} pt^{-5} + 12p^3
\]
\[
\frac{\partial f}{\partial t} = f_t = 24u^2 t^2 p + 5\sqrt{v} p^2 t^{-6} + 2u^3
\]

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

4. Find all the 1st order partial derivatives of the following function.

\[
f(u, v) = u^2 \sin(u + v^3) - \sec(4u) \tan^{-1}(2v)
\]

Solution

For this problem it looks like we’ll have two 1st order partial derivatives to compute.

Be careful with product rules with partial derivatives. For example the second term, while definitely a product, will not need the product rule since each “factor” of the product only contains \(u\)’s or \(v\)’s. On the other hand the first term will need a product rule when doing the \(u\) partial derivative since there are \(u\)’s in both of the “factors” of the product. However, just because we had to product rule with first term for the \(u\) partial derivative doesn’t mean that we’ll need to product rule for the \(v\) partial derivative as only the second “factor” in the product has a \(v\) in it.

Basically, be careful to not “overthink” product rules with partial derivatives. Do them when required but make sure to not do them just because you see a product. When you see a product look at the “factors” of the product. Do both “factors” have the variable you are differentiating with respect to or not and use the product rule only if they both do.

Here are the two 1st order partial derivatives for this problem.

\[
\frac{\partial f}{\partial u} = f_u = 2u \sin(u + v^3) + u^2 \cos(u + v^3) - 4\sec(4u) \tan(4u) \tan^{-1}(2v)
\]
\[
\frac{\partial f}{\partial v} = f_v = 3v^2 u^2 \cos(u + v^3) - \frac{2\sec(4u)}{1 + 4v^2}
\]

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

5. Find all the 1st order partial derivatives of the following function.
Calculus II

\[ f(x, z) = e^{-x} \sqrt{z^4 + x^2} - \frac{2x + 3z}{4z - 7x} \]

Solution
For this problem it looks like we’ll have two 1st order partial derivatives to compute.

Be careful with product rules and quotient rules with partial derivatives. For example the first term, while clearly a product, will only need the product rule for the x derivative since both “factors” in the product have x’s in them. On the other hand, the first “factor” in the first term does not contain a z and so we won’t need to do the product rule for the z derivative. In this case the second term will require a quotient rule for both derivatives.

Basically, be careful to not “overthink” product/quotient rules with partial derivatives. Do them when required but make sure to not do them just because you see a product/quotient. When you see a product/quotient look at the “factors” of the product/quotient. Do both “factors” have the variable you are differentiating with respect to or not and use the product/quotient rule only if they both do.

Here are the two 1st order partial derivatives for this problem.

\[
\frac{\partial f}{\partial x} = f_x = -e^{-x} \left( z^4 + x^2 \right)^{1/2} + x e^{-x} \left( z^4 + x^2 \right)^{1/2} - \frac{29z}{(4z - 7x)^2}
\]

\[
\frac{\partial f}{\partial z} = f_z = 2z^3 e^{-x} \left( z^4 + x^2 \right)^{1/2} + \frac{29x}{(4z - 7x)^2}
\]

Note that we did a little bit of simplification in the derivative work here and didn’t actually show the “first” step of the problem under the assumption that by this point of your mathematical career you can do the product and quotient rule and don’t really need us to show that step to you.

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

6. Find all the 1st order partial derivatives of the following function.

\[ g(s, t, v) = t^2 \ln(s + 2t) - \ln(3v)(s^3 + t^2 - 4v) \]

Solution
For this problem it looks like we’ll have three 1st order partial derivatives to compute.

Be careful with product rules with partial derivatives. The first term will only need a product rule for the t derivative and the second term will only need the product rule for the v derivative. Do not “overthink” product rules with partial derivatives. Do them when required but make sure to not do them just because you see a product. When you see a product look at the “factors” of the product. Do both “factors” have the variable you are differentiating with respect to or not and use the product rule only if they both do.

Here are the three 1st order partial derivatives for this problem.
Calculus II

\[ \frac{\partial g}{\partial s} = g_s = \frac{t^2}{s + 2t} - 3s^2 \ln(3v) \]
\[ \frac{\partial g}{\partial t} = g_t = 2t \ln(s + 2t) + \frac{2t^2}{s + 2t} - 2t \ln(3v) \]
\[ \frac{\partial g}{\partial v} = g_v = 4 \ln(3v) - \frac{s^3 + t^2 - 4v}{v} \]

Make sure you can differentiate natural logarithms as they will come up fairly often. Recall that, with the chain rule, we have,

\[ \frac{d}{dx} \left[ \ln \left( f(x) \right) \right] = \frac{f'(x)}{f(x)} \]

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

7. Find all the 1st order partial derivatives of the following function.

\[ R(x, y) = \frac{x^2}{y^2 + 1} - \frac{y^2}{x^2 + y} \]

Solution
For this problem it looks like we’ll have two 1st order partial derivatives to compute.

