Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Interpretations of Partial Derivatives

This is a fairly short section and is here so we can acknowledge that the two main interpretations of derivatives of functions of a single variable still hold for partial derivatives, with small modifications of course to account of the fact that we now have more than one variable.

The first interpretation we’ve already seen and is the more important of the two. As with functions of single variables partial derivatives represent the rates of change of the functions as the variables change. As we saw in the previous section, \( f'_x (x, y) \) represents the rate of change of the function \( f(x, y) \) as we change \( x \) and hold \( y \) fixed while \( f'_y (x, y) \) represents the rate of change of \( f(x, y) \) as we change \( y \) and hold \( x \) fixed.

**Example 1** Determine if \( f(x, y) = \frac{x^2}{y^3} \) is increasing or decreasing at \((2, 5)\),

(a) if we allow \( x \) to vary and hold \( y \) fixed.
(b) if we allow \( y \) to vary and hold \( x \) fixed.

**Solution**

(a) If we allow \( x \) to vary and hold \( y \) fixed.

In this case we will first need \( f'_x (x, y) \) and its value at the point.

\[
 f'_x (x, y) = \frac{2x}{y^3} \quad \Rightarrow \quad f'_x (2, 5) = \frac{4}{625} > 0
\]

So, the partial derivative with respect to \( x \) is positive and so if we hold \( y \) fixed the function is increasing at \((2, 5)\) as we vary \( x \).

(b) If we allow \( y \) to vary and hold \( x \) fixed.

For this part we will need \( f'_y (x, y) \) and its value at the point.

\[
 f'_y (x, y) = -\frac{3x^2}{y^4} \quad \Rightarrow \quad f'_y (2, 5) = -\frac{12}{625} < 0
\]

Here the partial derivative with respect to \( y \) is negative and so the function is decreasing at \((2, 5)\) as we vary \( y \) and hold \( x \) fixed.

Note that it is completely possible for a function to be increasing for a fixed \( y \) and decreasing for a fixed \( x \) at a point as this example has shown. To see a nice example of this take a look at the following graph.
This is a graph of a hyperbolic paraboloid and at the origin we can see that if we move in along the $y$-axis the graph is increasing and if we move along the $x$-axis the graph is decreasing. So it is completely possible to have a graph both increasing and decreasing at a point depending upon the direction that we move. We should never expect that the function will behave in exactly the same way at a point as each variable changes.

The next interpretation was one of the standard interpretations in a Calculus I class. We know from a Calculus I class that $f'(a)$ represents the slope of the tangent line to $y = f(x)$ at $x = a$. Well, $f_x(a,b)$ and $f_y(a,b)$ also represent the slopes of tangent lines. The difference here is the functions that they represent tangent lines to.

Partial derivatives are the slopes of traces. The partial derivative $f_x(a,b)$ is the slope of the trace of $f(x,y)$ for the plane $y = b$ at the point $(a,b)$. Likewise the partial derivative $f_y(a,b)$ is the slope of the trace of $f(x,y)$ for the plane $x = a$ at the point $(a,b)$.

**Example 2** Find the slopes of the traces to $z = 10 - 4x^2 - y^2$ at the point $(1,2)$.

**Solution**
We sketched the traces for the planes $x = 1$ and $y = 2$ in a previous section and these are the two traces for this point. For reference purposes here are the graphs of the traces.
Next we’ll need the two partial derivatives so we can get the slopes.

\[ f_x (x, y) = -8x \quad \quad \quad f_y (x, y) = -2y \]

To get the slopes all we need to do is evaluate the partial derivatives at the point in question.

\[ f_x (1, 2) = -8 \quad \quad \quad f_y (1, 2) = -4 \]

So, the tangent line at \((1, 2)\) for the trace to \(z = 10 - 4x^2 - y^2\) for the plane \(y = 2\) has a slope of -8. Also the tangent line at \((1, 2)\) for the trace to \(z = 10 - 4x^2 - y^2\) for the plane \(x = 1\) has a slope of -4.

Finally, let’s briefly talk about getting the equations of the tangent line. Recall that the equation of a line in 3-D space is given by a vector equation. Also to get the equation we need a point on the line and a vector that is parallel to the line.

The point is easy. Since we know the \(x\)-\(y\) coordinates of the point all we need to do is plug this into the equation to get the point. So, the point will be,

\[ (a, b, f (a, b)) \]

The parallel (or tangent) vector is also just as easy. We can write the equation of the surface as a vector function as follows,

\[ \vec{r} (x, y) = \langle x, y, z \rangle = \langle x, y, f (x, y) \rangle \]

We know that if we have a vector function of one variable we can get a tangent vector by differentiating the vector function. The same will hold true here. If we differentiate with respect to \(x\) we will get a tangent vector to traces for the plane \(y = b\) (i.e. for fixed \(y\)) and if we differentiate with respect to \(y\) we will get a tangent vector to traces for the plane \(x = a\) (or fixed \(x\)).
So, here is the tangent vector for traces with fixed \( y \).
\[
\vec{r}_x(x, y) = \langle 1, 0, f_x(x, y) \rangle
\]

We differentiated each component with respect to \( x \). Therefore the first component becomes a 1 and the second becomes a zero because we are treating \( y \) as a constant when we differentiate with respect to \( x \). The third component is just the partial derivative of the function with respect to \( x \).

For traces with fixed \( x \) the tangent vector is,
\[
\vec{r}_y(x, y) = \langle 0, 1, f_y(x, y) \rangle
\]

The equation for the tangent line to traces with fixed \( y \) is then,
\[
\vec{r}(t) = \langle a, b, f(a, b) \rangle + t \langle 1, 0, f_y(a, b) \rangle
\]
and the tangent line to traces with fixed \( x \) is,
\[
\vec{r}(t) = \langle a, b, f(a, b) \rangle + t \langle 0, 1, f_x(a, b) \rangle
\]

**Example 3** Write down the vector equations of the tangent lines to the traces to \( z = 10 - 4x^2 - y^2 \) at the point \( (1, 2) \).

**Solution**
There really isn’t all that much to do with these other than plugging the values and function into the formulas above. We’ve already computed the derivatives and their values at \( (1, 2) \) in the previous example and the point on each trace is,
\[
(1, 2, f(1, 2)) = (1, 2, 2)
\]

Here is the equation of the tangent line to the trace for the plane \( y = 2 \).
\[
\vec{r}(t) = \langle 1, 2, 2 \rangle + t \langle 1, 0, -8 \rangle = \langle 1 + t, 2, 2 - 8t \rangle
\]

Here is the equation of the tangent line to the trace for the plane \( x = 1 \).
\[
\vec{r}(t) = \langle 1, 2, 2 \rangle + t \langle 0, 1, -4 \rangle = \langle 1, 2 + t, 2 - 4t \rangle
\]