Preface

Here are my online notes for my Calculus III course that I teach here at Lamar University. Despite the fact that these are my “class notes”, they should be accessible to anyone wanting to learn Calculus III or needing a refresher in some of the topics from the class.

These notes do assume that the reader has a good working knowledge of Calculus I topics including limits, derivatives and integration. It also assumes that the reader has a good knowledge of several Calculus II topics including some integration techniques, parametric equations, vectors, and knowledge of three dimensional space.

Here are a couple of warnings to my students who may be here to get a copy of what happened on a day that you missed.

1. Because I wanted to make this a fairly complete set of notes for anyone wanting to learn calculus I have included some material that I do not usually have time to cover in class and because this changes from semester to semester it is not noted here. You will need to find one of your fellow class mates to see if there is something in these notes that wasn’t covered in class.

2. In general I try to work problems in class that are different from my notes. However, with Calculus III many of the problems are difficult to make up on the spur of the moment and so in this class my class work will follow these notes fairly close as far as worked problems go. With that being said I will, on occasion, work problems off the top of my head when I can to provide more examples than just those in my notes. Also, I often don’t have time in class to work all of the problems in the notes and so you will find that some sections contain problems that weren’t worked in class due to time restrictions.

3. Sometimes questions in class will lead down paths that are not covered here. I try to anticipate as many of the questions as possible in writing these up, but the reality is that I can’t anticipate all the questions. Sometimes a very good question gets asked in class that leads to insights that I’ve not included here. You should always talk to someone who was in class on the day you missed and compare these notes to their notes and see what the differences are.

4. This is somewhat related to the previous three items, but is important enough to merit its own item. THESE NOTES ARE NOT A SUBSTITUTE FOR ATTENDING CLASS!! Using these notes as a substitute for class is liable to get you in trouble. As already noted not everything in these notes is covered in class and often material or insights not in these notes is covered in class.
Higher Order Partial Derivatives

Just as we had higher order derivatives with functions of one variable we will also have higher order derivatives of functions of more than one variable. However, this time we will have more options since we do have more than one variable.

Consider the case of a function of two variables, \( f(x, y) \) since both of the first order partial derivatives are also functions of \( x \) and \( y \) we could in turn differentiate each with respect to \( x \) or \( y \). This means that for the case of a function of two variables there will be a total of four possible second order derivatives. Here they are and the notations that we’ll use to denote them.

\[
\begin{align*}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} \\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x \partial y} \\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} \\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}
\end{align*}
\]

The second and third second order partial derivatives are often called mixed partial derivatives since we are taking derivatives with respect to more than one variable. Note as well that the order that we take the derivatives in is given by the notation for each these. If we are using the subscripting notation, e.g. \( f_{xy} \), then we will differentiate from left to right. In other words, in this case, we will differentiate first with respect to \( x \) and then with respect to \( y \). With the fractional notation, e.g. \( \frac{\partial^2 f}{\partial y \partial x} \), it is the opposite. In these cases we differentiate moving along the denominator from right to left. So, again, in this case we differentiate with respect to \( x \) first and then \( y \).

Let’s take a quick look at an example.

**Example 1** Find all the second order derivatives for \( f(x, y) = \cos(2x) - x^2 e^{5y} + 3y^2 \).

**Solution**

We’ll first need the first order derivatives so here they are.

\[
\begin{align*}
f_x(x, y) &= -2 \sin(2x) - 2xe^{5y} \\
f_y(x, y) &= -5x^2 e^{5y} + 6y
\end{align*}
\]

Now, let’s get the second order derivatives.
\[ f_{xx} = -4\cos(2x) - 2e^{5y} \]
\[ f_{xy} = -10xe^{5y} \]
\[ f_{yx} = -10xe^{5y} \]
\[ f_{yy} = -25x^2e^{5y} + 6 \]

Notice that we dropped the \((x, y)\) from the derivatives. This is fairly standard and we will be doing it most of the time from this point on. We will also be dropping it for the first order derivatives in most cases.

Now let’s also notice that, in this case, \( f_{xy} = f_{yx} \). This is not by coincidence. If the function is “nice enough” this will always be the case. So, what’s “nice enough”? The following theorem tells us.