Be careful with quotient rules with partial derivatives. For example the first term, while clearly a quotient, will not require the quotient rule for the \( x \) derivative and will only require the quotient rule for the \( y \) derivative if we chose to leave the \( y^2 + 1 \) in the denominator (recall we could just bring it up to the numerator as \( (y^2 + 1)^{-1} \) if we wanted to). The second term on the other hand clearly has \( y \)'s in both the numerator and the denominator and so will require a quotient rule for the \( y \) derivative.

Here are the two 1st order partial derivatives for this problem.

\[ \frac{\partial R}{\partial x} = R_x = \frac{2x}{y^2 + 1} + \frac{2xy^2}{(x^2 + y)^2} \]
\[ \frac{\partial R}{\partial y} = R_y = -\frac{2yx^2}{(y^2 + 1)^2} - \frac{2yx^2 + y^2}{(x^2 + y)^2} \]

Note that we did a little bit of simplification in the derivative work here and didn’t actually show the “first” step of the problem under the assumption that by this point of your mathematical career you can do the quotient rule and don’t really need us to show that step to you.
Calculus II

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

8. Find all the 1st order partial derivatives of the following function.

\[ z = \frac{p^2 (r+1)}{t^3} + pr e^{2p+3r+4t} \]

Solution
For this problem it looks like we’ll have three 1st order partial derivatives to compute. Here they are,

\[
\frac{\partial z}{\partial p} = z_p = \frac{2p(r+1)}{t^3} + r e^{2p+3r+4t} + 2 pr e^{2p+3r+4t} \\
\frac{\partial z}{\partial r} = z_r = \frac{p^2}{t^3} + p e^{2p+3r+4t} + 3 pr e^{2p+3r+4t} \\
\frac{\partial z}{\partial t} = z_t = -\frac{3p^2(r+1)}{t^4} + 4 pr e^{2p+3r+4t}
\]

The notation used for the derivative doesn’t matter so we used both here just to make sure we’re familiar with both forms.

9. Find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \) for the following function.

\[ x^2 \sin \left( y^3 \right) + xe^{3z} - \cos \left( z^2 \right) = 3y - 6z + 8 \]

Step 1
Okay, we are basically being asked to do implicit differentiation here and recall that we are assuming that \( z \) is in fact \( z(x, y) \) when we do our derivative work.

Let’s get \( \frac{\partial z}{\partial x} \) first and that requires us to differentiate with respect to \( x \). Just recall that any product involving \( x \) and \( z \) will require the product rule because we’re assuming that \( z \) is a function of \( x \). Also recall to properly chain rule any derivative of \( z \) to pick up the \( \frac{\partial z}{\partial x} \) when differentiating the “inside” function.

Differentiating the equation with respect to \( x \) gives,

\[ 2x \sin \left( y^3 \right) + e^{3z} + 3 \frac{\partial z}{\partial x} xe^{3z} + 2z \frac{\partial z}{\partial x} \sin \left( z^2 \right) = -6 \frac{\partial z}{\partial x} \]
Solving for \( \frac{\partial z}{\partial x} \) gives,

\[
2x \sin(y^3) + e^{3z} = \left(-6 - 3xe^{3z} - 2z \sin(z^2)\right) \frac{\partial z}{\partial x} \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{2x \sin(y^3) + e^{3z}}{-6 - 3xe^{3z} - 2z \sin(z^2)}
\]

Step 2
Now we get to do it all over again except this time we’ll differentiate with respect to \( y \) in order to find \( \frac{\partial z}{\partial y} \). So, differentiating gives,

\[
3y^2x^2 \cos(y^3) + 3 \frac{\partial z}{\partial y} xe^{3z} + 2z \frac{\partial z}{\partial y} \sin(z^2) = y - 6 \frac{\partial z}{\partial y}
\]

Solving for \( \frac{\partial z}{\partial y} \) gives,

\[
3y^2x^2 \cos(y^3) - y = \left(-6 - 3xe^{3z} - 2z \sin(z^2)\right) \frac{\partial z}{\partial y} \quad \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{3y^2x^2 \cos(y^3) - y}{-6 - 3xe^{3z} - 2z \sin(z^2)}
\]