**Clairaut’s Theorem**

Suppose that \( f \) is defined on a disk \( D \) that contains the point \((a, b)\). If the functions \( f_{xy} \) and \( f_{yx} \) are continuous on this disk then,

\[ f_{xy}(a, b) = f_{yx}(a, b) \]

Now, do not get too excited about the disk business and the fact that we gave the theorem for a specific point. In pretty much every example in this class if the two mixed second order partial derivatives are continuous then they will be equal.

**Example 2** Verify Clairaut’s Theorem for \( f(x, y) = xe^{-x^2y^2} \).

**Solution**

We’ll first need the two first order derivatives.

\[ f_x(x, y) = e^{-x^2y^2} - 2x^2y^2e^{-x^2y^2} \]
\[ f_y(x, y) = -2xy^3e^{-x^2y^2} \]

Now, compute the two mixed second order partial derivatives.

\[ f_{xy}(x, y) = -2xy^2e^{-x^2y^2} - 4x^2y^3e^{-x^2y^2} + 4x^4y^3e^{-x^2y^2} = -6x^2y^2e^{-x^2y^2} + 4x^4y^3e^{-x^2y^2} \]
\[ f_{yx}(x, y) = -6x^2y^2e^{-x^2y^2} + 4x^4y^3e^{-x^2y^2} \]

Sure enough they are the same.

So far we have only looked at second order derivatives. There are, of course, higher order derivatives as well. Here are a couple of the third order partial derivatives of function of two variables.
\[ f_{xy} = (f_y)_x = \frac{\partial}{\partial x}\left( \frac{\partial^2 f}{\partial y \partial x} \right) = \frac{\partial^3 f}{\partial x^2 \partial y} \]

\[ f_{yx} = (f_x)_y = \frac{\partial}{\partial y}\left( \frac{\partial^2 f}{\partial x \partial y} \right) = \frac{\partial^3 f}{\partial x^2 \partial y} \]

Notice as well that for both of these we differentiate once with respect to \( y \) and twice with respect to \( x \). There is also another third order partial derivative in which we can do this, \( f_{xxy} \). There is an extension to Clairaut’s Theorem that says if all three of these are continuous then they should all be equal,

\[ f_{xxy} = f_{xyx} = f_{yxx} \]

To this point we’ve only looked at functions of two variables, but everything that we’ve done to this point will work regardless of the number of variables that we’ve got in the function and there are natural extensions to Clairaut’s theorem to all of these cases as well. For instance,

\[ f_{xz}(x, y, z) = f_{zx}(x, y, z) \]

provided both of the derivatives are continuous.

In general, we can extend Clairaut’s theorem to any function and mixed partial derivatives. The only requirement is that in each derivative we differentiate with respect to each variable the same number of times. In other words, provided we meet the continuity condition, the following will be equal

\[ f_{ssrssr} = f_{rrssrr} \]

because in each case we differentiate with respect to \( t \) once, \( s \) three times and \( r \) three times.

Let’s do a couple of examples with higher (well higher order than two anyway) order derivatives and functions of more than two variables.

**Example 3** Find the indicated derivative for each of the following functions.

(a) Find \( f_{xyzz} \) for \( f(x, y, z) = z^3y^2 \ln(x) \)  

(b) Find \( \frac{\partial^3 f}{\partial y \partial x^2} \) for \( f(x, y) = e^{xy} \)

**Solution**

(a) Find \( f_{xyzz} \) for \( f(x, y, z) = z^3y^2 \ln(x) \)

In this case remember that we differentiate from left to right. Here are the derivatives for this part.

\[
\begin{align*}
 f_x & = \frac{z^3y^2}{x} \\
 f_{xx} & = -\frac{z^3y^2}{x^2} \\
 f_{xy} & = -\frac{2z^3y}{x^2} \\
 f_{xxy} & = \frac{2z^3y}{x^2}
\end{align*}
\]
\[
\begin{align*}
    f_{xyz} &= -\frac{6z^2y}{x^2} \\
    f_{xyyz} &= -\frac{12zy}{x^2}
\end{align*}
\]

(b) Find \( \frac{\partial^3 f}{\partial y \partial x^2} \) for \( f(x, y) = e^{xy} \)

Here we differentiate from right to left. Here are the derivatives for this function.
\[
\begin{align*}
    \frac{\partial f}{\partial x} &= ye^{xy} \\
    \frac{\partial^2 f}{\partial x^2} &= y^2 e^{xy} \\
    \frac{\partial^3 f}{\partial y \partial x^2} &= 2ye^{xy} + xy^2 e^{xy}
\end{align*}
\]